DETERMINATION OF VERTICAL DEFLECTIONS USING
THE GLOBAL POSITIONING SYSTEM AND GEODETIC LEVELING

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Abstract: The capabilities of the Global Positioning System (GPS) for accurately determining geodetic quantities are well established. Nevertheless, no comparison between deflections of the vertical as determined through GPS with leveling and values conventionally computed by astrogeodetic methods has yet been published. This investigation demonstrates that accurate deflections of the vertical components ($\eta, \zeta$) can be obtained independently of classical astronomical observations by combining geodetic leveling with satellite GPS positioning. The approach uses a radial configuration of baselines to determine the "best" (in a least squares sense) values of ($\eta, \zeta$) at the central station.

Introduction

In recent years, several investigators have used GPS techniques and methods to determine geometric quantities, primarily baselines and ellipsoidal (geodetic) height differences [Engelis et al., 1984; Bock et al., 1985]. In conjunction with geodetic leveling, some computations of physical parameters such as geoid-ellipsoid separation (denoted $N_e$ and also referred to as undulation or geoid height) have been published [Engelis et al., 1985; Denker and Wenzel, 1987]. The major drawback with analyses of this type is that a set of independent "true" geoid heights are needed as a source of comparison. Unfortunately, undulations cannot be directly observed, and the gravimetrically computed values are not as accurate as the leveled orthometric heights or the GPS determined ellipsoidal heights. Relative gravimetric geoid undulations obtained using standard computational methods (geopotential models and Stokes' integration, Fast Fourier Transforms (FFT) and/or least squares collocation) and terrestrial gravity are at best accurate to a few centimeters. Estimates are 2.5 to 5 ppm (parts-per-million) of the baseline distance depending on the number of refinements incorporated [Kearsley, 1988]. However, recent advancements in GPS instrumentation and software provide repeatability of relative ellipsoidal heights of about 1 part in 10^7 [Dong and Bock, 1989]. Orthometric height differences are measured even more precisely with high order geodetic leveling. This one order of magnitude improvement of observed over gravimetrically predicted quantities becomes more critical when small angles such as vertical deflections are analyzed.

The deflection of the vertical is the angle at the point of observation between the normal to the adopted reference surface (generally an ellipsoid of revolution) and the direction of the plumb line. Like the undulation, it can be predicted using gravimetric methods (e.g., Vening Meinesz's equations complemented by earth gravitational models). Nevertheless, astrogeodetic methods are considered more accurate. When deflections of the vertical are predicted using gravity and the most modern and sophisticated algorithms, the reported agreement with the observed astrogeodetic deflections is not better than 1" [Kearsley et al., 1985]. Furthermore, to attain these results extensive gravity coverage and regional detailed digitized topographic height information are imperative.

The full relationship between the GPS determined local geodetic frames and the "natural", directly observed, local astronomic frames, is an important problem (e.g., it establishes unequivocally the direction of the gravity vector). The three differential rotations involved in this transformation, respectively around the first (pointing east), second (north), and third (zenith) axes, are the two components of the deflection of the vertical along the geodetic meridian and prime vertical, denoted respectively $\xi$ and $\eta$, and the so called Laplace's correction (i.e., the difference between the astronomic and geodetic azimuths). Soler et al. [1988] compared GPS and astronomically determined azimuths and concluded that GPS determined geodetic azimuths can be accurately converted to astronomic azimuths once the deflections of the vertical are known. See also Evans et al., [1989].

This paper compares, for the first time, vertical deflections determined by GPS and spirit leveling with classical astrogeodetic deflections. The investigation demonstrates the feasibility of determining deflections of the vertical at least as accurate as astrogeodetic deflections although completely independent of stellar observations. A major advantage of this alternative procedure will be the reduction of time spent in the field, thus, increasing cost effectiveness. GPS is an all weather operation and its practicality can be dramatically increased once kinematic surveys are perfected. Also, relative leveling requirements may already exist or can be measured in a matter of hours if an appropriate short radial network is established.

Accurate and uncomplicated determination of vertical deflections, when done periodically, may become a significant tool in tectonic and geo-

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dynamic studies. They complement the three-dimensional representation of regional crustal distortions. Repeated GPS baselines could define the area's strain field. Comparison of differences in ellipsoidal and/or orthometric heights will provide the uplift or subsidence of the region. Finally, changes in the deflection components ($\eta, \xi$) are the only geodetic parameters able to disclose the direction of possible shallow underground mass displacements.

