NOAA Technical Report NOS 67 NGS 3

# Algorithms for Computing the Geopotential Using a Simple-Layer Density Model

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U.S. DEPARTMENT OF COMMERCE
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# Algorithms for Computing the Geopotential Using a Simple-Layer Density Model

Foster Morrison

Geodetic Research and Development Laboratory
National Geodetic Survey

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March 1977

**U.S. DEPARTMENT OF COMMERCE**
**Juanita M. Kreps, Secretary**
**National Oceanic and Atmospheric Administration**
Robert M. White, Administrator
National Ocean Survey
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# Algorithms for Computing the Geopotential Using a Simple-Layer Density Model

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ABSTRACT.—Several algorithms have been developed for computing the gravitational attraction of a simple-density layer; these are numerical integration, Taylor series, and mixed analytic and numerical integration of a special approximation. A computer program has been written to combine these techniques for computing the higher frequency components of the gravitational acceleration of an artificial Earth satellite. A total of 1640 equal-area, constant surface density ($5°\times5°$) blocks on an oblate spheroid is used. The special approximation is used in the subsatellite region, Taylor series in a surrounding zone, and numerical quadrature in the remaining regions. The relative sizes of these zones are readily changed. An auxiliary program can generate all the parameters for different equal-area block configurations. Different orders may be used in the numerical quadrature done in connection with the special approximation. Numerical tests comprising integrations of equations of satellite motion and static gravity simulations indicate the simple-density layer model is not only feasible, but highly practical and very easy to use.

## I. INTRODUCTION

Choosing to use a density layer model to represent a gravitational potential field does not specify an algorithm for the computation. Using an analytic solution to a boundary value problem, such as spherical or spheroidal harmonics, specifies the general outline of the algorithm. Quite a bit of discretion remains to the user as to what recursion relations (if any) to apply and what normalization is most efficient: a considerable literature has developed through inquiries into these questions.

Density layer models provide us with many more options. For one thing, we must specify the surface, or boundary, upon which the density is to be defined. The shape of the surface is related to the choice of coordinates, which is another option. Finally, there is a choice of methods of representing the density on the surface. Intricate surfaces and exotic coordinates both would require more information storage and computation to obtain the gravity vector at a specified point. The choice of analytic or discrete representations of the density upon the surface will affect the efficiency and speed of one's calculations, also.

To a large extent, the application of the gravity model is a decisive factor in its choice. A model suitable for the combination of gravimetry, and satellite derived data, which may be dynamical, geometric, or both, might be too cumbersome for computing the ephemeris of a satellite.

## II. METHODS IN USE

Working directly from optical satellite observations, Koch and Morrison (1970) derived a density layer model for the geopotential. In theory, the surface layer was on the Earth's surface, while in fact the mathematical model was equivalent to point masses fixed onto an approximate surface (Morrison 1971). Improved solutions (Koch and Witte 1971), and (Koch 1974) utilizing additional data have retained this type of algorithm.

Vinti (1971), on the other hand, has used a density layer model with a surface consisting of the smallest sphere enclosing the Earth. Vinti's model avoids the occurrence of impulses on satellite orbits coming close to any of the point masses, but it is computationally no different from using the spherical harmonic model for the geopotential. Moreover, having the surface layer above the Earth's actual surface yields a model unsuitable for computing mean gravity anomalies at the surface or for doing combination solutions.

## III. APPROXIMATION BY TAYLOR SERIES

The fundamental formula for density layer models is

$$U=G\int_{\sigma}\int \frac{\chi d\sigma}{r^*} \tag{1}$$

where $\mathbf{r}^*=\mathbf{r}-\mathbf{r}_s$ (See appendix I for explanation of undefined terms). If the surface, $\sigma$, is a sphere, we may use

$$d\sigma=r^2\cos\phi d\lambda d\phi$$

and compute (1) as an iterated integral

$$U=G\int_0^{2\pi}\int_{-\pi}^{+\pi}\frac{\chi(\phi,\lambda)r^2\cos\phi d\lambda d\phi}{r^*}. \tag{2}$$

Even if the surface is not exactly a sphere, (2) may be used provided the surface is starlike with respect to the origin. (For a starlike surface a straight line starting at the origin intersects the surface only once, i.e., the radial coordinate of the surface is a single valued function of $\phi$ and $\lambda$.) Even if the surface has high frequency ripples and a resulting very large surface area, the total mass of the surface will be very little affected because the surface density is divided by the slope of the topography. An analogous expression could be derived for spheroidal coordinates. Another alternative is to remove $r^2$ from the numerator of (2) and use density per unit of solid angle: the position of the layer would enter only through $r^*$. For the numerical tests made, only (2) was used.

Given (2), there are various options for expressing the integrand. The simplest, perhaps, is to write

$$h(\phi,\lambda)=\frac{G\chi(\phi,\lambda)r^2\cos\phi}{r^*} \tag{3}$$

and expand $h$ into a Taylor series. (See appendix II.) Since the domain of validity of such a series would be restricted, expansions would have to be done within different regions on the Earth, usually tesserae bounded by lines of latitude and longitude. The result is of the form

$$U=\sum_i(\Delta T)_i+U_0 \tag{4}$$

$$(\Delta T)_i\cong\int\int[h_0+h_\phi\phi+h_\lambda\lambda+\tfrac{1}{2}(h_{\phi\phi}\phi^2+2h_{\phi\lambda}\phi\lambda+h_{\lambda\lambda}\lambda^2)]d\phi d\lambda \tag{5}$$

if we truncate the series at the second-order terms. The function $U_0$ is a spherical harmonic expansion used for the central force term, oblateness and long-wavelength parts of the potential. Eq. (5), an integral of a polynomial, is evaluated quite easily. The derivatives are somewhat complicated, but straightforward.

To obtain the gravity force, the gradient operator is applied

$$\mathbf{g}=\nabla U, \tag{6}$$

$$\nabla=\left(\mathbf{i}\frac{\partial}{\partial x_s}+\mathbf{j}\frac{\partial}{\partial y_s}+\mathbf{k}\frac{\partial}{\partial z_s}\right)^T$$

Numerical tests have been made at altitudes of 300 and 1000 km, using 15°×15° and 5°×5° patches. Results are satisfactory for the 300-km altitude only when the smaller patches are used. This is due to the fact that the Taylor series has many properties of the point mass. Specifically, these properties are

$$h_0=O(1/r^*)$$

$$h_\phi=O(1/r^{*2}),$$

$$h_{\phi\phi}=O(1/r^{*3}),\text{ etc.}$$

Nevertheless, the method is fast and accurate for altitudes above 1000 km and softens the impulses of a purely point mass approximation.

Spherical coordinates are used for the sake of rapid computation, and for the area differential the spherical approximation

$$d\sigma=r^2\cos\phi\, d\lambda d\phi+O(e^2)$$

suffices. Simplifications occur if $h$ is expanded about a point whose latitude is the average of the north and south edges of the block and similarly for the longitude. (See fig. 1.) When $h$ is expanded to the second order and integrated. the odd order terms drop out

$$(\Delta T)_i = \phi_H \lambda_H \left[ h_0 + \frac{1}{24} \left( h_{\phi\phi} \phi^2_H + h_{\lambda\lambda} \lambda^2_H \right) \right] \qquad (7)$$

where $\phi_H$ and $\lambda_H$ are the extent of the block in latitude and longitude.

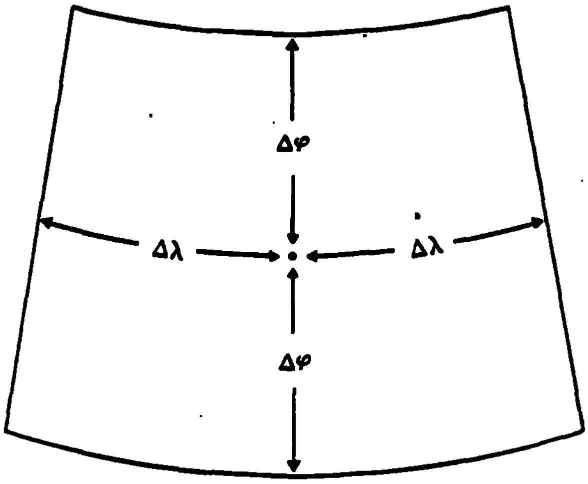


FIGURE 1.—The block over which ΔT is evaluated. The origin is in the "center."

## IV. THE METHOD OF SINGULARITY-MATCHING

The limitations of the Taylor series approximations to $U$ are caused by the presence of the square root function in the denominator of (2)

$$r^* = \sqrt{(\mathbf{r} - \mathbf{r}_s)^2}.$$

here $(\mathbf{r} - \mathbf{r}_s)^2$ is an analytic function of the coordinates everywhere. but $r^*$ is not at $r^* = 0$. and $1/r^*$ is not well approximated by Taylor series in that region.

The method of singularity-matching provides approximation techniques for computing such things as convergent improper integrals where exact analytic solutions cannot be found. The troublesome integrand $h(x)$ is factored by an elementary function $p(x)$ with the same type of singularities so that one may write

$$h(x) = p(x)\, f(x). \qquad (8)$$

Now $f(x)$ may be expanded as a Taylor series. The desired results may be obtained provided the expressions $x^n p(x)$, $n = 0, \ldots, N$ can be integrated analytically. Often this can be done by repeated application of integration by parts.

Computing the potential of a surface layer is complicated by the facts that (1) is a double integral and. in most cases. $r^*$ is close to but not exactly zero. We begin by converting (1) to an iterated integral in the spherical coordinates $\phi$ and $\lambda$. as was done for a Taylor series expansion. For the factor with a singularity we choose

$$p(\lambda) = (D + E\lambda + F\lambda^2)^{-1/2},$$

where $D$. $E$ and $F$ are functions of $\phi$. For practical applications at satellite altitudes. it is simplest to use the Taylor series expansion for $r^{*2}$ to compute $D, E$ and $F$: then we have

$$f(\lambda) = G\chi r^2(\phi) \cos\phi = \text{"constant"}$$

which greatly simplifies the results. The integration with respect to $\phi$ now may be done by a conventional numerical quadrature. Newton-Cotes formulas of order 3 through 9 were programmed (Abramowitz and Stegun 1964).

All of the integrals with respect to $\lambda$ are of the simple form

$$\int h(\lambda) d\lambda = \int \frac{(A + B\lambda + C\lambda^2) d\lambda}{\sqrt{D + E\lambda + F\lambda^2}} \qquad (9)$$

which can be expressed in elementary function for various values of $D$, $E$ and $F$, e.g.. see Peirce and Foster (1956). It is more efficient to branch to different coding than to compute (9) as a function of a complex variable.

Various improvements and generalizations can be made over what has been programmed. For low altitudes. the matching of the singularity in $p(\lambda)$ and $1/r^*$ should be precise. Gaussian quadrature should be faster and more accurate than Newton-Cotes. Some of the integration with respect to $\phi$ also could be done analytically to save time and reduce error.

Density variations within a block and the departure of the approximation $p(\lambda)$ from $1/r^*$ can be modeled by expanding

$$f(\lambda) = \frac{G\chi r^2 \cos\phi}{p(\lambda) r^*}$$

in a Taylor series. No matter what order Taylor series is used. the integral may be done quite simply (Gradshteyn and Ryzhik 1965. p. 80). so any level of approximation can be achieved. Even surfaces of complex shape could be modeled if it were really necessary.

If we define

$$f = G\chi r^2 \cos\phi \tag{10}$$

$$g = (\mathbf{r} - \mathbf{r}_s)^2$$

$$p = g^{-1/2} \tag{11}$$

and use the approximations

$$f = A + B\lambda + C\lambda^2$$

$$\begin{aligned} A &= f_0 + f_\phi \phi + f_{\phi\phi}\phi^2/2 \\ B &= f_\lambda + f_{\lambda\phi}\phi \\ C &= \tfrac{1}{2} f_{\lambda\lambda} \end{aligned} \tag{12}$$

$$g = D + E\lambda + F\lambda^2$$

$$\begin{aligned} D &= g_0 + g_\phi \phi + g_{\phi\phi}\phi^2/2 \\ E &= g_\lambda + g_{\lambda\phi}\phi \\ F &= \tfrac{1}{2} g_{\lambda\lambda} \end{aligned} \tag{13}$$

we can perform the integration with respect to $\lambda$ in closed form. If a higher approximation to $g$ is required, the integrals will be elliptic and a degree of simplicity will disappear. An alternative to the elliptic integrals is to subdivide the blocks. Doing one of the iterated integrals analytically removes the problem of approximating $1/r^*$ by a Taylor series. The integration with respect to $\phi$ precedes smoothly by numerical quadratures. Some part of the quadrature with respect to $\phi$ could be done analytically.

Numerical tests of the potential function have proved very promising at the 300 km altitude even with 15° patches. These results are far more promising than those obtained with the direct numerical integration algorithm (Koch 1971).

If $F \neq 0$, we may use in evaluating (9)

$$\int \frac{A + B\lambda + C\lambda^2}{\sqrt{D + E\lambda + F\lambda^2}}\, d\lambda \tag{14}$$

$$= \frac{B\sqrt{X}}{F} + C\left[\frac{\lambda}{2F} - \frac{3}{4}\frac{E}{F^2}\right]\sqrt{X} + Q\int \frac{d\lambda}{\sqrt{X}},$$

where $X = D + E\lambda + F\lambda^2$ and

$$Q = A - \frac{BE}{2F} - \frac{(3E^2 - 4DF)C}{8F^2}.$$

If the quantity $q \geq 0$, $q = 4\,DF - E^2$, then

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}} \sinh^{-1}\left(\frac{2F\lambda + E}{\sqrt{q}}\right). \tag{15}$$

To have $q \geq 0$, it is necessary to have $F > 0$, since $D \geq 0$ for this application. If $F < 0$, one can use

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{-F}} \sin^{-1}\left(\frac{-2F\lambda - E}{\sqrt{-q}}\right) \tag{16}$$

It may happen that $F > 0$, but $q < 0$. Such may be the case if $F$ is relatively small and $E$ large. In this case one has

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}} \log\left(\sqrt{X} + \lambda\sqrt{F} + \frac{E}{2\sqrt{F}}\right). \tag{17}$$

These well known formulas (Weast and Selby, 1967; Gradshteyn and Ryzhik, 1965) are sufficient for all the cases that occur in this application, provided $F$ is not too small.

If $F$ is exactly zero, which may happen in an actual computation, (14) simplifies to

$$\int \frac{A + B\lambda + C\lambda^2}{\sqrt{D + E\lambda}}\, d\lambda = \left(\sqrt{D + E\lambda}\right) \tag{18}$$

$$\times \left[\frac{2A}{E} - \frac{2B(2D - E\lambda)}{3E^2} + \frac{2C(8D^2 - 4DE\lambda + 3E^2\lambda^2)}{15E^3}\right]$$

Equation (18) is the application of equations 101, 102, 103 of Peirce and Foster (1956).

When both $E$ and $F$ are very small compared to $D$, (14) is computed best by expanding the denominator in the binomial series and integrating term by term. This procedure is identical in the formal sense to the Taylor series approximation, but there are differences in the numerical application. For one thing, the quantities $D$, $E$, and $F$, required to perform the tests indicating whether (14), (15), (16), (17), or (18) should be applied, are used at once. Also, the binomial series expansion might not be used for every value of the latitude in the second iterated integral, which is done numerically. The results are

$$\begin{aligned} \int_{-\delta\lambda}^{\delta\lambda} (A + B\lambda + C\lambda^2) X^{-1/2}\, d\lambda = D^{-1/2}\Big\{ & 2A\delta\lambda \\ & + \frac{2}{3}\delta\lambda^3\left[C - \frac{1}{2}BE^* + \frac{3}{8}AE^{*2} - \frac{1}{2}AF^*\right] \\ & + \frac{\delta\lambda^5}{5}\left[\frac{3}{4}CE^{*2} + \frac{3}{4}AF^{*2} + \frac{3}{2}BE^*F^* - CF^*\right] \\ & + \frac{3}{28}\delta\lambda^7 CF^{*2} + O(\delta\lambda^9)\Big\}, \end{aligned} \tag{19}$$

where $E^* = E/D$, $F^* = F/D$.

Equation (19) is given as a definite integral since all the even powers of $\delta\lambda$ have zero coefficients with the limits taken thus: and these limits are the ones used. All the constants of integration have been omitted from (14)–(18).

There is a definite advantage in interchanging the orders of differentiation and the integration implicit for $U$ in (6). The gradient operator is applied inside the integration (9): the gradients of $D$, $E$, and $F$ in (14) are required. The quantities $A$, $B$, and $C$ do not depend on the point at which the potential or gravity is being evaluated. so they act as constants in the differentiation. Gradients are computed for equations (14) through (18). which in turn are integrated by the same numerical quadrature formula used for the potential.

$$\begin{aligned}\nabla(\Delta U) &= \nabla \int\int f g^{-1/2}\, d\phi\, d\lambda \\ &= \int d\phi \left\{ \nabla \int f g^{-1/2}\, d\lambda \right\}.\end{aligned} \tag{20}$$

When $F$ is small. (14) is plagued by small divisors. Upon taking gradients. the divisors become smaller still. so it is necessary to use (18) or (19) to prevent excessive rounding errors.