Test site and data collection

The test area selected for this feasibility study is located at the Naval Surface Warfare Center (NSWC) in Dahlgren, VA. The planning, observational schedule, and data collection were completed through the joint efforts of the NSWC, the National Geodetic Survey (NGS), and the Defense Mapping Agency (DMA). In addition to GPS and leveling observations, conventional terrestrial measurements of gravity, distances (using Electronic Distance Measurement Instruments, i.e., EDMs) and astronomical positions and azimuths (using first order theodolites) were also performed within the test area [Evans et al., 1989].

Eight days of GPS data were collected starting on August 17, 1987. Five Texas Instrument TI4100 geodetic receivers, with the GEodetic Satellite Receiver (GESAR) operating software, were employed to obtain both pseudorange and dual carrier phase measurements on L1 and L2 frequencies. This equipment tracks up to four satellites simultaneously. In order to optimize the station-satellite geometric configuration an observation span of six hours, which allowed a total of six different satellites to be observed, was implemented. Specifically, satellites with Pseudo-Random-Noise (PRN) code numbers 3, 6, 8, 11, 12, and 13 were utilized. With the advent of a full GPS constellation and new instruments which can track all satellites in view, the observation time span per session can possibly be reduced considerably. Figure 1 depicts the four GPS baselines used in this investigation. They range in length from 1.2 to 4.8 km and radiate from the central reference station MBRE where the deflections were sought. Bench marks at individual baseline end points were connected by double-run first order spirit leveling. The leveling circuit was surveyed on Aug. 28-29 by NGS personnel and is shown with dotted lines in Figure 1. The total loop distance is 18.830 km and the misclosure 0.14 mm. Classical measurements of astronomic longitude ($\lambda$) and latitude ($\phi$) at station MBRE required to compute the astrogeodetic vertical deflections were observed with a Wild T4 universal theodolite by DMA's Geodetic Survey Squadron (GSS) from Cheyenne, WY, in April 1988.

Methodology

The mathematical model selected herein is conceptually very simple. The fundamental aim is to establish a functional relationship between the unknown parameters ($\eta, \xi$) and the GPS and leveling observables.

The undulation total differential can be written

$$\frac{dN_q}{N_q} = \frac{\partial N_q}{\partial \lambda} d\lambda + \frac{\partial N_q}{\partial \phi} d\phi$$

(1)

where $\lambda$ and $\phi$ are, respectively, the geodetic longitude and latitude. The components of the deflection of the vertical ($\eta, \xi$) when referred to an ellipsoid of revolution of semiaxial axes $a$ and flattening $f$ are defined by the following:

$$\eta = -\frac{1}{(N+h)\cos\phi} \frac{\partial N_q}{\partial \lambda}$$

(2)

$$\xi = -\frac{1}{(N+h)\phi} \frac{\partial N_q}{\partial \phi}$$

(3)

where

$N = a/W$, principal radius of curvature in the

![Fig. 1. GPS baselines and leveling loop.](image-url)
plane of the prime vertical; 
\[ M = a(1-e^2)^{\lambda^2}, \text{ principal radius of curvature in} \]
the plane of the meridian; 
\[ \phi = (1-e^2 \sin^2 \phi)^{1/2}, \]
\[ e^2 = 2c^2. \]
Substituting the partial derivatives from (2) and 
(3) into (1), and assuming \( \beta = H \) (this equality is 
not rigorously correct but the errors introduced 
are negligible), where \( H \) is the leveled orthometric 
height, we find the relationship 
\[ dh - dH = (N + h) \cos \lambda \Delta \eta - (N + h) \Delta \phi \xi. \]  
(4)

Expression (4) can be written in its implicit 
form \( F(X, L, 0) = 0 \) and thus be adopted as a general 
least squares mathematical model, where \( dw, dh, d\lambda \) 
and \( d\phi \) are the unknowns (L), and \( \eta \) and \( \xi \) 
the unknowns (X). Several advantages are implicit in 
the above model. One is its linearity on the 
parameters; consequently, iterations are unnecessary. 
In addition, because normal heights (H' as 
opposed to H) were used, \( N \) really represents 
the height anomaly \( \xi \) (\( \xi \neq H' \)), and the deflections 
are obtained at the earth's surface and not on the 
geoid. Hence, the unreduced one to one comparisons with 
astrogateodetic deflections are feasible. Notice 
that \( \Delta \lambda, \Delta \phi, dh \), denote the differences in geodetic 
coordinates between every end point in the radial 
network and the central station. These values are 
strictly geometric quantities and are determined through 
GPS observations. Finally, \( dh \) is the 
difference in observed leveled heights between the 
two ends of a radial baseline. This is a physical 
quantity defining approximately the separation 
between the geopotential surfaces of the two 
stations at the end point.

Because a least squares estimation is invoked, in 
actuality the best "average" values of \( (\eta, \xi) \) at 
the central station in terms of the height anomaly 
differences around it were estimated. Alternately, 
the components of the deflection of the 
vertical along the E-W and N-S directions are 
calculated in terms of the deflections along arbitrary 
azimuths. Clearly, the computed deflections are 
closer to the astrogateodetic "point value" for 
shorter baselines. The main factors affecting the 
accuracy of the determined deflections as estimates 
of "point" rather than "average" deflections 
are the length of the baselines and the terrain 
roughness.

Results

All GPS solutions were obtained using precise 
post-fitted ephemeris derived at NSWC and 
referred to the World Geodetic System of 1984 (WGS84). 
Geocentric point positions of the central station 
MBRE were determined using pseudoranges, which 
avoid the integer cycle ambiguity problem inherent 
in phase observables, and the sequential filter 
algorithm developed at NSWC [Hermann, NSWC Internal 
Report, 1988]. Table 1 shows the "natural" 
("true" observed physical quantities) and geodetic 
(strictly geometric) coordinates of the base 
station MBRE referred to the Geodetic Reference System 
1980 (GRS80) ellipsoid.

Table 1 suggests that absolute determination of 
curvilinear "datum" coordinates can be currently 
determined with GPS, hence minimizing weather 
constraints, about ten times more accurately than 
with classical astronomical observations. From 
the tabulated values the two components of the astro-

<table>
<thead>
<tr>
<th>Natural Coordinates</th>
<th>Geodetic coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta ) 282° 58' 30.48E0°.5</td>
<td>( \lambda ) 282° 58' 23.86E0°.04</td>
</tr>
<tr>
<td>( \phi ) 38° 19' 33.71E0°.5</td>
<td>( \phi ) 38° 19' 36.23E0°.03</td>
</tr>
<tr>
<td>H (8.560E0.01) m</td>
<td>h (-25.4E1.0) m</td>
</tr>
</tbody>
</table>

\( ^a \) Determined through astronomical (A, \( \Phi \)) and leveling (H) observations. 
\( ^b \) Determined through GPS observations. Refer to the 
GRS80 ellipsoid and WGS84 coordinate system.

géodetic deflection of the vertical can be derived using 
the well known expressions 
\[ \eta = (A - \lambda) \cos \phi ; \quad \xi = \phi - \phi. \]  
(5)

Several assumptions are implicit in the above 
equations. The Cartesian geodetic coordinate system 
\( (u, v, w) \) associated with the GRS80 ellipsoid and to 
which \( \lambda \) and \( \phi \) refer, coincides or is parallel to 
the GPS realized WGS84 geocentric coordinate system 
\( (x, y, z) \). This \( (x, y, z) \) reference frame must be 
equal to the one implied in the reduction of 
aromatic observations, thus these must be 
accurately corrected for polar motion and UT1-UTC. 
The linear components of \( \eta \) and \( \xi \) refer to a local geodetic 
frame \( (e, n, u) \), being respectively positive 
toward east and north. Mathematical relationships 
between these three coordinate systems are reviewed 
by Soler and Hothen [1988].