To optimize the numerical integration method. Koch (1971) and Fröhlich and Koch (1974) have studied different configurations of the evaluation points. Recently. Fröhlich (1975) published some algorithms of very high accuracy.

## V. PROGRAMMING OF THE ALGORITHMS

To construct a program suitable for application to anticipated high precision satellite altimetry data. the computer programs for these algorithms were changed to use a scheme of 1640 equal-area blocks on an oblate spheroid: these blocks are about 5°×5° at the equator and fairly uniform in shape everywhere (see figs. 2 and 3). Blocks near the poles are somewhat extended in longitude. for example. those at the poles cover 90° in longitude. For this reason these blocks are subdivided into roughly 5° longitude bands and the algorithms are applied within each subblock generated. This is illustrated in figure 4. To save computer time, different algorithms are used depending on the distance from the satellite to the center of the block.

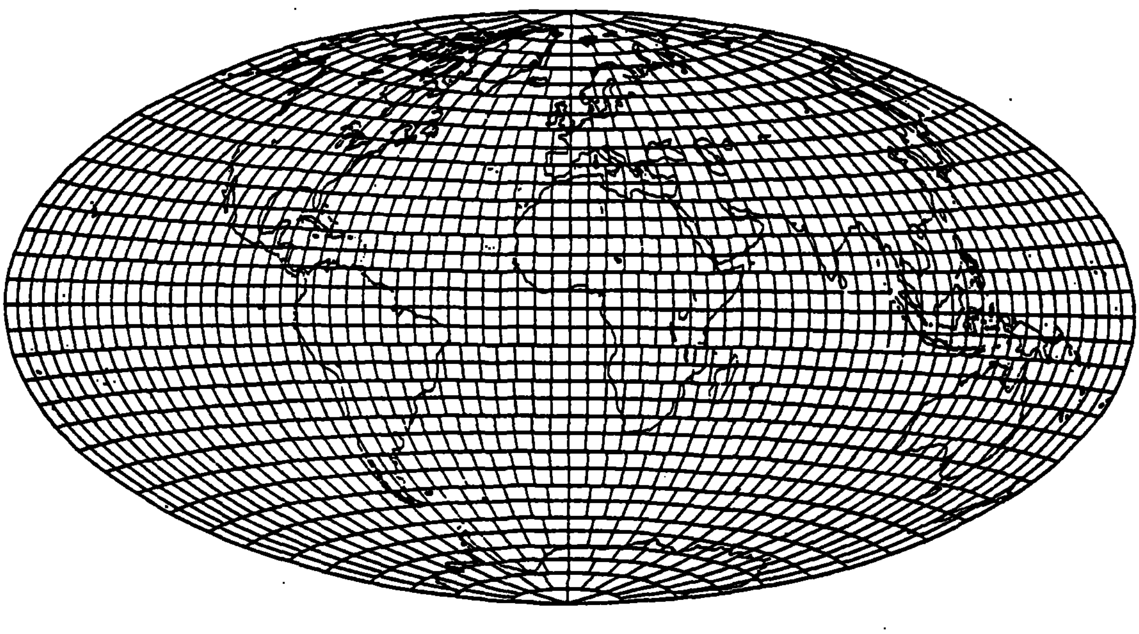

FIGURE 2.—1640 five-degree equal-area blocks. Aitoff equal-area projection.



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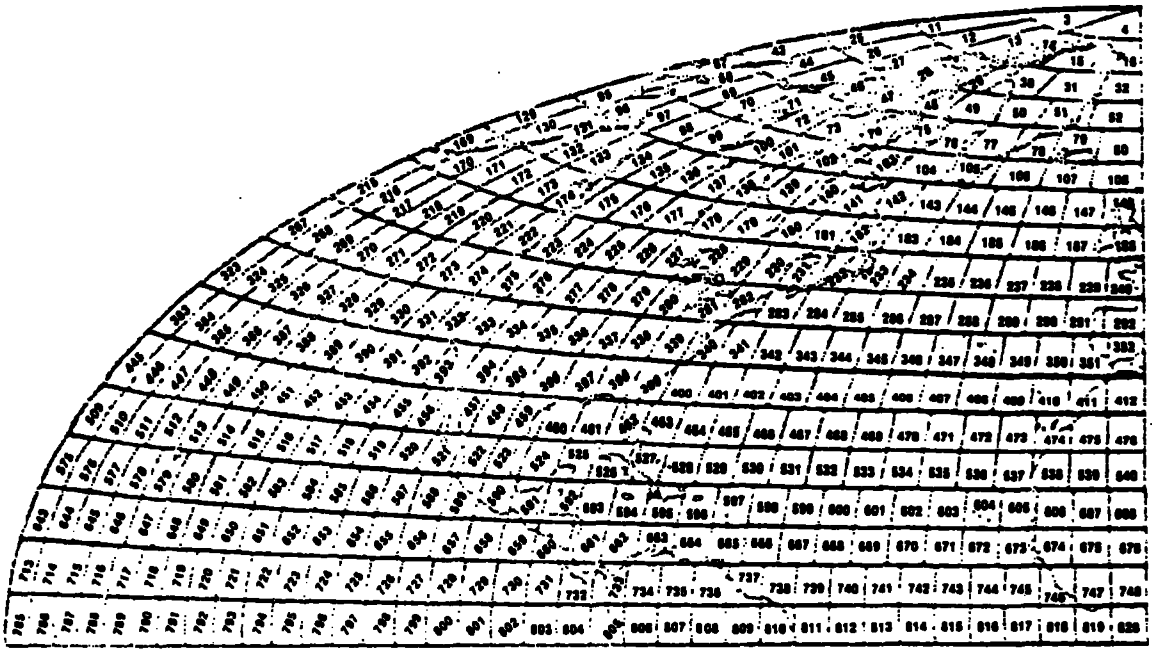

FIGURE 3a. – Equal-area blocks with index numbers. NW quadrant.

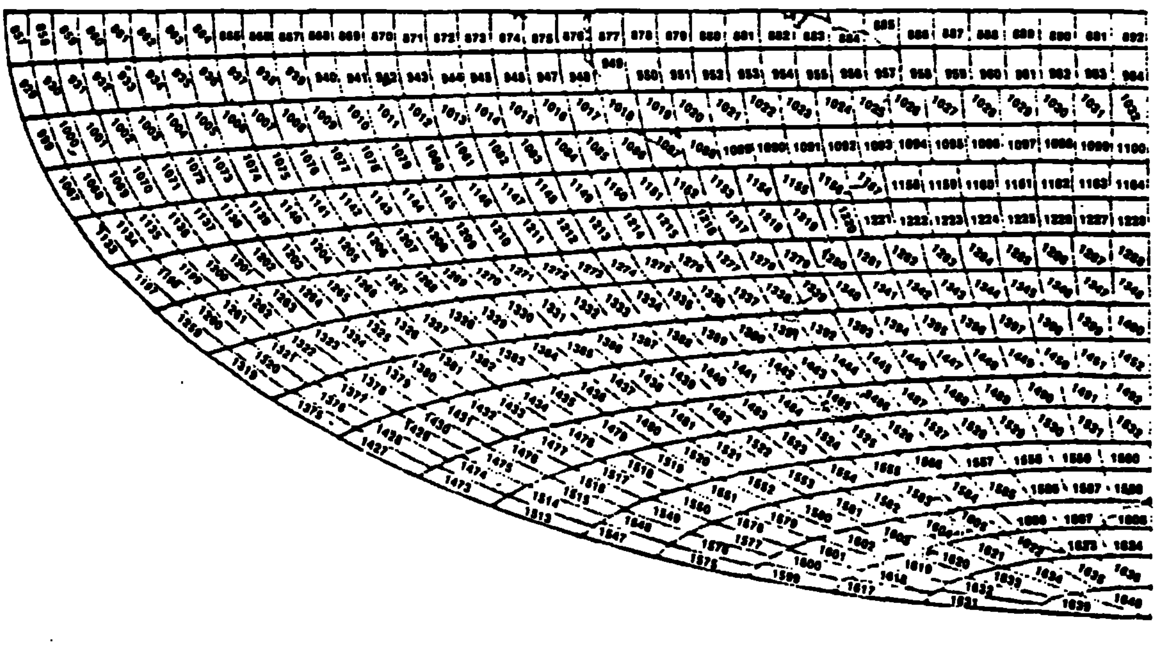

FIGURE 3b. – Equal-area blocks with index numbers. SW quadrant.

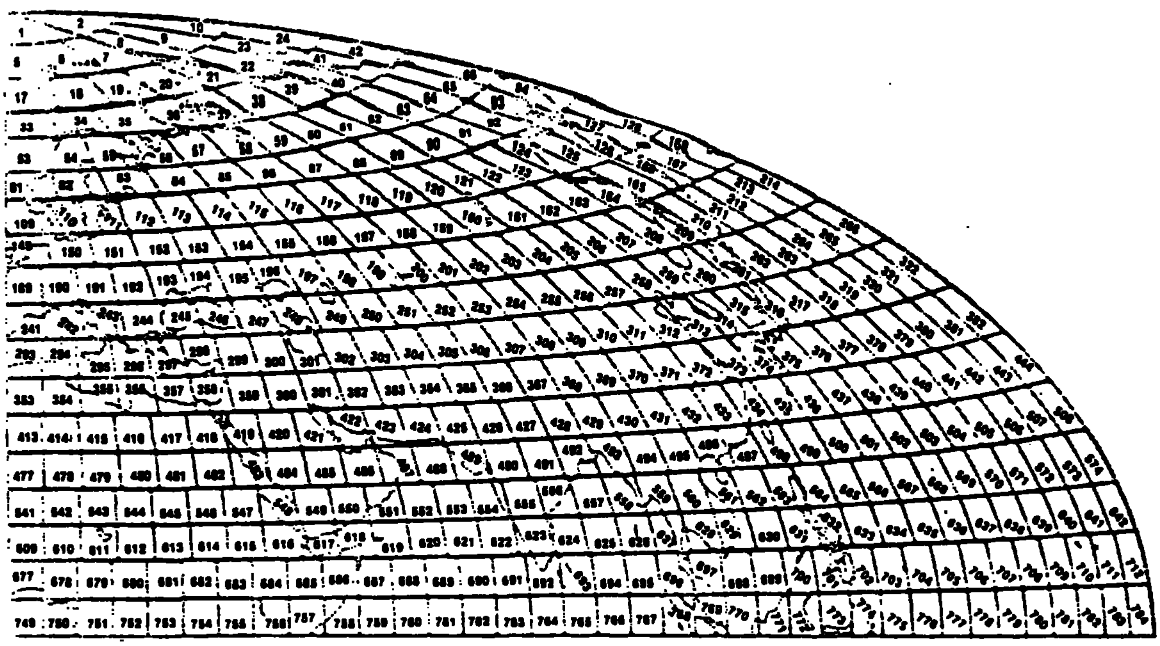

FIGURE 3c.—Equal-area blocks with index numbers. NE quadrant.

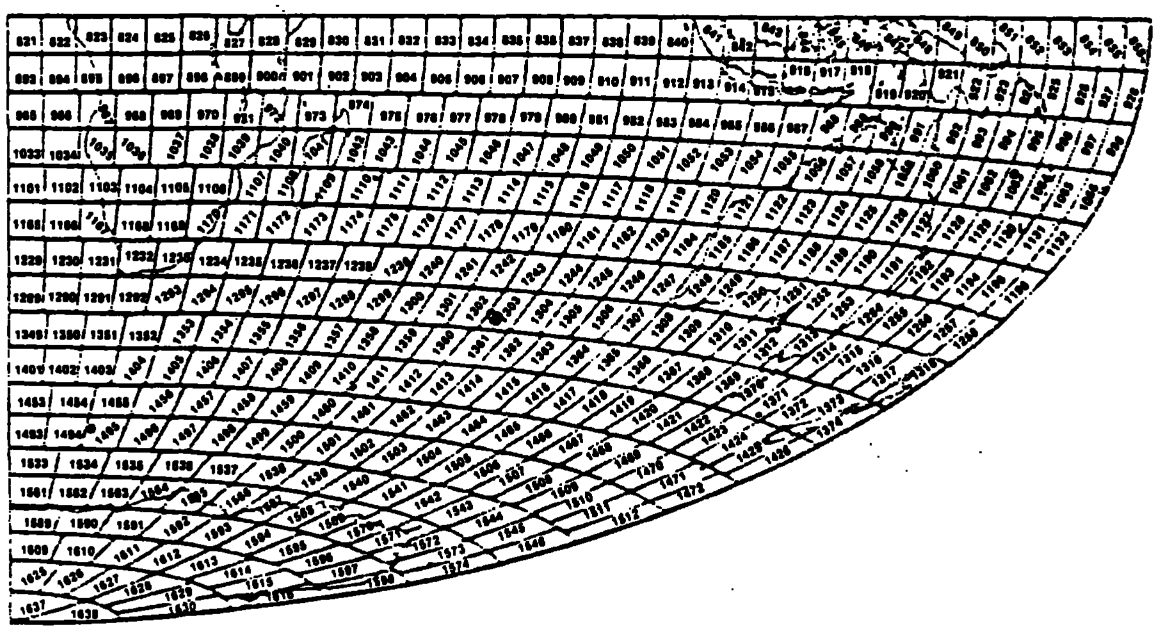

FIGURE 3d.—Equal-area blocks with index numbers. SE quadrant.

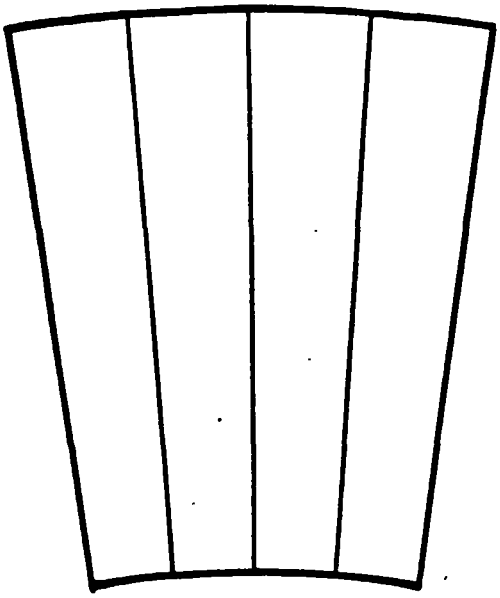

FIGURE 4.—Subdivision of blocks near pole into blocks equal in longitude and latitude.

Density is assumed constant within each block in this production version of the program. so that

$$B=C=0 \tag{21}$$

in expression (14). The method using (14) is employed for the closest blocks. Taylor series for a zone surrounding that. and numerical integration for the farthest blocks. No loss of accuracy is suffered if the distances of $R/4$ and $R$, ($R$=Earth radius), are used to define these zones. Accuracy is maintained from the lowest possible altitudes for orbiting satellites out to infinity.

For operational purposes and to facilitate incorporation into large-scale orbital analysis programs. some optimization has been done. The equal-area block system has an eight-fold symmetry which has been used to reduce storage requirements. Certain sines and cosines are handled as complex exponential functions. which can save time and memory. Some loops were written out explicitly to sacrifice memory for central processor time. It was found that the Taylor series and numerical quadrature algorithms can be computed using the short single-precision word length of the IBM 360 with double-precision accumulation of the geopotential and gravity. The more complicated formulas which arise from applying (2) require double precision at almost every step. but just for certain troublesome satellite locations. More detailed study is required on this question. though the use of double precision is a satisfactory if not optimal solution.

Some time is saved by direct application of the Newton-Raphson algorithm instead of the compiler subroutine to compute the square root needed to evaluate the distance from the satellite to a given block. Since the blocks whose contributions are evaluated successively are always adjacent. very accurate first approximations to this distance are available. except for the case of the first block. of course. The compiler function is used to initialize the process.

## VI. PROGRAM TESTING

Timing and debugging tests were made using 90 satellite positions—half at 300 km. most of the rest at 1000 km and one each at 10R. 100R. 1000R. 1.0000R. and 1.000.000R. If the density on an oblate spheroid is chosen so that

$$\chi=\chi(\phi)=1/r^2(\phi) \tag{22}$$

the potential will be nearly that due to a point mass equal to the mass of the spheriod and located at its center. This was determined through comparisons with a spherical shell of constant density. If we set $U=\mu/r_s$ in (1) and attempt to solve the integral equation for $\chi$. we can observe that $d\sigma = r^2(\phi)\cos\phi d\phi d\lambda$. which in turn implies that (22) is an approximate solution good to an order of about $e^2$. A higher order analytic solution to the problem would be useful for testing, but none has as yet been found either through attempts at solution or in the literature.

The program has been inserted as a force model in two orbit computation programs. The first is a small program on the CDC 6600 that generates only orbital elements. The geopotential modeling program was then converted for use on the IBM 360. optimized. and inserted into the well-known GEODYN program. which is now operational on the NOAA IBM 360-195 at Suitland, Md.

Program linkage and compatibility have been tested by using (22) as the density distribution and integrating a two-body problem perturbed only by the computational errors in the surface density model program. The results in Keplerian elements $a$, $e$, and $I$ are given in figures 5, 6, and 7. Both the semi-major axis. $a$, and the eccentricity. $e$, exhibit periodic changes only. at least for the 9 hour period for which the integration was done. Since the mean value of $a$ obviously is not the same as the initial value. drift occurs in the mean anomaly. In practice. the process of orbital adjustment would rectify this problem. A very large secular or long-period perturbation is present in the inclination. $I$. Runs made (on the CDC 6600) with an inclination of 59.4° did not exhibit this effect. so it is most likely the result of resonance caused by the truncation errors in the force model. A force model with zero total density and with no errors equivalent to low order harmonics would be much less prone to cause such problems (Morrison 1972). This is a very severe test in many ways. The model will be used with a spherical harmonic model of, perhaps, degree and order eight, so it will be used to model the last few parts per million of the gravity field. A thorough analysis of the dynamical effects of these computational errors on an actual satellite orbit remains to be done. A qualitative analytic analysis of the problem has been done by Morrison (1972). More computer simulations and a numerical verification of that study would be desirable.

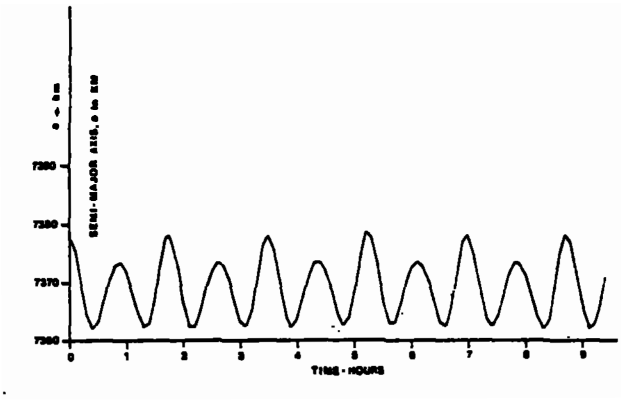

FIGURE 5.—The semi-major axis of the orbit of a particle moving in a central force field modeled by the surface density algorithm and equation (22).

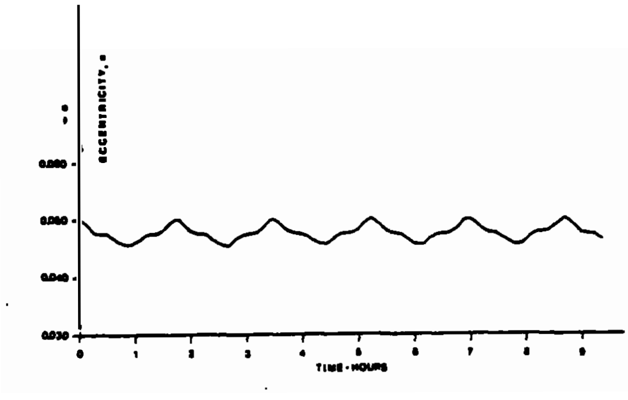

FIGURE 6.—The eccentricity of the orbit of the particle.

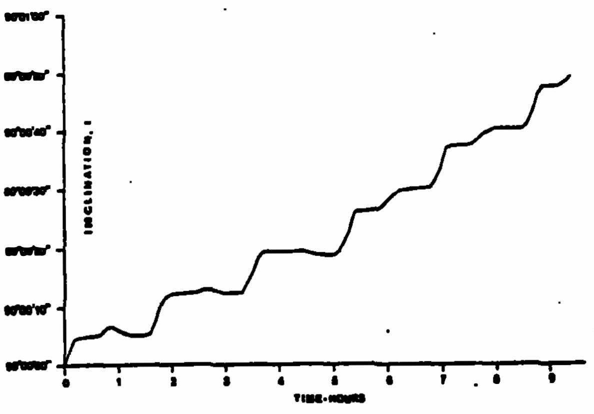

FIGURE 7.—The inclination of the orbit of the particle.

## VII. APPLICATIONS

Most of the work in developing the density layer method of geopotential modeling has been devoted to finding approximation techniques for computing (1). Inasmuch as this is part of the art of computing, as opposed to a purely mathematical problem, it is "solved" in a satisfactory way, but time and experience will bring improvements of some sort. More analysis and programming are needed for adjusting the density values from combinations of various kinds of data.

The present program, using 5° blocks, is suitable for using satellite tracking, satellite altimetry, and gravity anomalies which have been averaged over areas of about 5° "squares." The model corresponds, more or less, to a spherical harmonic expansion to degree and order 36. A study of the frequency response of the model is included in Morrison (1976). Since the data quantities and density values can be related through linear integral transforms, either exactly or to a high degree of approximation, the data can be combined in a consistent and optimum way by covariance methods. The density layer method has been used by advocates of discrete variable representations, whereas the techniques known as least-squares collocation and spherical harmonic sampling functions have used, basically, orthogonal function interpolation. The motivation to develop the discrete approach has been mostly intuitive, based on the observation that when you have to add a new term to your expansion in eigenfunctions for every new data point, you are making unnecessary work for yourself and your computer, and a discrete variable method is better. It seems best to represent the long-wave portions of the geopotential in spherical harmonics and the short-wave parts with the surface coating. The advantage of the surface coating over a discrete variable representation of the gravity is that the upward continuation transformation is much simpler. i.e.. (1) is simpler than the Stokes' integral and more flexible as far as choosing a reference surface.

Even for 5° blocks. the application of the covariance methods for interpolation is not completely straightforward. Williamson and Gaposchkin (1973) have shown that gravity is not stationary. not even for 1° and 5° blocks. Point anomalies should be much less stationary than these means. A problem of interest to some geodesists is the interpolation of deflections of the vertical. point anomalies. and gravity gradients. To do this will require a program with a much smaller block size than 5°.

## VIII. CONSTRUCTING THE EQUAL-AREA BLOCKS

Equal-area blocks have been a popular means for determining a uniform sampling of data distributed over spherelike domains (Rapp. 1971). Making the blocks exactly equal in area is fairly simple for spheres or oblate spheroids. and eliminates the need to store and reference a lot of weight factors in operations such as numerical integration.

The area of the zone between the equator and latitude $\phi$ on an oblate spheroid is given by Unguendoli (1972):

$$A(\phi) = 2\pi a^2(1-e^2)\int_0^{\phi} \frac{\cos\xi\, d\xi}{(1-e^2\sin^2\xi)^2} \quad (23)$$

where $a$ = semi-major axis. and $e$ = meridian eccentricity. There is no need to resort to series to evaluate (23). since the integrand is a rational function if we use the substitution

$$x = e\sin\xi. \quad (24)$$

The result is

$$A(\phi) = \pi a^2(1-e^2)\left[\frac{\sin\phi}{1-e^2\sin^2\phi} + \frac{1}{2e}\log\frac{1+e\sin\phi}{1-e\sin\phi}\right] \quad (25)$$

Strictly speaking. (25) is not defined for $e = 0$. but

$$\lim_{e\to 0} A(\phi) = 2\pi a^2\sin\phi, \quad (26)$$

which is the correct result for a sphere. For small values of $e$ it is necessary to evaluate the logarithm portion of the expression by series and cancel the divisor $e$ to avoid loss of numerical significance.

To establish a system of equal-area blocks. one needs first to pick a block size. say 5°×5°. Then a number of zones are chosen. $J = 18 = 90°/5°$ for one hemisphere (the other hemisphere can be obtained by reflection). One next chooses the number of blocks in a zone. starting from

$$n(i) = \left[\frac{360°}{5°}\cos\langle\phi_i\rangle\right], \quad (27)$$

where the zones are numbered starting from the equator: $\langle\phi_i\rangle$ = mean latitude in zone $i$ (and for 5° any other appropriate value may be substituted); the [] symbolizes truncation to the nearest integer. The area of the zone from 0 to $\phi_j$ is then

$$A(\phi_j) = \frac{\sum_{i=1}^{j} n(i)}{\sum_{i=1}^{J} n(i)} A(90°). \quad (28)$$

To find $\phi_j$ one substitutes $A(\phi_j)$ from (25) into (28) and uses the Newton-Raphson iteration. since the transcendental equation is not readily solved. For a starting value the spherical approximation may be used.

$$(\phi_j)_0 = \sum^{j} n(i) \Big/ \sum^{J} n(i). \quad (29)$$

The derivative needed is quite simple.

$$dA/d\phi = 2\pi\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}. \quad (30)$$

When a set of blocks is generated. they may be tested for "squareness" using a criterion suggested by Paul (1973)

$$R_i = \frac{\text{length of side along mean parallel}}{\text{the same along a meridian}} \approx 1. \quad (31)$$

If the values of $R_i$ are not satisfactory one may change the values of $n(i)$ in a trial-and-error way to see if improvement is possible. Setting up a rigorous adjustment of the $n(i)$ by some criterion of obtaining a better approximation in (31) does not seem worth the effort.

For the poles there are some complications. The blocks at the polar zone are best divided into four. and the next group into 12 (see figs. 3 and 8) and so on. as

$$\begin{aligned} n(J) &= 4 \\ n(J-1) &= 12 \\ &\cdots \\ n(J-k) &= 4(k+1)^2 - n(J-k+1) \\ &= 4(2k+1) \\ k &\ll J. \end{aligned}$$

This is because the zones are nearly concentric rings at the poles and for those regions

$$R_{J-K} \approx \pi/4$$

gives a better criterion for "squareness."

For blocks of 5° only, the two zones nearest either pole follow this rule. (See figs. 2 and 3, and tables 3 and 4.)

The 5° equal-area block model chosen follows the well-known Zhonogolovitch 10° blocks as closely as was considered practical. The zone boundaries near 10°, 20°, 30°, and 60° are the same. Table 1 gives the parameters for the Zhongolovitch blocks on a sphere, table 2 for a spheroid.

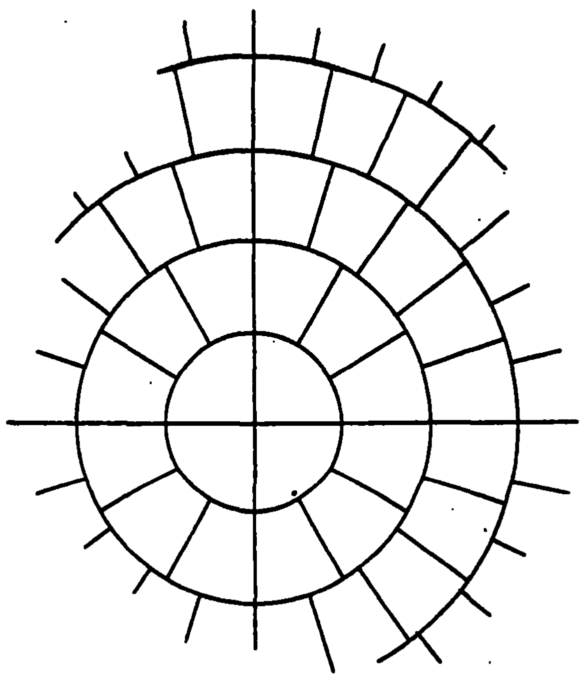

FIGURE 8.—Idealized set of equal-area blocks at pole.

TABLE 1.—*Zhongolovitch 10° equal-area blocks for the sphere*

| $a = 1.0000000000$ | | $b = 1.0000000000$ | | $e^2 = 0.000000000000$ | Sectors = 1 | Area = 0.03064968442527 | |
|---|---|---|---|---|---|---|---|
| J | N | PHI-2 | PHIBAR | DPHI | DLAMBDA | RJ | PHI-2 SPH |
| | | ° | ° | ° | ° | | ° |
| 1 | 36 | 10.114144088 | 5.057072044 | 10.114144088 | 10.000000000 | 0.984865728 | 10.114144088 |
| 2 | 34 | 19.966058297 | 15.040101193 | 9.851914209 | 10.588235294 | 1.037923106 | 19.966058297 |
| 3 | 32 | 29.838766211 | 24.902412254 | 9.872707913 | 11.250000000 | 1.033560986 | 29.838766211 |
| 4 | 30 | 40.083433915 | 34.961100063 | 10.244667704 | 12.000000000 | 0.959962351 | 40.083433915 |
| 5 | 25 | 49.982997154 | 45.033215534 | 9.899563239 | 14.400000000 | 1.027967879 | 49.982997154 |
| 6 | 21 | 60.260840340 | 55.121918747 | 10.277843186 | 17.142857143 | 0.953783344 | 60.260840340 |
| 7 | 15 | 70.298788126 | 65.279814233 | 10.037947786 | 24.000000000 | 0.999854867 | 70.298788126 |
| 8 | 9 | 80.185857012 | 75.242322569 | 9.887068886 | 40.000000000 | 1.030564385 | 80.185857012 |
| 9 | 3 | 90.000000000 | 85.092928506 | 9.814142988 | 120.000000000 | 1.045917823 | 90.000000000 |
| Total | 205 | | | | | | |

TABLE 2.—*Zhongolovitch 10° equal-area blocks for the spheroid*

| $a$ = 1.0000000000 | | $b$ = 0.9966470765 | | $e^2$ = 0.006694605000 | Sectors = 1 | Area = 0.03058119674137 | |
|---|---|---|---|---|---|---|---|
| J | N | PHI-2 | PHIBAR | DPHI | DLAMBDA | RJ | PHI-2 SPH |
| | | ° | ° | ° | ° | | ° |
| 1 | 36 | 10.158613279 | 5.079306640 | 10.158613279 | 10.000000000 | 0.980520745 | 10.092007853 |
| 2 | 34 | 20.048575322 | 15.103594300 | 9.889962043 | 10.588235294 | 1.033621594 | 19.924951454 |
| 3 | 32 | 29.949659806 | 24.999117564 | 9.901084484 | 11.250000000 | 1.029789802 | 29.783459904 |
| 4 | 30 | 40.209905973 | 35.079782889 | 10.260246167 | 12.000000000 | 0.957114533 | 40.020265940 |
| 5 | 25 | 50.109306146 | 45.159606059 | 9.899400173 | 14.400000000 | 1.025712019 | 49.919815831 |
| 6 | 21 | 60.371219506 | 55.240262826 | 10.261913360 | 17.142857143 | 0.952431222 | 60.205546609 |
| 7 | 15 | 70.380067424 | 65.375643465 | 10.008847918 | 24.000000000 | 0.999117443 | 70.258024753 |
| 8 | 9 | 80.228851341 | 75.304459382 | 9.848783917 | 40.000000000 | 1.030310595 | 80.164278502 |
| 9 | 3 | 90.000000000 | 85.114425671 | 9.771148659 | 120.000000000 | 1.045929007 | 90.000000000 |
| Total | 205 | | | | | | |

The following legend is used for all tables in this text:

$a$ = Semi-major axis
$b$ = Semi-minor axis
$e$ = Eccentricity of the spheroid
PHI 2 = Latitude of northern edge of block
PHIBAR = Average latitude of block
DPHI = Extent of block in latitude
DLAMBDA = Extent of block in longitude
RJ = $R_i$ = See (31)
PHI-2 SPH = Spherical latitude for PHI 2
J = Index of zones in one hemisphere
N = Number of blocks in one sector of a zone