The observable, \( dh \), is a byproduct of relative 
position solutions of the four remote stations 
(S02M, CHUR, HERD and SHKB) with respect to MBRE. 
They were determined using carrier phase measure-
ments and the PHASER software developed at NSWC which 
is based on a non-differenced phase model [Goad, 
1985]. The geocentric position of the satellites 
was constrained to the ephemeris values, an 
assumption not very crucial considering the short 
length of the baselines. All session reductions were 
obtained from multi-baseline solutions using single 
frequency (LL) measurements sampled every 30 
seconds and constraining the ambiguity biases to 
integers. A total of ten observation equations 
(i.e., eight degrees of freedom) corresponding to 
different combinations of repeated baselines and 
sessions were included in the adjustment. The 
standard errors for \( \Delta \lambda, \Delta \phi, \) and \( dh \) can be 
computed from the original GPS results (i.e., \( \sigma_{\lambda}, \sigma_{\phi}, \sigma_{dh} \)) 
after propagating errors. However, although no observations were rejected some \( dh \) observables were 
weighted differently according to their repeata-
ability. The final standard errors applied to \( dh \) 
ranged from 10 mm (3 observations) to 2 mm (3 obser-
vations) with a mean of 4.5 mm for the 10 observa-
tions. The sigmas for \( dh \) observables were 
computed according to the formula \( \text{rms}\)K where \( \text{K} \) 
is the length of the leveled line in km. The adjust-
ment "a posteriori" standard error of unit weight 
was 0.96, which is very close to the expected value of 
1.0.

The final derived deflection components are 
shown in Table 2. Notice the remarkable agreement 
between the two sets of independently computed vertical 
deflections. Moreover, the standard errors for the 
astrogateodetic \( \eta \) and \( \xi \) are about four times the 
values obtained in the least squares adjustment for 
the combined GPS and leveling method. This may 
indicate that the standard errors estimated from
TABLE 2. Vertical deflection components at MBRE.

<table>
<thead>
<tr>
<th></th>
<th>Astrogeodetic</th>
<th>GPS &amp; leveling</th>
<th>Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>η</td>
<td>5°.1920°.5</td>
<td>5°.2820°.10</td>
<td>-0°.09</td>
</tr>
<tr>
<td>ξ</td>
<td>-2°.5820°.5</td>
<td>-2°.7620°.14</td>
<td>0°.18</td>
</tr>
</tbody>
</table>

first-order astronomic observations (see Table 1) are generally much smaller than the often quoted value of 0°.5. Due to the orientation of the baseline (one is located south of MBRE) the determined value of ξ is less reliable than that of η. Even with this limitation, the results corroborated the potential of GPS and leveling for this type of computation. To simplify future operational procedures as much as possible, only four or five baselines oriented quasi-symmetrically along the four quadrants will suffice to provide a strong geometry. One important factor, though, is to achieve repeatability of ellipsoidal height differences. Three several independent sessions (a minimum of three) should be scheduled for each baseline arrangement in order to detect possible outliers (e.g., errors in measured instrument height) or discern the proper weighting scheme for the observables of each individual baseline. The only observational errors which directly affect the solutions are errors in the measured height differences dh and dH. Errors as large as 10 m in horizontal position (λ, φ) caused insignificant changes in the parameters (η, ξ) or their variances. It is extremely difficult to simulate astrogeodetic deflections consistent with a set of given orthometric heights and GPS observables, thus an exhaustive error analysis is precluded. Based strictly on the geometry and observations of this particular survey, it was estimated that if the standard errors of all dh observations, as originally used in the adjustment, were systematically increased by 1 cm it would still be possible to recover the deflections at the 0°.4 level, which is well inside the predicted accuracy of presently applied methods.

Summary

Excellent agreement between astrogeodetic and GPS and leveling determined deflections of the vertical is reported for this feasibility test. Thus, the conclusion is reached that we can expand still further the geodetic applications of the GPS. What to date was a complicated and tedious astronomic operation, requiring inherent skills and limiting considerably the number of deflection-control points, can now be replaced by simple procedures and concepts based on new GPS techniques and standard spirit leveling methods.

The area where this test was performed is not mountainous. Thus, it may be argued that the reported agreement is a consequence of the local smoothness of the geoid and may not be a valid indication of the accuracy to be expected under more typical conditions. However, astrogeodetic deflections were also computed at CHUR (4.8 km from MBRE and 27 m higher) and differences of 1°.18 and -0°.48 respectively in η and ξ with respect to the values at MBRE were noted. These changes in vertical deflections from point to point are significant and much higher than the differences presented in Table 2 between the two observational methods. Encouraged by the results of this study, more tests in different geographic regions with varied terrain characteristics are clearly needed to rigorously compare the two methods at the 0°.2 level.

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