TABLE 3.—*Five-degree blocks used for geopotential modeling for the sphere*

| $a$ = 1.0000000000 | | $b$ = 1.0000000000 | | $e^2$ = 0.000000000000 | Sectors = 4 | Area = 0.07662421106316 | |
|---|---|---|---|---|---|---|---|
| J | N | PHI-2 | PHIBAR | DPHI | DLAMBDA | RJ | PHI-2 SPH |
| | | ° | ° | ° | ° | | ° |
| 1 | 18 | 5.037335850 | 2.518667925 | 5.037335850 | 5.000000000 | 0.991629292 | 5.037335850 |
| 2 | 18 | 10.114144088 | 7.575739969 | 5.076808238 | 5.000000000 | 0.976274270 | 10.114144088 |
| 3 | 17 | 14.983246004 | 12.548695046 | 4.869101916 | 5.294117647 | 1.061314857 | 14.983246004 |
| 4 | 17 | 19.966058297 | 17.474652151 | 4.982812293 | 5.294117647 | 1.013442458 | 19.966058297 |
| 5 | 16 | 24.803794163 | 22.384926230 | 4.837735865 | 5.625000000 | 1.075117645 | 24.803794163 |
| 6 | 16 | 29.838766211 | 27.321280187 | 5.034972048 | 5.625000000 | 0.992560304 | 29.838766211 |
| 7 | 15 | 34.801265705 | 32.320015958 | 4.962499494 | 6.000000000 | 1.021753384 | 34.801265705 |
| 8 | 15 | 40.083433915 | 37.442349810 | 5.282168210 | 6.000000000 | 0.901863137 | 40.083433915 |
| 9 | 13 | 45.017042172 | 42.550238043 | 4.933608258 | 6.923076923 | 1.033751458 | 45.017042172 |
| 10 | 13 | 50.419641072 | 47.718341622 | 5.402598900 | 6.923076923 | 0.862118040 | 50.419641072 |
| 11 | 10 | 55.035992386 | 52.727816729 | 4.616351314 | 9.000000000 | 1.180676862 | 55.035992386 |
| 12 | 10 | 60.260840340 | 57.648416363 | 5.224847954 | 9.000000000 | 0.921752781 | 60.260840340 |
| 13 | 7 | 64.480541509 | 62.370690925 | 4.219701169 | 12.857142857 | 1.413012628 | 64.480541509 |
| 14 | 7 | 69.485799940 | 66.983170724 | 5.005258431 | 12.857142857 | 1.004376110 | 69.485799940 |
| 15 | 5 | 73.940639486 | 71.713219713 | 4.454839546 | 18.000000000 | 1.267816931 | 73.940639486 |
| 16 | 4 | 78.662967149 | 76.301803318 | 4.722327663 | 22.500000000 | 1.128293217 | 78.662967149 |
| 17 | 3 | 84.338423235 | 81.500695192 | 5.675456086 | 30.000000000 | 0.781245113 | 84.338423235 |
| 18 | 1 | 90.000000000 | 87.169211617 | 5.661576765 | 90.000000000 | 0.785078675 | 90.000000000 |

Total = 205 × 4 = 1640

**TABLE 4.** — *Five-degree blocks used for geopotential modeling for the spheroid*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a = 1.0000000000$ | | $b = 0.9966470765$ | | $e^2 = 0.006694605000$ | Sectors = 4 | Area = 0.07645299185343 | |

| J | N | PHI-2 | PHIBAR | DPHI | DLAMBDA | RI | PHI-2 SPH |
|---|---|---|---|---|---|---|---|
| | | ° | ° | ° | ° | ° | ° |
| 1 | 18 | 5.059836888 | 2.529918444 | 5.059836888 | 5.000000000 | 0.987210982 | 5.026137369 |
| 2 | 18 | 10.158613279 | 7.609225084 | 5.098776391 | 5.000000000 | 0.971992256 | 10.092007853 |
| 3 | 17 | 15.047475056 | 12.603044168 | 4.888861777 | 5.294117647 | 1.056801567 | 14.951263466 |
| 4 | 17 | 20.048575322 | 17.548025189 | 5.001100265 | 5.294117647 | 1.009328610 | 19.924951454 |
| 5 | 16 | 24.901674946 | 22.475125134 | 4.853099625 | 5.625000000 | 1.071017871 | 24.755008330 |
| 6 | 16 | 29.949659806 | 27.425667376 | 5.047984859 | 5.625000000 | 0.989068218 | 29.783459904 |
| 7 | 15 | 34.921630212 | 32.435645009 | 4.971970406 | 6.000000000 | 1.018502939 | 34.741194898 |
| 8 | 15 | 40.209905973 | 37.565768092 | 5.288275761 | 6.000000000 | 0.899333629 | 40.020265940 |
| 9 | 13 | 45.145343214 | 42.677624593 | 4.935437241 | 6.923076923 | 1.031256833 | 44.952912706 |
| 10 | 13 | 50.545592266 | 47.845467740 | 5.400249052 | 6.923076923 | 0.860386605 | 50.356634602 |
| 11 | 10 | 55.156394527 | 52.850993396 | 4.610802261 | 9.000000000 | 1.178755745 | 54.975721014 |
| 12 | 10 | 60.371219506 | 57.763807016 | 5.214824979 | 9.000000000 | 0.920586272 | 60.205546609 |
| 13 | 7 | 64.580143945 | 62.475681725 | 4.208924439 | 12.857142857 | 1.411668910 | 64.430619998 |
| 14 | 7 | 69.569857941 | 67.075000943 | 4.989713996 | 12.857142857 | 1.003702695 | 69.443646351 |
| 15 | 5 | 74.008705762 | 71.789281851 | 4.438847821 | 18.000000000 | 1.267271917 | 73.906491693 |
| 16 | 4 | 78.712308656 | 76.360507209 | 4.703602894 | 22.500000000 | 1.128022569 | 78.638205110 |
| 17 | 3 | 84.363550821 | 81.537929738 | 5.651242164 | 30.000000000 | 0.781180387 | 84.325809778 |
| 18 | 1 | 90.000000000 | 87.181775410 | 5.636449179 | 90.000000000 | 0.785081504 | 90.000000000 |

Total = 205 × 4 = 1640

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## APPENDIX I. NOTATION

Most notations in this paper are standard: symbols undefined in the text are:

| | |
|---|---|
| G | gravitational constant |
| $\mathbf{r}$ | position vector on the Earth |
| $\mathbf{r}_s$ | position of a point where potential is computed |
| $x_s, y_s, z_s$ | components of $\mathbf{r}_s$ |
| $T$ | anomalous gravitational potential |
| $U$ | gravitational potential |
| $x, \xi$ | dummy variables |
| $\lambda$ | longitude |
| $\phi$ | latitude (spherical coordinate) |
| $\chi$ | density |
| $\sigma$ | surface area |

## APPENDIX II. DETAILED MATHEMATICAL DEVELOPMENTS

This section is comprised of a detailed presentation of all formulas used. In some cases, options not used in the programs are described so that this material will be suitable for a number of programs tailored for specific purposes.

### 1. THE TAYLOR SERIES METHOD

#### A. Development of the Potential

$$T=\sum \Delta T$$

$$\Delta T=\int_{-\Delta\phi}^{\Delta\phi}\int_{-\Delta\lambda}^{\Delta\lambda}\frac{\chi(\phi,\lambda)r^2\cos\phi}{r^*}\,d\lambda d\phi \qquad (1.1)$$

where

$$\mathbf{r}^*=\mathbf{r}-\mathbf{r}_s$$

and $r^*=\|\mathbf{r}^*\|$

The integrand is abbreviated to

$$f=\frac{\chi r^2\cos\phi}{r^*} \qquad (1.2)$$

and

$$\frac{\partial f}{\partial\phi}=-\frac{\chi r^2\sin\phi}{r^*}-\frac{\partial r^*}{\partial\phi}\frac{\chi r^2\cos\phi}{r^{*2}}+\frac{\cos\phi}{r^*}\left[r^2\frac{\partial\chi}{\partial\phi}+2r\chi\frac{\partial r}{\partial\phi}\right] \qquad (1.3)$$

$$\frac{\partial f}{\partial\lambda}=-\frac{\partial r^*}{\partial\lambda}\frac{\chi r^2\cos\phi}{r^{*2}}+\frac{\cos\phi}{r^*}\left[r^2\frac{\partial\chi}{\partial\lambda}+2r\chi\frac{\partial r}{\partial\lambda}\right] \qquad (1.4)$$

$$\frac{\partial^2 f}{\partial\phi^2}=2\left(\frac{r}{r^*}\right)^2\chi\sin\phi\frac{\partial r^*}{\partial\phi}-2\frac{r^2}{r^*}\sin\phi\frac{\partial\chi}{\partial\phi}-4\chi\frac{r}{r^*}\sin\phi\frac{\partial r}{\partial\phi}-\frac{r^2}{r^*}\chi\cos\phi+2\chi\frac{r^2}{r^{*3}}\cos\phi\left(\frac{\partial r^*}{\partial\phi}\right)^2-\chi\left(\frac{r}{r^*}\right)^2\cos\phi\frac{\partial^2 r^*}{\partial\phi^2}-2\frac{\cos\phi}{r^{*2}}\frac{\partial r^*}{\partial\phi}\left[r^2\frac{\partial\chi}{\partial\phi}+2r\chi\frac{\partial r}{\partial\phi}\right]+\frac{\cos\phi}{r^*}\left[4r\frac{\partial r}{\partial\phi}\frac{\partial\chi}{\partial\phi}+r^2\frac{\partial^2\chi}{\partial\phi^2}+2\chi\left(\frac{\partial r}{\partial\phi}\right)^2+2r\chi\frac{\partial^2 r}{\partial\phi^2}\right] \qquad (1.5)$$

$$\frac{\partial^2 f}{\partial\phi\partial\lambda}=2\chi\frac{r^2}{r^{*3}}\cos\phi\frac{\partial r^*}{\partial\phi}\frac{\partial r^*}{\partial\lambda}+\chi\left(\frac{r}{r^*}\right)^2\sin\phi\frac{\partial r^*}{\partial\lambda}-\chi\left(\frac{r}{r^*}\right)^2\cos\phi\frac{\partial^2 r^*}{\partial\phi\partial\lambda}-\frac{r}{r^{*2}}\cos\phi\left\{\left[r\frac{\partial\chi}{\partial\phi}+2\chi\frac{\partial r}{\partial\phi}\right]\frac{\partial r^*}{\partial\lambda}+\left[r\frac{\partial\chi}{\partial\lambda}+2\chi\frac{\partial r}{\partial\lambda}\right]\frac{\partial r^*}{\partial\phi}\right\}-\frac{r}{r^*}\sin\phi\left[r\frac{\partial\chi}{\partial\lambda}+2\chi\frac{\partial r}{\partial\lambda}\right]+\frac{\cos\phi}{r^*}\left[2r\frac{\partial r}{\partial\phi}\frac{\partial\chi}{\partial\lambda}+r^2\frac{\partial^2\chi}{\partial\phi\partial\lambda}+2r\chi\frac{\partial^2 r}{\partial\phi\partial\lambda}+2\chi\frac{\partial r}{\partial\phi}\frac{\partial r}{\partial\lambda}+2r\frac{\partial\chi}{\partial\phi}\frac{\partial r}{\partial\lambda}\right] \qquad (1.6)$$

$$\frac{\partial^2 f}{\partial\lambda^2}=2\chi\frac{r^2}{r^{*3}}\cos\phi\left(\frac{\partial r^*}{\partial\lambda}\right)^2-\chi\frac{r^2}{r^{*2}}\cos\phi\frac{\partial^2 r^*}{\partial\lambda^2}-2\frac{\cos\phi}{r^{*2}}\frac{\partial r^*}{\partial\lambda}\left[2r\chi\frac{\partial r}{\partial\lambda}+r^2\frac{\partial\chi}{\partial\lambda}\right]+\frac{\cos\phi}{r^*}\left[4r\frac{\partial r}{\partial\lambda}\frac{\partial\chi}{\partial\lambda}+r^2\frac{\partial^2\chi}{\partial\lambda^2}+2r\chi\frac{\partial^2 r}{\partial\lambda^2}+2\chi\left(\frac{\partial r}{\partial\lambda}\right)^2\right] \qquad (1.7)$$

$$\mathbf{r}^*=\mathbf{r}-\mathbf{r}_s \qquad (1.8)$$

$$r^*=\|\mathbf{r}^*\|$$

$$=(\mathbf{r}^*\cdot\mathbf{r}^*)^{1/2} \qquad (1.9)$$

$$\mathbf{r}_s=(x_s,\ y_s,\ z_s)^T \qquad (1.10)$$

The vector $\mathbf{r}_s$ is the satellite position, $\mathbf{r}$ the coordinates of the increment $d\sigma$ on the reference surface.

$$\mathbf{r}=r(\cos\phi\cos\lambda,\ \cos\phi\sin\lambda,\ \sin\phi)^T \qquad (1.11)$$

$$\frac{\partial r^*}{\partial \phi}=\frac{\mathbf{r}^*}{r^*}\cdot\frac{\partial \mathbf{r}}{\partial \phi} \tag{1.12}$$

$$\frac{\partial r^*}{\partial \lambda}=\frac{\mathbf{r}^*}{r^*}\cdot\frac{\partial \mathbf{r}}{\partial \lambda} \tag{1.13}$$

$$\frac{\partial \mathbf{r}}{\partial \phi}=r\,(-\sin\phi\cos\lambda,-\sin\phi\sin\lambda,\cos\phi)^T$$

$$+\frac{\partial r}{\partial \phi}(\cos\phi\cos\lambda,\cos\phi\sin\lambda,\sin\phi)^T \tag{1.14}$$

$$\frac{\partial \mathbf{r}}{\partial \lambda}=r(-\cos\phi\sin\lambda,\cos\phi\cos\lambda,0)^T$$

$$+\frac{\partial r}{\partial \lambda}\frac{\mathbf{r}}{r} \tag{1.15}$$

$$\frac{\partial^2 r^*}{\partial \phi^2}=\frac{1}{r^*}\left[r^2+\mathbf{r}^*\cdot\frac{\partial^2 \mathbf{r}}{\partial \phi^2}-\left(\frac{\partial r^*}{\partial \phi}\right)^2\right] \tag{1.16}$$

$$\frac{\partial^2 r^*}{\partial \phi\partial \lambda}=\frac{1}{r^*}\left[\frac{\partial \mathbf{r}}{\partial \phi}\cdot\frac{\partial \mathbf{r}}{\partial \lambda}+\mathbf{r}^*\cdot\frac{\partial^2 \mathbf{r}}{\partial \phi\partial \lambda}-\frac{\partial r^*}{\partial \phi}\frac{\partial r^*}{\partial \lambda}\right] \tag{1.17}$$

$$\frac{\partial^2 r^*}{\partial \lambda^2}=\frac{l}{r^*}\left[\left(\frac{\partial \mathbf{r}}{\partial \lambda}\right)^2+\mathbf{r}^*\cdot\frac{\partial^2 \mathbf{r}}{\partial \lambda^2}-\left(\frac{\partial r^*}{\partial \lambda}\right)^2\right] \tag{1.18}$$

$$\frac{\partial^2 \mathbf{r}}{\partial \phi^2}=-\mathbf{r}+4\frac{\mathbf{r}}{r}\frac{\partial^2 r}{\partial \phi^2}.$$

$$+2\frac{\partial r}{\partial \phi}(-\sin\phi\cos\lambda,-\sin\phi\sin\lambda,\cos\phi)^T \tag{1.19}$$

$$\frac{\partial^2 \mathbf{r}}{\partial \phi\partial \lambda}=r(\sin\phi\sin\lambda,-\sin\phi\cos\lambda,0)^T \tag{1.20}$$

$$+\frac{\partial r}{\partial \phi}(-\cos\phi\sin\lambda.\cos\phi\cos\lambda,0)^T$$

$$\frac{\partial^2 \mathbf{r}}{\partial \lambda^2}=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\mathbf{r} \tag{1.21}$$

Most terms with $\partial^2 r/\partial\lambda^2$ are omitted.

Substitute (1.2) into (1.1) and expand by Taylor's theorem:

$$\Delta T=\int_{-\Delta\phi}^{\Delta\phi}\int_{-\Delta\lambda}^{\Delta\lambda}\left\{f_0+\frac{\partial f}{\partial \phi}(\phi-\phi_0)+\frac{\partial f}{\partial \lambda}(\lambda-\lambda_0)\right.$$

$$+\frac{1}{2}\left[\frac{\partial^2 f}{\partial \phi^2}(\phi-\phi_0)^2+2\frac{\partial^2 f}{\partial \phi\partial \lambda}(\phi-\phi_0)(\lambda-\lambda_0)\right.$$

$$\left.\left.+\frac{\partial^2 f}{\partial \lambda^2}(\lambda-\lambda_0)^2\right]+\ldots\right\}d\lambda d\phi. \tag{1.22}$$

All the terms containing *odd* powers of $(\phi-\phi_0)$ or $(\lambda-\lambda_0)$ integrate out to zero, which includes all the third-order terms. Hence, we can find

$$\Delta T=\Delta\phi\Delta\lambda\left[4f_0+\frac{2}{3}\frac{\partial^2 f}{\partial \phi^2}\Delta\phi^2+\frac{2}{3}\frac{\partial^2 f}{\partial \lambda^2}\Delta\lambda^2\right.$$

$$\left.+O(\Delta\phi^4,\Delta\lambda^4)\right]. \tag{1.23}$$

## B. The Expressions for the Gravity Vector

The formula

$$\Delta\mathbf{g}=\nabla T \tag{1.24}$$

is fundamental.

Substitute (1.23) and (1.1) into (1.24)

$$\Delta\mathbf{g}=\nabla\sum\Delta T \tag{1.25}$$

$$=\sum\left[4\nabla f_0+\frac{2}{3}\nabla f_{\phi\phi}\Delta\phi^2+\frac{2}{3}\nabla f_{\lambda\lambda}\Delta\lambda^2\right]\Delta\phi\Delta\lambda \tag{1.25a}$$

$$\nabla f_0=\chi r^2\cos\phi\,\frac{\mathbf{r}^*}{r^{*3}} \tag{1.26}$$

The only factor of (1.2) that is not constant with respect to $\nabla$ is $1/r^*$, and $\nabla$ operates on (1.5) and (1.7) similarly. Actually, (1.6) is not needed and (1.3) and (1.4) are only steps to derive (1.5) and (1.7).

Gradients of derivatives of $r^*$

$$\nabla\left(\frac{\partial r^*}{\partial \phi}\right)=-\frac{1}{r^*}\frac{\partial \mathbf{r}}{\partial \phi}+\frac{\mathbf{r}^*}{r^{*2}}\frac{\partial r^*}{\partial \phi} \tag{1.27}$$

$$\nabla\left(\frac{\partial r^*}{\partial \lambda}\right)=-\frac{1}{r^*}\frac{\partial \mathbf{r}}{\partial \lambda}+\frac{\mathbf{r}^*}{r^{*2}}\frac{\partial r^*}{\partial \lambda} \tag{1.28}$$

$$\nabla\left(\frac{\partial^2 r^*}{\partial \phi^2}\right)=\frac{\mathbf{r}^*}{r^{*2}}\left(\frac{\partial^2 r^*}{\partial \phi^2}\right)-\frac{1}{r^*}\left[\frac{\partial^2 \mathbf{r}}{\partial \phi^2}\right.$$

$$\left.+2\frac{\partial r^*}{\partial \phi}\nabla\left(\frac{\partial r^*}{\partial \phi}\right)\right] \tag{1.29}$$

$$\nabla\left(\frac{\partial^2 r^*}{\partial \lambda^2}\right)=\frac{\mathbf{r}^*}{r^{*2}}\left(\frac{\partial^2 r^*}{\partial \lambda^2}\right)-\frac{1}{r^*}\left[\frac{\partial^2 \mathbf{r}}{\partial \lambda^2}\right.$$

$$\left.+2\frac{\partial r^*}{\partial \lambda}\nabla\left(\frac{\partial r^*}{\partial \lambda}\right)\right] \tag{1.30}$$

We may factor out seven distinct functions of $r^*$ from $f_{\phi\phi}$ and $f_{\lambda\lambda}$. It is convenient for program coding to write these as elements of $4\times 4$ matrices $\Phi$ and $\Lambda$.

$$\Phi_{11}=1/r^* \tag{1.31a}$$

$$\Phi_{12}=\frac{1}{r^{*2}}\frac{\partial r^*}{\partial\phi} \tag{1.31b}$$

$$\Phi_{13}=\frac{1}{r^{*2}}\frac{\partial^2 r^*}{\partial\phi^2} \tag{1.31c}$$

$$\Phi_{14}=\frac{1}{r^{*3}}\left(\frac{\partial r^*}{\partial\phi}\right)^2 \tag{1.31d}$$

$$\Lambda_{11}=\Phi_{11} \tag{1.32a}$$

$$\Lambda_{12}=\frac{1}{r^{*2}}\frac{\partial r^*}{\partial\lambda} \tag{1.32b}$$

$$\Lambda_{13}=\frac{1}{r^{*2}}\frac{\partial^2 r^*}{\partial\lambda^2} \tag{1.32c}$$

$$\Lambda_{14}=\frac{1}{r^{*3}}\left(\frac{\partial r^*}{\partial\lambda}\right)^2 \tag{1.32d}$$

Then we can write

$$(f_{\phi\phi}, \nabla f_{\phi\phi})^T=\Phi\,\alpha \tag{1.33}$$

$$(f_{\lambda\lambda}, \nabla f_{\lambda\lambda})^T=\Lambda\,\beta. \tag{1.34}$$

with $\alpha$ and $\beta$ being 4-vectors:

$$\alpha_1=-2\frac{\partial\chi}{\partial\phi}r^2\sin\phi-4\chi r\sin\phi\frac{\partial r}{\partial\phi}-\chi r^2\cos\phi + \cos\phi\left[4r\frac{\partial r}{\partial\phi}\frac{\partial\chi}{\partial\phi}+r^2\frac{\partial^2\chi}{\partial\phi^2}+2\chi\left(\frac{\partial r}{\partial\phi}\right)^2 + 2r\chi\frac{\partial^2 r}{\partial\phi^2}\right] \tag{1.35a}$$

$$\alpha_2=2\chi r^2\sin\phi-2\cos\phi\left[r^2\frac{\partial\chi}{\partial\phi}+2r\chi\frac{\partial r}{\partial\phi}\right] \tag{1.35b}$$

$$\alpha_3=-\chi r^2\cos\phi \tag{1.35c}$$

$$\alpha_4=2\chi r^2\cos\phi \tag{1.35d}$$

$$\beta_1=\cos\phi\left[4r\frac{\partial r}{\partial\lambda}\frac{\partial\chi}{\partial\lambda}+r^2\frac{\partial^2\chi}{\partial\lambda^2}-2r\chi\frac{\partial^2 r}{\partial\lambda^2} + 2\chi\left(\frac{\partial r}{\partial\lambda}\right)^2\right] \tag{1.36a}$$

$$\beta_2=2\cos\phi\left[2r\chi\frac{\partial r}{\partial\lambda}+r^2\frac{\partial\chi}{\partial\lambda}\right] \tag{1.36b}$$

$$\beta_3=\alpha_3 \tag{1.36c}$$

$$\beta_4=\alpha_4 \tag{1.36d}$$

The remaining rows of $\Phi$ and $\Lambda$ are obtained by applying the operator $\nabla$ to the first row of each

$$(\Phi_{21},\Phi_{31},\Phi_{41})^T=\nabla\left(\frac{1}{r^*}\right)=\frac{\mathbf{r}^*}{r^{*3}} \tag{1.37a}$$

$$(\Lambda_{21},\Lambda_{31},\Lambda_{41})=(\Phi_{21},\Phi_{31},\Phi_{41}) \tag{1.38a}$$

$$(\Phi_{22},\Phi_{32},\Phi_{42})^T=\frac{2}{r^*}\frac{\partial r^*}{\partial\phi}\nabla\left(\frac{1}{r^*}\right) + \frac{1}{r^{*2}}\nabla\left(\frac{\partial r^*}{\partial\phi}\right) \tag{1.37b}$$

$$(\Lambda_{22},\Lambda_{32},\Lambda_{42})^T=\frac{2}{r^*}\frac{\partial r^*}{\partial\lambda}\nabla\left(\frac{1}{r^*}\right) + \frac{1}{r^{*2}}\nabla\left(\frac{\partial r^*}{\partial\lambda}\right) \tag{1.38b}$$

$$(\Phi_{23},\Phi_{31},\Phi_{43})^T=\frac{2}{r^*}\frac{\partial^2 r^*}{\partial\phi^2}\nabla\left(\frac{1}{r^*}\right) + \frac{1}{r^{*2}}\nabla\left(\frac{\partial^2 r^*}{\partial\phi^2}\right) \tag{1.37c}$$

$$(\Lambda_{23},\Lambda_{33},\Lambda_{43})^T=\frac{2}{r^*}\frac{\partial^2 r^*}{\partial\lambda^2}\nabla\left(\frac{1}{r^*}\right) + \frac{1}{r^{*2}}\nabla\left(\frac{\partial^2 r^*}{\partial\lambda^2}\right) \tag{1.38c}$$

$$(\Phi_{24},\Phi_{34},\Phi_{44})^T=\frac{3}{r^{*2}}\left(\frac{\partial r^*}{\partial\phi}\right)^2\nabla\left(\frac{1}{r^*}\right) + \frac{2}{r^{*3}}\frac{\partial r^*}{\partial\phi}\nabla\left(\frac{\partial r^*}{\partial\phi}\right) \tag{1.37d}$$

$$(\Lambda_{24},\Lambda_{34},\Lambda_{44})^T=\frac{3}{r^{*2}}\left(\frac{\partial r^*}{\partial\lambda}\right)\nabla\left(\frac{1}{r^*}\right) + \frac{2}{r^{*3}}\left(\frac{\partial r^*}{\partial\lambda}\right)\nabla\left(\frac{\partial r^*}{\partial\lambda}\right). \tag{1.38d}$$

### C. Special Procedure for the Cosine Factor in (1.2) for Large Block Sizes.

For larger block sizes one may wish to use an approximation higher than second-order to cos $\phi$ in (1.2).

Let

$$f = H\cos\phi \quad (1.39)$$

$$H = \frac{\chi r^2}{r^*} \quad (1.40)$$

The expansion in Taylor series is now done as

$$f = \cos\phi\,\{H_0 + (\phi-\phi_0)\,H_\phi + (\lambda-\lambda_0)\,H_\lambda + \frac{1}{2}\,[(\phi-\phi_0)^2\,H_{\phi\phi} + 2H_{\phi\lambda}\,(\phi-\phi_0)(\lambda-\lambda_0) + (\lambda-\lambda_0)^2\,H_{\lambda\lambda}] + \ldots\}\,. \quad (1.41)$$

Integrating the zero-order term

$$\iint H_0 \cos\phi\, d\phi\, d\lambda = 2\Delta\lambda H_0 \int_{-\Delta\phi}^{\Delta\phi} \cos\phi\, d\phi$$

$$= 4\Delta\lambda H_0 \cos\phi_0 \sin\Delta\phi\,. \quad (1.42)$$

Comparing this to (1.23) we can observe $f_0 = H_0\cos\phi_0$ and $\sin\Delta\phi$ is fairly small (less than $7\frac{1}{2}°$), we can use

$$\sin\Delta\phi = \Delta\phi - \frac{1}{6}\,\Delta\phi^3 + \frac{\Delta\phi^5}{120} - \frac{\Delta\phi^7}{5040} + \ldots \quad (1.43)$$

Since

$$\sin\Delta\phi \approx \Delta\phi - \frac{\Delta\phi^3}{6} \quad (1.43)$$

was assumed in (1.22). we can add the remaining terms of (1.43) to get a correction to (1.23)

$$\delta_1(\Delta T) = \frac{\Delta\phi^5\Delta\lambda f_0}{30}\left(1 - \frac{\Delta\phi^2}{42}\right)\,. \quad (1.44)$$

None of the *factors* in the correction depends on the satellite distance $r^*$: they depend only on the block size used.

Another correction of the same order as (1.44) may be obtained from the consideration of the first-order terms of (1.41). The term containing $(\lambda-\lambda_0)H_\lambda$ integrates out to zero, but $\frac{1}{2}H_\phi(\phi-\phi_0)\cos\phi$ does not. Hence. we may write

$$\Delta T_1 = \frac{1}{2}\iint H_\phi(\phi-\phi_0)\cos\phi\, d\phi\, d\lambda$$

$$= \Delta\lambda H_\phi\left\{\int_{\phi_0-\Delta\phi}^{\phi_0+\Delta\phi}\phi\cos\phi\, d\phi - \phi_0\int_{\phi-\Delta\phi}^{\phi+\Delta\phi}\cos\phi\, d\phi\right\}\,. \quad (1.45)$$

The integrals in (1.45) are standard forms and simplify to

$$\Delta T_1 = 2\Delta\lambda H_\phi \sin\phi_0(\Delta\phi\cos\Delta\phi - \sin\Delta\phi)\,. \quad (1.46)$$

The trigonometric functions of $\Delta\phi$ in (1.46) can be expanded in Taylor series: the terms in $\Delta\phi$ drop out:

$$\Delta T_1 = -\frac{2}{3}\,\Delta\lambda\Delta\phi^3 H_\phi \sin\phi_0 \times\left[1 - \frac{1}{10}\,\Delta\phi^2 + O\,(\Delta\phi^4)\right]\,. \quad (1.47)$$

Since the first term of (1.47) is already included in (1.23). the correction is

$$\delta_2\,(\Delta T) = \frac{1}{15}\,\Delta\lambda\Delta\phi^5 H_\phi \sin\phi_0. \quad (1.48)$$

All other corrections would be of even higher order and have been neglected since they will not improve the accuracy of the computation.

### D. Corrections to the Gravity Vector

Applying the operator $\nabla$ to (1.44) yields

$$\nabla[\delta_1(\Delta T)] = \frac{\Delta\phi^5\Delta\lambda}{30}\left(1 - \frac{\Delta\phi^2}{42}\right)\nabla f_0\,. \quad (1.49)$$

For (1.48) the result is

$$\nabla\,[\delta_2(\Delta T)] = \frac{\Delta\phi^5\Delta\lambda}{15}\sin\phi_0\nabla H_\phi. \quad (1.50)$$

$$H_\phi = -\frac{\chi r^2}{r^{*2}}\frac{\partial r^*}{\partial\phi}$$

$$= -\chi r^2\Phi_{12}. \quad (1.51)$$

$$=\Delta\lambda H_\phi = -\chi r^2(\Phi_{21}, \Phi_{31}, \Phi_{41})^T. \quad (1.52)$$

## 2. THE METHOD OF SINGULARITY-MATCHING

### A. Computing the Potential

Instead of (1.2) we use

$$\Delta T = G\iint \frac{f}{\sqrt{g}}\, d\lambda\, d\phi \quad (2.1)$$

where $f$ is now defined as

$$f = \chi r^2\cos\phi \quad (2.2)$$

and

$$g = r^{*2} \quad (2.3)$$

To compute the integral (2.1) expand $f$ and $g$ in Taylor series about $(\phi_0, \lambda_0)$

$$f = f_0 + f_\phi \Delta\phi + f_\lambda \Delta\lambda + \frac{1}{2}\,[f_{\phi\phi}\Delta\phi^2 + 2f_{\phi\lambda}\Delta\phi\Delta\lambda + f_{\lambda\lambda}\Delta\lambda^2] + \ldots \tag{2.4}$$

$$g = g_0 + g_\phi \Delta\phi + g_\lambda \Delta\lambda + \frac{1}{2}\,[g_{\phi\phi}\Delta\phi^2 + 2g_{\phi\lambda}\Delta\phi\Delta\lambda + g_{\lambda\lambda}\Delta\lambda^2] + \ldots \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.1) leads to

$$\Delta T = \iint \frac{A + B\Delta\lambda + C\Delta\lambda^2}{\sqrt{D + E\Delta\lambda + F\Delta\lambda^2}}\, d\lambda d\phi \tag{2.6}$$

with

$$A = f_0 + f_\phi \Delta\phi + \frac{1}{2} f_{\phi\phi}\Delta\phi^2. \tag{2.7a}$$

$$B = f_\lambda + f_{\phi\lambda}\Delta\phi, \tag{2.7b}$$

$$C = \frac{1}{2} f_{\lambda\lambda}. \tag{2.7c}$$

$$D = g_0 + g_\phi \Delta\phi + \frac{1}{2} g_{\phi\phi}\Delta\phi^2, \tag{2.7d}$$

$$E = g_\lambda + g_{\phi\lambda}\Delta\phi. \tag{2.7e}$$

$$F = \frac{1}{2} g_{\lambda\lambda}, \tag{2.7f}$$

$$f_0 = \chi r^2 \cos\phi. \tag{2.8}$$

$$f_\phi = \frac{\partial\chi}{\partial\phi} r^2 \cos\phi + 2r\frac{\partial r}{\partial\phi}\chi\cos\phi - \chi r^2 \sin\phi. \tag{2.9}$$

$$f_{\phi\phi} = \cos\phi\left[r^2\frac{\partial^2\chi}{\partial\phi^2} + 4\frac{\partial r}{\partial\phi}\frac{\partial\chi}{\partial\phi} + 2r\chi\frac{\partial^2 r}{\partial\phi^2} + 2\chi\left(\frac{\partial r}{\partial\phi}\right)^2\right] - \sin\phi\left[2r^2\frac{\partial\chi}{\partial\phi} - 4r\chi\frac{\partial r}{\partial\phi}\right] - f_0. \tag{2.10}$$

$$f_\lambda = \cos\phi\left[2r\chi\frac{\partial r}{\partial\lambda} + r^2\frac{\partial\chi}{\partial\lambda}\right] \tag{2.11}$$

$$f_{\lambda\lambda} = \cos\phi\left[2\chi\left(\frac{\partial r}{\partial\lambda}\right)^2 + 2r\chi\frac{\partial^2 r}{\partial\lambda^2} + 4r\frac{\partial r}{\partial\lambda}\frac{\partial\chi}{\partial\lambda} + r^2\frac{\partial^2\chi}{\partial\lambda^2}\right]. \tag{2.12}$$

$$f_{\lambda\phi} = -\sin\phi\left[2r\chi\frac{\partial r}{\partial\lambda} + r^2\frac{\partial\chi}{\partial\lambda}\right] + \cos\phi\left[2\chi\frac{\partial r}{\partial\phi}\frac{\partial r}{\partial\lambda} + 2r\left(\chi\frac{\partial^2 r}{\partial\phi\partial\lambda} + \frac{\partial r}{\partial\lambda}\frac{\partial\chi}{\partial\phi} + \frac{\partial r}{\partial\phi}\frac{\partial\chi}{\partial\lambda}\right) + r^2\frac{\partial^2\chi}{\partial\lambda\partial\phi}\right]. \tag{2.13}$$

$$g_0 = r^{*2}. \tag{2.14}$$

$$g_\phi = 2r^*\frac{\partial r^*}{\partial\phi}. \tag{2.15}$$

$$g_\lambda = 2r^*\frac{\partial r^*}{\partial\lambda}. \tag{2.16}$$

$$g_{\phi\phi} = 2\left[r^*\frac{\partial^2 r^*}{\partial\phi^2} + \left(\frac{\partial r^*}{\partial\phi}\right)^2\right] \tag{2.17}$$

$$g_{\phi\lambda} = 2\left[\frac{\partial r^*}{\partial\lambda}\frac{\partial r^*}{\partial\phi} + r^*\frac{\partial^2 r^*}{\partial\phi\partial\lambda}\right] \tag{2.18}$$

and

$$g_{\lambda\lambda} = 2\left[\left(\frac{\partial \mathbf{r}}{\partial\lambda}\right)^2 + \mathbf{r}^*\cdot\frac{\partial^2\mathbf{r}}{\partial\lambda^2}\right] \tag{2.19}$$

where (1.17) is applied. The derivatives of $r^*$ and $\mathbf{r}$ are obtained from (1.12), (1.13), (1.14), (1.15), (1.16) and (1.20).

The computational strategy will be to integrate (2.6) analytically with respect to $\lambda$. This will eliminate the improper condition from the integral. The integration with respect to $\phi$ then can be done numerically in a completely straightforward manner. The possibility of doing some or all of the integration with respect to $\phi$ by analytic means will be treated elsewhere. Special procedures for very low altitudes also will be treated elsewhere. These formulas will be suitable for satellite altitudes, typically 200 km or more.

We now define

$$(\Delta T)_i = \int_{\lambda_0-\Delta\lambda}^{\lambda_0+\Delta\lambda} (A_i + B_i\Delta\lambda + C_i\Delta\lambda^2) \times (D_i + E_i\Delta\lambda + F_i\Delta\lambda^2)^{-1/2} d\lambda \tag{2.20}$$

where the subscript $i$ indicates $A$, $B$, etc. are evaluated at sequentially spaced intervals of $\phi$

$$\phi_0 - \Delta\phi = \phi_1 < \phi_2 < \phi_3 \ldots < \phi_{n-1} < \phi_n$$
$$= \phi_0 + \Delta\phi$$

$$\phi_{i-1} - \phi_i = \delta\phi_i$$

so that (2.6) can be computed from (2.19) by numerical quadrature formulas. For simplicity the subscripts will be omitted in the detailed formulas that follow.

Let us adopt the abbreviation

$$X = F\Delta\lambda^2 + E\Delta\lambda + D \qquad (2.21)$$

and also drop the $\Delta$ symbols, so we can write

$$X = F\lambda^2 + E\lambda + D$$
$$(\Delta T)_i = \int (A + B\lambda + C\lambda^2)X^{-1/2}d\lambda. \quad (2.22)$$

All this does is move the working origin to the "center" of each equal-area block. Now we can apply standard forms to obtain

$$(\Delta T)_i = A\int\frac{d\lambda}{\sqrt{X}} + B\int\frac{\lambda d\lambda}{\sqrt{X}} + C\int\frac{\lambda^2 d\lambda}{\sqrt{X}}$$
$$= A\int\frac{d\lambda}{\sqrt{X}} + B\left[\frac{\sqrt{X}}{F} - \frac{E}{2F}\int\frac{d\lambda}{\sqrt{X}}\right] \quad (2.23)$$
$$+ C\left[\left(\frac{\lambda}{2F} - \frac{3E}{4F^2}\right)\sqrt{X}\right.$$
$$\left. + \frac{3E^2 - 4DF}{8F^2}\int\frac{d\lambda}{\sqrt{X}}\right]$$

(Peirce and Foster 1956, eq. 174 and 177).
By collecting terms we obtain

$$(\Delta T)_i = \frac{B\sqrt{X}}{F} + C\left[\frac{\lambda}{2F} - \frac{3}{4}\frac{E}{F^2}\right]\sqrt{X} + Q\int\frac{d\lambda}{\sqrt{X}} \quad (2.24)$$

with

$$Q = A - \frac{BE}{2F} + \frac{(3E^2 - 4DF)C}{8F^2}. \qquad (2.25)$$

The expression $\int X^{-1/2}\,d\lambda$ can be considered to define a function with two poles: one at each root of $X$. Hence, for all values of $D$, $E$, $F$ it can be expressed as a single elementary function of a complex variable, at least when we exclude negative values of the square root. Since we have the special case that $D$, $E$, $F$ are all real and that the result is real and the integral is taken only on the real line, it is more efficient computationally to express $\int X^{-1/2}\,d\lambda$ as different real functions of a real variable and use the one designated by appropriate functions of $D$, $E$, $F$. Using the appropriate complex function would about double the amount of computation.

The auxiliary functions needed are $F$ and

$$q = 4DF - E^2. \qquad (2.26)$$

The three useful basic forms are

$$\int\frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}}\sinh^{-1}\left(\frac{2F\lambda + E}{\sqrt{q}}\right), \qquad (2.27)$$
$$F > 0,\ q > 0;$$

$$\int\frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{-F}}\sin^{-1}\left(\frac{2F\lambda + E}{\sqrt{-q}}\right), \qquad (2.28)$$
$$F < 0,\ q < 0.$$

and

$$\int\frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}}\log\left(\sqrt{X} + \lambda\sqrt{F} + \frac{E}{2\sqrt{F}}\right), \qquad (2.29)$$
$$F > 0.$$

Form (2.29) is very handy for evaluating definite integrals, since the difference of logarithms is the logarithm of the quotient of the arguments

$$\log x - \log y = \log\frac{x}{y}; \qquad (2.30)$$

hence

$$\log\left[\kappa(\lambda)\right]_{-\Delta\lambda}^{\Delta\lambda} = \log\left[\frac{\kappa(\Delta\lambda)}{\kappa(-\Delta\lambda)}\right], \qquad (2.31)$$

where $\kappa(\lambda)$ is a function of $\lambda$. Similar, convenient forms can be obtained for (2.27) and (2.28) by applying appropriate identities.

The inverse hyperbolic sine may be expressed in logarithms as

$$\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right), \qquad (2.32)$$

so (2.31) may be applied to evaluate a definite integral of the form (2.27). Definite integrals of the form (2.28) may be evaluated with the identity

$$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$$
$$= \tan^{-1}\left[\frac{x\sqrt{1-y^2} - y\sqrt{1-x^2}}{xy + \sqrt{1-x^2}\sqrt{1-y^2}}\right]. \qquad (2.33)$$

In some cases $F=0$, or $F$ may be so small that (2.27), (2.28), and (2.29) may cause numerical problems. If $E$ is not similarly small the approximation

$$\Delta T \approx \int\int \frac{A+B\Delta\lambda+C\Delta\lambda^2}{\sqrt{D+E\Delta\lambda}}\,d\lambda d\phi \qquad (2.34)$$

may be used in place of (2.6). The analog of (2.23) is, then (Peirce and Foster, 1956, eq. 100, 101, 102):

$$(\Delta T)_i = \sqrt{D+E\lambda}\left[\frac{2A}{E} - \frac{2B(2D-E\lambda)}{3E^2} + \frac{2C(8D^2-4DE\lambda+3E^2\lambda^2)}{15E^3}\right]\Bigg|_{-\Delta\lambda}^{\Delta\lambda} \qquad (2.35)$$

Both $E\Delta\lambda$ and $F\Delta\lambda^2$ may be small compared to $D$, so that the binomial series is the most accurate method to use.

$$(D+E\lambda+F\lambda^2)^{-1/2} = D^{-1/2}\left[1-\frac{E\lambda+F\lambda^2}{2D} + \frac{3}{8}\left(\frac{E\lambda+F\lambda^2}{D}\right)^2 - \frac{5}{16}\left(\frac{E\lambda+F\lambda^2}{D}\right)^3 + \ldots\right]. \qquad (2.36)$$

Computing $(\Delta T)_i$ is done by multiplying and integrating power series in the usual way

$$\int_{-\Delta\lambda}^{\Delta\lambda}(A+B\lambda+C\lambda^2)\,X^{-1/2}\,d\lambda = D^{-1/2}\Big\{2A\Delta\lambda + \frac{2}{3}\Delta\lambda^3\left[C-\frac{BE}{2D}+\frac{3AE^2}{D^2}-\frac{AF}{2D}\right] + \frac{\Delta\lambda^5}{5}\left[\frac{3}{4}\frac{CE^2}{D^2}+\frac{3}{4}\frac{AF^2}{D^2}+\frac{3}{2}\frac{BEF}{D^2}-\frac{CF}{D}\right] + \frac{3}{28}\Delta\lambda^7\frac{CF^2}{D^2}+O(\Delta\lambda^9)\Big\}. \qquad (2.37)$$

The use of binomial series might be feasible over an entire block, but in some cases $E$ and $F$ will be small only for a limited band in latitude and the expansion of $\Delta T$ would not be accurate if done in terms of $\phi$ as well as $\lambda$.

## B. Gravity From the Singularity-Matching Method

Let us determine the gradients of the factors in (2.24):

$$\nabla Q = -\frac{C}{2F}\nabla D - \left(\frac{3}{4}\frac{EC}{F^2}-\frac{B}{2F}\right)\nabla E - \left(\frac{BE-CD}{2F^2}-\frac{3CE^2}{4F^3}\right)\nabla F \qquad (2.38)$$

$$\nabla D = -2\mathbf{r}^* - 2\frac{\partial \mathbf{r}}{\partial\phi}\Delta\phi - \frac{\partial^2\mathbf{r}}{\partial\phi^2}\Delta\phi^2 \qquad (2.39)$$

$$\nabla E = -2\frac{\partial\mathbf{r}}{\partial\lambda} - 2\frac{\partial^2\mathbf{r}}{\partial\phi\,\partial\lambda}\Delta\phi \qquad (2.40)$$

$$\nabla F = -\frac{\partial^2\mathbf{r}}{\partial\lambda^2}. \qquad (2.41)$$

The derivatives of $\mathbf{r}$ are given in (1.14), (1.15), (1.19), (1.20), and (1.21).

The gradient of (2.24) is given by

$$\nabla(\Delta T)_i = \nabla\left\{\frac{B\sqrt{X}}{F} + C\left[\frac{\lambda}{2F}-\frac{3}{4}\frac{E}{F^2}\right]\sqrt{X}\right\} + (\nabla Q)\int\frac{d\lambda}{\sqrt{X}} + Q\nabla\int\frac{d\lambda}{\sqrt{X}}. \qquad (2.42)$$

The first part of (2.42) is written symbolically because for the case of constant density blocks on an oblate spheroid $B=C=0$; moreover, for that case $\nabla Q=0$. Then (2.42) reduces to

$$\nabla(\Delta T)_i = Q\nabla\int\frac{d\lambda}{\sqrt{X}}. \qquad (2.42A)$$

If (2.27) or (2.28) is used, it is convenient to define

$$Y = (2F\lambda+E)\,|q|^{-1/2} \qquad (2.43)$$

and obtain

$$\nabla\int\frac{d\lambda}{\sqrt{X}} = -\frac{1}{2}F^{-3/2}(\nabla F)\sinh^{-1}Y + [F(1+Y^2)]^{-1/2}\nabla Y. \qquad (2.44)$$

or

$$\nabla\int\frac{d\lambda}{\sqrt{X}} = \frac{1}{2}(-F)^{-3/2}(\nabla F)\sin^{-1}Y + [-F(1-Y^2)]^{-1/2}\nabla Y, \qquad (2.45)$$

with

$$\nabla Y = |q|^{-1/2}(2\lambda\nabla F+\nabla E) - \frac{1}{2}(2F\lambda+E)\,|q|^{-3/2}\nabla|q| \qquad (2.46)$$

$$\nabla|q| = \frac{q}{|q|}\nabla q \qquad (2.47)$$

$$\nabla q = 4(F\nabla D + D\nabla F) - 2E\nabla E. \qquad (2.48)$$

If (2.29) is used, define

$$Z = X^{1/2} + \lambda F^{1/2} + 2EF^{-1/2} \qquad (2.49)$$

and obtain

$$\nabla \int \frac{d\lambda}{\sqrt{X}} = -\frac{1}{2} F^{-3/2} (\nabla F) \log Z + \frac{\nabla Z}{\sqrt{FZ}} \quad (2.50)$$

with

$$\nabla Z = \frac{1}{2} X^{-1/2} \nabla X + \frac{\lambda}{2} F^{-1/2} \nabla F$$
$$+ 2F^{-1/2} \nabla E - EF^{-3/2} \nabla F \quad (2.51)$$

$$\nabla X = \nabla D + \lambda \nabla E + \lambda^2 \nabla F. \quad (2.52)$$

Where $B = C = 0$, (2.35) simplifies to

$$(\Delta T)_i = \frac{2A}{E} \sqrt{D + E\lambda} \quad (2.53)$$

and

$$\nabla (\Delta T)_i = -\frac{2A\nabla E}{E^2} \sqrt{D + E\lambda}$$
$$+ \frac{A}{E} (\lambda \nabla E)(D + E\lambda)^{-1/2}. \quad (2.54)$$

For $B = C = 0$ (2.37) simplifies to

$$\int_{-\Delta\lambda}^{\Delta\lambda} AX^{-1/2} d\lambda = D^{-1/2} W \quad (2.55)$$

$$W = 2A\Delta\lambda + \frac{2}{3} \Delta\lambda^3 \left[ \frac{3AE^2}{D^2} - \frac{AF}{2D} \right] + \frac{3}{20} \Delta\lambda^5 \frac{AF^2}{D^2} \quad (2.56)$$

$$\nabla \int_{-\Delta\lambda}^{\Delta\lambda} AX^{-1/2} d\lambda = D^{-1/2} \nabla W - \frac{1}{2} W D^{-3/2} \nabla D \quad (2.57)$$

$$\nabla W = \frac{2}{3} \Delta\lambda^3 \left[ 6A \frac{E}{D} \nabla (E/D) - \frac{A}{2} \nabla (F/D) \right]$$
$$+ \frac{3}{10} \Delta\lambda^5 \frac{AF}{D} \nabla (F/D) \quad (2.58)$$

$$\nabla (E/D) = (D\nabla E - E\nabla D)/D^2 \quad (2.59)$$

$$\nabla (F/D) = (D\nabla F - F\nabla D)/D^2. \quad (2.60)$$

## 3. POINT MASS AND NUMERICAL CUBATURE ALGORITHMS

The simplest possible numerical cubature scheme was used by Koch and Morrison (1970) and Koch and Witte (1971). The constant density blocks were divided into four sub-blocks and the distance $r^*$ from the sub-block to the satellite was used: each sub-block was weighted by its area. Obviously, this is the same as using four point masses. Any conventional numerical cubature would correspond to some array of point masses.

One sets up the array of sampling points in the $\phi$, $\lambda$ coordinate system and applies the numerical cubature formula to

$$T = \int \int \frac{\chi d\sigma}{r^*} \quad (3.1)$$

to obtain

$$\Delta T_n = \sum_{i=1}^{N} \frac{w_i \chi(\phi_i, \lambda_i) \cos \phi_i \Delta\phi\Delta\lambda}{\sqrt{(x_i - x_s)^2 + (y_i - y_s)^2 + (z_i - z_s)^2}} \quad (3.2)$$

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = r_i(\phi_i, \lambda_i) \begin{pmatrix} \cos \lambda_i \cos \phi_i \\ \sin \lambda_i \cos \phi_i \\ \sin \phi_i \end{pmatrix} \quad (3.3)$$

$$T = \sum_n \Delta T_n, \quad (3.4)$$

where $w_i$ are the cubature formula weights.

For the formula actually used, $N = 1$, and $w_1 \Delta\phi\Delta\lambda \cos \phi_i =$ area of the block or sub-block. Hence, (3.2) simplifies to

$$\Delta T_n = A_n \chi_n / r_n^* \quad (3.5)$$

and

$$\nabla (\Delta T_n) = A_n \chi_n \nabla (1/r_n^*) \quad (3.6)$$

with

$$\nabla (1/r_n^*) = \frac{1}{r_n^{*3}} \begin{pmatrix} x_n - x_s \\ y_n - y_s \\ z_n - z_s \end{pmatrix}. \quad (3.7)$$

# 4. ANNOTATED COMPUTER LISTING

```
PROGRAM   DETOUR                    CDC 6600 FTN V3.0-324  OPT=1  03/26/76  13.05.18.     PAGE    1
          PROGRAM DETOUR(INPUT,OUTPUT)
C
C...      PROGRAM DETOUR IS THE DRIVING ROUTINE.
C
          COMMON/GEOCON/ALFA(4)
          COMMON/LYNX/BETA(21)
          COMMON/DIFCOR/GAMMA(7)
          COMMON/CONST/EPSILON(180)
          COMMON/XDT/ZETA(5)
          COMMON/VRBL/ETA(12)
          COMMON/EARTH/THETA(1640)
          COMMON/KEYSYM/IOTA(182)
          COMMON/QUAD/KAPPA(25)
          COMMON/SCRTCH/LAMBDA(6)
          COMMON/POINTS/MU(5300)
          COMMON/AXEL/NU(5)
          CALL SETUP
          CALL TTEST
          STOP6600
          END
```

```
      SUBROUTINE TTEST
C
C...   TTEST COMPUTES THE POTENTIAL AND ATTRACTION OF A POINT MASS AND
C      CALLS DTTDX TO COMPUTE THEM FOR THE APPROXIMATE DISTRIBUTION ON AN
C      OBLATE SPHEROID.
C
      COMMON/GEOCON/DYR, SR, ECCSQ, XMU
      COMMON/LYNX/X(6), UVW(6), ROTRAT, PI, PIO180, TWOPI, FOURPI,
     1     THETA, SINTH, COSTH, FLMJD
      COMMON/DIFCOR/COSB, SINB, RAD, RADXY, RAD2, RADXY2, RAD3
      COMMON/AXEL/IGRAV, DX(3), ITEM
      COMMON/VRBL/XS(3), RTM(3,3)
      COMMON/XDT/TT, DTDX(3), FACTOR
      DIMENSION UTOPIA(3), UE(4), UEG(3), UES(3), UED(3)
      EQUIVALENCE (YS,XS(2)), (ZS,XS(3)), (UE(2),UEG(1)),(UTOPIA,DX)
      KOUNT = 0
C
   96 CONTINUE
C
      KOUNT = 0
  100 FORMAT(F16.9, 4X, 3F15.8)
C...  READ TIME, ASTRONOMICAL LATITUDE, LONGITUDE, RADIAL COORDINATE.
   10 READ 100, TIMES,PHIS,XLONGS,RS
      IF(TIMES.LT.0.0) CALL EXIT
C...  CONVERT ANGLES TO RADIANS.
      PHIZ=PHIS*0.017453292519943
      XLONGZ=XLONGS*0.017453292519943
      COSFI=COS(PHIZ)
      RS = RS*DYR
      RS3= RS**3
C...  COMPUTE EARTH-FIXED COORDINATES.
      XS = RS*COSFI*COS(XLONGZ)
      YS = RS*COSFI*SIN(XLONGZ)
      SINFI = SIN(PHIZ)
      ZS = RS*SINFI
      UVW(1) = XS
      UVW(2) = YS
      X(3) = ZS
      FACTOR = 1.0
      CALL SECOND(TT1)
      CALL DTTDX
C...  VALUES FROM SUBROUTINE DTTDX.
C...  TT                  POTENTIAL
C...  DTDX                ATTRACTION COMPONENTS.
C...  ANGLE IS THE
      CALL SECOND(TT2)
      TEA = TT2 - TT1
      GEE2 = DTDX(1)**2 + DTDX(2)**2 + DTDX(3)**2
      GEE = SQRT(GEE2)
      IF(KOUNT.NE.0) GO TO 22
      PRINT 101
      PRINT 104
   22 CONTINUE
  101 FORMAT(1H1,10X,1HR,8X,3HPHI,7X,6HLAMBDA,11X,1HU,16X,2HUX,16X,2HUV,
     1     16X,2HUZ,16X, 1HG )
```

```
  102 FORMAT(4X,F10.6,2F12.6,5E18.10 )
  106 FORMAT(1X,E13.6,2F12.6,5E18.10 )
      IF(RS.GT.999.9999995) GO TO 25
      PRINT 102 ,RS, PHIS, XLONGS, TT, DTDX(1), DTDX(2), DTDX(3), GEE
      GO TO 26
   25 PRINT 106 ,RS, PHIS, XLONGS, TT, DTDX(1), DTDX(2), DTDX(3), GEE
   26 CONTINUE
      KOUNT=KOUNT + 1
C
C...  COMPUTE VALUES FOR A POINT MASS OR UNIFORM SPHERICAL DISTRIBUTION.
C...  UTAH              POTENTIAL
C...  UTOPIA            ATTRACTION COMPONENTS
      UTAH = XMU/RS
      UTOPIA(1) =-(XMU*XS(1))/RS3
      UTOPIA(2) = -(XMU*YS)/RS3
      UTOPIA(3) = -(XMU*ZS)/RS3
      GP2 = UTOPIA(1)**2 + UTOPIA(2)**2 + UTOPIA(3)**2
      GEEP = SQRT(GP2)
      PRINT 103, UTAH, (UTOPIA(KKK),KKK=1,3), GEEP
  103 FORMAT(38X,5E18.10 )
      KOUNT = KOUNT + 1
C
C...  COMPUTE TRUNCATION ERRORS OF VALUES FROM DTTDX.
      UE(1) = UTAH - TT
      UE(2) = UTOPIA(1) - DTDX(1)
      UE(3) = UTOPIA(2) - DTDX(2)
      UE(4) = UTOPIA(3) - DTDX(3)
      GE = GEEP - GEE
      PRINT 103, (UE(KKK), KKK = 1, 4), GE
      KOUNT = KOUNT + 1
      IF(GEE.LE.1.0E-42) GO TO 10
      ERROR2 = UEG(1)**2 + UEG(2)**2 + UEG(3)**2
      ERROR = SQRT(ERROR2)
      DO 30 IVY = 1, 3
      UES(IVY) = (UTOPIA(IVY)/GEEP + DTDX(IVY)/GEE)/2.0
      UED(IVY) = (UTOPIA(IVY)/GEEP - DTDX(IVY)/GEE)/2.0
   30 CONTINUE
      ANUM = UED(1)**2 + UED(2)**2 + UED(3)**2
      ADEN = UES(1)**2 + UES(2)**2 + UES(3)**2
C...  ANGLE             ANGLE IN DEGREES BETWEEN THE THEORETICAL AND
C                       COMPUTED ATTRACTION VECTORS.
      ANGLE = 114.5915590261*ATAN(SQRT(ANUM/ADEN))
      PRINT 105, ERROR, ANGLE, TEA
      KOUNT = KOUNT + 2
  105 FORMAT(38X,9HABS(DG) = ,E13.5,5X,12HDEFLECTION =,F13.9,5H DEG.
     1  5X, 7HTIME =  ,F8.4,5H SEC.  /)
  104 FORMAT(/)
      IF(KOUNT .GT.52) KOUNT = 0
      GO TO 10
      END
```

```
      SUBROUTINE SETUP
      COMMON/GEOCON/DYR, SR, ECCSQ, XMU
      COMMON/LYNX/X(6), UVW(6), ROIRAT, PI, PIO180, TWOPI, FOURPI,
     1    THETA, SINTH, COSTH, FLMJD
      COMMON/DIFCOR/COSB, SINB, RAD, RADXY, RAD2, RADXY2, RAD3
      COMMON/CONST/ D1(36), D2(36), D22(36), D11(36), D21(36)
      COMMON/XDT/ TT,DTDX(3)
      COMMON/VRBL/ XS(3),RTM(3,3)
      COMMON/EARTH/ RHO(1640)
      COMMON/KEYSYM/KEY(36), KEEY(36), KEYSUM(37), KEESUM(37), WTK(36)
      COMMON/QUAD/INTGT,INTL, DEEPHI, DEEFEE(10), DEFESQ(10)
      COMMON/SCRTCH/US(3) ,U(3)
      COMMON/POINTS/FCPHI(36,2), FCLAM(2552,2), R(36),DR(36), D2R(36)
      COMMON/AXEL/IGRAV,DX(3), IIEM
      DIMENSION PHISPH(18)
C...  N.B. THE CDC 6600 FORTRAN IV ALLOWS ONE TO LOAD COMMON BY DATA.
C...  PHISPH               SPHERICAL LAT. OF N. BORDER OF N. HEMISPHERE BLOCKS
      DATA                                                           PHISPH/
     1  0.0877226457377,0.1761387651605,0.2609487747858,0.3477560061365,
     2  0.4320564016823,0.5198194379129,0.6063482369929,0.6984854081328,
     3  0.7845763350315,0.8788890739735,0.9595073402986,1.0507850152541,
     4  1.1245264580253,1.2120202733884,1.2899116186106,1.3724955970138,
     5  1.4717630250231,1.5707963267949/
C...  R                    SPHEROID RADIUS AT CENTER OF EACH BLOCK.
      DATA                                                           R     /
     1  0.9966552494476,0.9967202763778,0.9968349994554,0.9969770143458,
     2  0.9971594289735,0.9973678447199,0.9976067771352,0.9978763272499,
     3  0.9981642186147,0.9984662417476,0.9987602423835,0.9990411766387,
     4  0.9992933418563,0.9995134867237,0.9996975056559,0.9998416344681,
     5  0.9999416897678,0.9999935212250/
C...  DR                   DERIVATIVE OF R WRT LAT.
      DATA                                                           DR    /
     1  -.0003293204460,-.0009760124276,-.0015372659667,-.0019913712656,
     2  -.0024055254852,-.0027483404743,-.0030235933640,-.0032257790967,
     3  -.0033320203234,-.0033373476021,-.0032347008642,-.0030296614629,
     4  -.0027354104516,-.0023632173291,-.0019227104544,-.0014240619733,
     5  -.0008775797852,-.0002948443230/
C...  D2R                  SECOND DERIVATIVE OF R.
      DATA                                                           D2R   /
     1  0.0033173064115,0.0031892505111,0.0029639096906,0.0026838183208,
     2  0.0023229744340,0.0019107279252,0.0014363152502,0.0009005506196,
     3  0.0003265757432,-.0002770035489,-.0008660974843,-.0014306000230,
     4  -.0019384104682,-.0023828167453,-.0027547886644,-.0030467097636,
     5  -.0032493581603,-.0033545246541/
      DATA (FOURPI = 12.566370614359173)
C
C...  THE KEY SYSTEM FOR REFERENCING THE BLOCKS AND SUB-BLOCKS.
      DATA KEY/4,12,16,20,28,28,40,40,52,52,60,60,64,64,68,68,72,72
     1        ,72,72,68,68,64,64,60,60,52,52,40,40,28,28,20,16,12,4/
      DATA KEEY/18,6,4,3,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,1,1
     1  ,1,1,1,2,2,2,2,2,2,3,4,6,18/
      DATA KEYSUM/0,4,16,32,52,80,108,148,188,240,292,352,412,476,540,
     1       608,676,748,820,892,964,1032,1100,1164,1228,1288,1348,1400,
     21452,1492,1532,1560,1588,1608,1624,1636,1640/
      DATA KEESUM/0,72,144,208,268,324,380,460,540,644,748,808,868,932
```

```
              1     ,996,1064,1132,1204,1276,1348,1420,1488,1556,1620,1684
              2     ,1744,1804,1908,2012,2092,2172,2228,2284,2344,2408,2480,2552/
      C
      C...  THE EQUATION NUMBERS IN THE MATHEMATICAL DESCRIPTION ARE GIVEN IN
      C                                                        COLUMNS 72.....80
      C
      C...  THETA            GREENWICH HOUR ANGLE, SET TO 0.0 IN THIS TEST.
            THETA = 0.0
            SINTH = 0.0
            COSTH = 1.0
            XMU = 1.0
      C...  INTGT            NUMBER OF POINTS IN NEWTON-COTES INTEGRATION.
            INTGT = 7
            INTL = INTGT - 1
            XINTL = FLOAT(INTL)
            DYR = 1.0
            PI = FOURPI/4.0
            TWOPI = FOURPI/2.0
      C
      C...  COMPUTE THE AREA FOR ANY SUB-BLOCK IN ALL 36 ZONES.
            DO 62 IY = 1, 36
            WTK(IY) = 12.5382906627047(1640.*FLOAT(KEEY(IY)))
         62 CONTINUE
      C     ***         ****
            DO 60 IY = 1, 18
            IDX = 19 - IY
            PHIBRS = 0.5*PHISPH(IDX)
            D1(IY) = 0.5*PHISPH(IDX)
            IDS = 37 - IY
            R(IDS) = R(IY)
            DR(IDS) = -DR(IY)
            D2R(IDS) = D2R(IY)
            IF(IDX.GT.1) D1(IY) = D1(IY) - 0.5*PHISPH(IDX-1)
            IF(IDX.GT.1) PHIBRS = PHIBRS + 0.5*PHISPH(IDX-1)
            FCPHI(IY,1) = COS(PHIBRS)
            FCPHI(IY,2) = SIN(PHIBRS)
            FCPHI(IDS,1) = FCPHI(IY,1)
            FCPHI(IDS,2) = -FCPHI(IY,2)
            D11(IY) = D1(IY)**2
            IWTOP = KEY(IY)*KEEY(IY)
            D2(IY) = PI/IWTOP
            D22(IY) = D2(IY)**2
            D21(IY) = D1(IY)*D2(IY)
            D1(IDS) = D1(IY)
            D2(IDS) = D2(IY)
            D11(IDS) = D11(IY)
            D21(IDS) = D21(IY)
            D22(IDS) = D22(IY)
            DO 60 IW = 1, IWTOP
            ARGL = FLOAT(2*IW - 1)*D2(IY)
            IDXL = KEESUM(IY) + IW
            IDXL2 = KEESUM(38-IY) - IW + 1
            FCLAM(IDXL,1) = COS(ARGL)
            FCLAM(IDXL,2) = SIN(ARGL)
            FCLAM(IDXL2,1) = FCLAM(IDXL,1)
```

```
      FCLAM(IDXL2,2) = -FCLAM(IDXL,2)
   60 CONTINUE
C...  THIS SECTION LOADS THE DENSITY VALUES TO MODEL APPROXIMATELY A
C     CENTRAL FORCE FIELD BY A COATING ON AN OBLATE SPHEROID.
      DO 75 I = 1, 36
      CHI = 1.0/R(I)**2
      R(I) = R(I)*DYR
      DR(I) = DR(I)*DYR
      D2R(I) = D2R(I)*DYR
      J1 = KEYSUM(I) + 1
      J2 = KEYSUM(I+1)
      DO 75 J = J1,J2
C...  LOAD DENSITY VALUES INTO COMMON.
      RHO(J) = CHI/FOURPI                                          (22)
   75 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DTTDX
C
C...  DTTDX COMPUTES THE POTENTIAL AND ATTRACTION DUE TO A SURFACE
C     DENSITY ON AN OBLATE SPHEROID. THE DENSITY IS PARAMETERIZED BY THE
C     USE OF 1640 EQUAL AREA BLOCKS, EACH ABOUT 5 DEGREES ON A SIDE.
C---
      COMMON/GEOCON/DYR, SR, ECCSQ, XMU
      COMMON/CONST/ D1(36), D2(36), D22(36), D11(36), D21(36)
      COMMON/LYNX/X(6), UVW(6), ROTRAT, PI, PI0180, TWOPI, FOURPI,
     1    THETA, SINTH, COSTH, FLMJD
      COMMON/DIFCOR/COSB, SINB, RAD, RADXY, RAD2, RADXY2, RAD3
      COMMON/EARTH/ RHO(1640)
      COMMON/KEYSYM/ KEY(36), KEEY(36), KEYSUM(37), KEESUM(37),WTK(36)
      COMMON/XDT/TFCN(4) , FACTOR
      COMMON/VRBL/ XS(3),RTM(3,3)
      COMMON/QUAD/INTGT,INTL, DEEPHI, DEEFEE(10),DEFESQ(10)
      COMMON/SCRTCH/US(3) ,U(3)
      COMMON/POINTS/FCPHI(36,2), FCLAM(2552,2), R(36), DR(36), D2R(36)
C---
      DIMENSION DSRP(3),DSRL(3),DSR2P(3),DSR2L(3), FPHI(4), FZERO(4)
      DIMENSION FCOMP(4,2), PL(4,4,2), ALBT(4,2), GRRTQ(3), RSTAR(3)
      DIMENSION    FFF(10), D(4), E(4), F(4), D2RDPL(3), GARGG(2,3)
      DIMENSION DRDFI(3),DRDLM(3),D2RDF2(3),D2RDL2(3), ALPHA(6,10)
      DIMENSION GTOTAL(4), FDF(10,4),'GRAD(2,3), GARG(2,3), GRQQQ(3)
      DIMENSION GRADF(10,3), GRT(2,3), GRQRTF(3), GRASH(2,3),GRRTF(3)
C---
      EQUIVALENCE(PL(1,2,1),R2RF), (PL(1,2,2),R2RL), (FONE,AF1)
      EQUIVALENCE (FFF(1),FDF(1,1)), (GRADF(1,1),FDF(1,2)), (GO,RSTAR2)
      EQUIVALENCE(PL(1,3,1),R2R2F), (PL(1,3,2),R2R2L), (FO,FB)
      EQUIVALENCE(A,ABC,Q),  (F(1),G22A), (U(3),W)
C
C...  COEFFICIENTS FOR NEWTON-COTES NUMERICAL QUADRATURE              ORDERS
      DATA ALPHA/1.0,4*0.0,1.0,1.0,1.0,3*0.0,2.0,1.0,2.0,3*0.0,3.0,    1,2,3
     1    3.0,9.0,3*0.0,8.0, 14.0,64.0,12.0,2*0.0,45.0,               4,5
     2    95.0,375.0,250.0,2*0.0,288.0, 41.0,216.0,27.0,136.0,0.0,140.0, 6,7
     3    5257.0,25039.0,9261.0,20923.0,0.0,17280.0,                  8
     4    3956.0,23552.0,-3712.0,41984.0, -9080.,14175.0,             9
     5    25713.0,141669.0,9720.0,174096.0,52002.0,89600.0/           10
      DATA(ALBT(1,2) = 0.0), (ALBT(2,2) = 0.0)
      DATA(DRDLM(3)=0.0), (D2RDL2(3)=0.0), (F(4)=0.0)
      DATA (D2RDPL(3)=0.0)
C
C...  THE EQUATION NUMBERS IN THE MATHEMATICAL DESCRIPTION ARE GIVEN IN
C                                                       COLUMNS 72.....80
C
      INTGTY = INTGT + 1
      INTACT = INTGTY / 2
      TAG2 = DYR
      TAG = 0.25*DYR
      ALPHER = FLOAT(INTGT-1)*ALPHA(6,INTGT)
C...  US              SATELLITE POSITION IN EARTH-FIXED COORDINATES.
      US(1) = UVW(1)
      US(2) = UVW(2)
      US(3) = X(3)
      DO 101 NO = 1, 4
```

```
          TFCN(NO) = 0.0
      101 CONTINUE
C...      CYCLE THROUGH THE 36 ZONES.
          DO 999 I = 1, 36
          JTOP = KEY(I)
          KTOP = KEEY(I)
          FAKTOR = (2.0*D1(I))/ALPHER
          DEFI = (2.0*D1(I))/FLOAT(INTGT-1)
          DO 105 II = 1, INTGT
          DEEFEE(II) = -D1(I) + FLOAT(II-1)*DEFI
          DEFESQ(II) = DEEFEE(II)**2
      105 CONTINUE
          DELLAM = D2(I)
          DLAM2 = D22(I)
          COSPHI = FCPHI(I,1)
          SINPHI = FCPHI(I,2)
          R1 = DR(I)
          R2 = D2R(I)
          TWOR2 = R2 + R2
          ARE = R(I)
          RCOSFI = ARE*COSPHI
          RCF2 = RCOSFI**2
          ARE2 = ARE**2
C...      U                        COORDINATES OF CENTER OF BLOCK
          U(3) = ARE*SINPHI                                              (1.11)
          RSTAR(3) = U(3) - US(3)                                        (1.8)
          RST3SQ = RSTAR(3)**2
C...      CYCLE OVER THE BLOCKS IN A ZONE.
          DO 999  J = 1, JTOP
C...      JRHO                     INDEX TO BLOCKS = 1,...,1640.
          JRHO = KEYSUM(I) + J
          JLAMO = KEESUM(I) + (J-1)*KEEY(I)
          RHX = RHO(JRHO)
C...      CYCLE OVER THE SUB-BLOCKS WITHIN A BLOCK.
          DO 999 K = 1, KTOP
C...      JLAM                     INDEX TO SUB-BLOCKS = 1,...,2552.
          JLAM = JLAMO + K
          COSLAM = FCLAM(JLAM,1)
          SINLAM = FCLAM(JLAM,2)
          U(1) = RCOSFI*COSLAM                                           (1.11)
          U(2) = RCOSFI*SINLAM                                           (1.11)
          RSTAR(1) = U(1) - US(1)                                        (1.8)
          RSTAR(2) = U(2) - US(2)                                        (1.8)
          RSTAR2 = RSTAR(1)**2 + RSTAR(2)**2 + RST3SQ
          RST  =SQRT(RSTAR2)
          RST1=1.0/RST
          RST2=1.0/RSTAR2
          RST3 = RST1*RST2
C...      IF(RST.GT.TAG2) USE THE ONE-POINT NUMERICAL CUBATURE.
          IF(RST.GT.TAG2) GO TO 701
          RST23 = 3.0 * RST2
          CFCL = COSPHI*COSLAM
          CFSL = COSPHI*SINLAM
          DRDFI(1)= -W*COSLAM + R1*CFCL                                  (1.14)
          DRDFI(2)= -W*SINLAM + R1*CFSL                                  (1.14)
```

```
              DRDFI(3) = RCOSFI +R1*SINLAM                                          (1.16)
              DRDLM(1) = -U(2)                                                      (1.15)
              DRDLM(2) =  U(1)                                                      (1.15)
        C     DRDLM(3) =  0.0                                                       (1.15)
              D2RDF2(1) = -U(1) + TWOR2*CFCL                                        (1.19)
              D2RDF2(2) = -U(2) + TWOR2*CFSL                                        (1.19)
              D2RDF2(3) = -U(3) + TWOR2*SINPHI                                      (1.19)
              D2RDL2(1) = -U(1)                                                     (1.21)
              D2RDL2(2) = -U(2)                                                     (1.21)
              DRFC = RSTAR(1)*DRDFI(1) + RSTAR(2)*DRDFI(2) + RSTAR(3)*DRDFI(3)      (1.16)
              G1 = DRFC + DRFC                                                      (2.15)
              DRSDFI = RST1*DRFC                                                    (1.12)
              DRFC2 = -RSTAR(1)*U(2) + RSTAR(2)*U(1)                                (2.16)
              G2 = DRFC2 + DRFC2                                                    (2.16)
              DRSDLM = RSTT*DRFC2                                                   (1.13)
              GAA = RSTAR(1)*U(T) + RSTAR(2)*U(2)
              G22A = RCF2 - GAA                                                     (2.19)
              G11A = ARE2 - GAA -W*RSTAR(3)                                         (2.17)
              FA = RHX*ARE2
              FB = FA*COSPHI
              FC = FA*SINPHI
              TWOR= ARE+ARE
              FOUR = TWOR + TWOR
        C     IF(RST.GT.TAG) USE THE TAYLOR SERIES METHOD.
              IF(RST.GT.TAG) GO TO 601
        C
        C...  SINGULARITY-MATCHING METHOD.
        C
              D2RDPL(1) = -DRDFI(2)                                                 (1.20)
              D2RDPL(2) =  DRDFI(1)                                                 (1.20)
        C     D2RDPL(3) = 0.0                                                       (1.20)
              DOTT1 = -RSTAR(1)*DRDFI(2) + RSTAR(2)*DRDFI(1)
              DOTT2 = -DRDFI(1)*U(2) + DRDFI(2)*U(1)
              DRS2DB = RST1*(DOTT1 + DOTT2 - DRSDFI*DRSDLM)                         (1.17)
              G21 = 2.0*(DRSDFI*DRSDLM + RST*DRS2DB)                                (2.18)
        C     F0 = RHO(I,J)*ARE2*COSPHI                                             (2.8)
              F1 = COSPHI*TWOR*RHX*DR(I) - FC                                       (2.9)
              F11 = COSPHI*(TWOR*D2R(I) + 2.0*RMX*DR(I)**2)                         (2.10)
              F11 = F11 + SINPHI*FOUR*DR(I)*RHX                                     (2.10)
              F11 = F11 - FB                                                        (2.10)
              F11A = 0.5*F11
        C...  CYCLE THROUGH THE NUMERICAL QUADRATURE.
              DO 445 II = 1, INTGT
              A = F0 + F1*DEEFEE(II) + F11A*DEFESQ(II)                              (2.7A)
              D(1) = G0 + G1*DEEFEE(II) + G11A*DEFESQ(II)                           (2.7D)
              E(1) = G2 + G21*DEEFEE(II)                                            (2.7E)
        C     F(1) = G22/2.0                                                        (2.7F)
              ESQ = E(1)**2
              DO 441 IB = 2, 4
              IBL = IB - 1
              D(IB) = -(2.0*RSTAR(IBL) + 2.0*DRDFI(IBL)*DEEFEE(II)                  (2.39)
             1          + D2RDF2(IBL)*DEFESQ(II) )
              E(IB) = -2.0*(DRDLM(IBL) + D2RDPL(IBL)*DEEFEE(II))                    (2.40)
        C     F(IB) = -D2RDL2(IBL)                                                  (2.41)
          441 CONTINUE
```

```
      F(2)=U(1)                                                          (2.41)
      F(3)=U(2)                                                          (2.41)
 1010 FORMAT(9E13.5)
      AD1 = ABS(D(1))
      AF1 = ABS(F(1))
      IF(AF1.GT.0.001*AD1) GO TO 443
      AE1 = ABS(E(1))
      IF(AE1.GT.10000.0*AF1.AND.AE1.GT.0.01*AD1) GO TO 439
C
C...  THIS SECTION USES THE BINOMIAL SERIES.
      DHF = 1.0/SQRT(D(1))
      DL2 = DELLAM**2
      DL3 = DELLAM*DL2
      DL5 = DL3*DL2
      BIGED = E(1)/D(1)
      FED = F(1)/D(1)
      ED2 = BIGED**2
      FED2 = FED**2
      DL323 = 0.6666666666667*DL3
      FCT2 = A*(2.0*DELLAM + DL323*(0.375*ED2 - 0.5*FED) +0.15*DL5*FED2)
      DSQ = D(1)**2
      FFF(II) = DHF*FCT2                                                 (2.37)
      DO 438 IB = 1, 3
      IBP = IB + 1
      EDPR  = (D(1)*E(IBP)-E(1)*D(IBP))/DSQ                              (2.59)
      FEDPR = (D(1)*F(IBP) - F(1)*D(IBP))/DSQ                            (2.60)
      GRADF(II,IB) = (-0.5*D(IBP)*DHF*FCT2)/D(1)                         (2.57)
     1     + DHF*(DL323*(0.75*A*BIGED*EDPR - 0.5*A*FEDPR)
     2     + 0.30*A*FED*FEDPR*DL5)
  438 CONTINUE
      GO TO 445
C
C...  THIS SECTION ASSUMES F = 0.
  439 CONTINUE
 1001 FORMAT(3I5,6E16.7)
      FFF(II) = 0.0
      GRADF(II,1) = 0.0
      GRADF(II,2) = 0.0
      GRADF(II,3) = 0.0
      ECUBE = ESQ*E(1)
      DO 442 IBIS = 1, 2
      FC1 = 2*IBIS-3
      FC2 = -FC1
      DLL = FC2*DELLAM
      FCT1 = SQRT(D(1) + E(1)*DLL)
C     ABC = A
      FCT2 = (ABC+ABC)/E(1)
      FFF(II) = FFF(II) + FCT1*FCT2*FC2                                  (2.35)
      DO 442 IB = 1, 3
      IBP = IB + 1
      EDP = E(1)*D(IBP)
      DEP = D(1)*E(IBP)
      DFCT1 = (0.5*( D(IBP) + E(IBP)*DLL))/FCT1
      DFCT2 = -( (ABC+ABC)*E(IBP))/ESQ
      GRADF(II,IB) = GRADF(II,IB) + FC2*(FCT1*DFCT2 + FCT2*DFCT1)        (2.54)
```

```
  442 CONTINUE
      GO TO 445
C
  443 CONTINUE
      FSQ = F(1)**2
      FCUBE = F(1)*FSQ
C     Q= A                                                              (2.25)
C     FONE = ABS(F(1))
      FONERT = SQRT(FONE)
      RTF = SIGN(FONERT, F(1))
      QRTF = Q/RTF
      QQ = 4.0*D(1)*F(1) - ESQ                                          (2.26)
      F32 = FONE*FONERT
      FX32 = SIGN(FONERT,F(1))
      FF32 = 1.0/FX32
      QQQ = ABS(QQ)
      RTQ = SQRT(QQQ)
      HALF = SIGN(0.5 , F(1))
      TWORTQ = 2.0*RTQ
      DO 1445 IB = 1, 3
      IBP = IB + 1
      GRQIB = 4.0*(D(IBP)*F(1) + D(1)*F(IBP)) - 2.0*E(1)*E(IBP)
      GRQQQ(IB) = GRQIB*SIGN(1.0,QQ)
      GRRTF(IB) = (HALF*F(IBP))/FONERT
      GRRTQ(IB) = GRQQQ(IB)/TWORTQ
      GRQRTF(IB) = -(0.5*Q*F(IBP))/F32
 1445 CONTINUE
      RADO = D(1) + F(1)*DLAM2
      ED = E(1)*DELLAM
      RAD1 = RADO + ED
      RAD2 = RADO - ED
      ARGO = 2.0*F(1)*DELLAM
      ARG1 = (E(1) + ARGO)/RTQ                                          (2.43)
      ARG2 = (E(1) - ARGO)/RTQ                                          (2.43)
      DO 1446 IB = 1, 3
      IBP = IB + 1
      GRD = D(IBP) + F(IBP)*DLAM2
      EDD = E(IBP)*DELLAM
      GRAD(1,IB) = GRD +EDD
      GRAD(2,IB) = GRD - EDD
      GARGO = 2.0*F(IBP)*DELLAM
      GARG(1,IB) =  (RTQ*(E(IBP)+GARGO) - (E(1)+ARGO)*GRRTQ(IB))/QQQ    (2.46)
      GARG(2,IB) =  (RTQ*(E(IBP)-GARGO) - (E(1)-ARGO)*GRRTQ(IB))/QQQ    (2.46)
 1446 CONTINUE
      RT1 = SQRT(RAD1)
      RT2 = SQRT(RAD2)
      RTO1 = 0.5/RT1
      RTO2 = 0.5/RT2
      DO 1447 IB = 1, 3
      GRT(1,IB) = RTO1*GRAD(1,IB)
      GRT(2,IB) = RTO2*GRAD(2,IB)
 1447 CONTINUE
C...
      IF(QQ.LT.0.0) GO TO 446
      BARG1 = SQRT(1.0 + ARG1**2)
```

```
      BARG2 = SQRT(1.0 + ARG2**2)
      DASH = ALOG((ARG1 + BARG1)/(ARG2 + BARG2))                        (2.32)
  448 CONTINUE
      FARG1 = 1.0/BARG1
      FARG2 = 1.0/BARG2
      DO 1448 IB = 1, 3
      GRASH(1,IB) = FARG1*GARG(1,IB)
      GRASH(2,IB) = FARG2*GARG(2,IB)
 1448 CONTINUE
      GO TO 447
C...
  446 CONTINUE
      IF(ABS(ARG1).LE.1.0.AND.ABS(ARG2).LE.1.0) GO TO 449
      IF(F(1).GT.0.0) GO TO 452
  453 CONTINUE
 1002 FORMAT(6X,6E20.10)
      PRINT 1002, QQ, D(1), E(1), F(1)
      GO TO 475
  452 CONTINUE
      A1 = 2.0*(RTF*RT1 + F(1)*DELLAM) + E(1)
      A2 = 2.0*(RTF*RT2 - F(1)*DELLAM) + E(1)
      TEST = A1/A2
      IF(TEST.GT.0.0) GO TO 451
      PRINT 1003, A1,A2
 1003 FORMAT(3X, 4HARGG  , 2E20.12)
      GO TO 453
C...
  451 CONTINUE
      DASH = ALOG(TEST)                                                 (2.31)
      DO 454 IB = 1, 3
      IBP = IB + 1
      FIDL = F(IBP)*DELLAM
      GARGG(1,IB) = 2.0*(GRRTF(  IB)*RT1 + RTF*GRT(1,IB) + FIDL)*E(IBP)
      GARGG(2,IB) = 2.0*(GRRTF(  IB)*RT2 + RTF*GRT(2,IB) - FIDL)*E(IBP)
      GRASH(1,IB) = GARGG(1,IB)/A1
      GRASH(2,IB) = GARGG(2,IB)/A2
  454 CONTINUE
      GO TO 447
C...
  449 CONTINUE
      BARG1 = SQRT(1.0 - ARG1**2)
      BARG2 = SQRT(1.0 - ARG2**2)
      XNUM = ARG1*BARG2 - ARG2*BARG1
      DENO = ARG1*ARG2 + BARG1*BARG2
      DASH = ATAN2(XNUM,DENO)                                           (2.33)
      GO TO 448
C...
  447 CONTINUE
      FFF(II) = QRTF*DASH                                               (2.24)
      DO 444 IB = 1, 3
      GRADF(II,IB) =                                                    (2.44)
     1          GRQRTF(IB)*DASH + QRTF*(GRASH(1,IB) - GRASH(2,IB))      (2.45)
  444 CONTINUE
C
  445 CONTINUE
```

```
      C...  NEWTON-COTES NUMERICAL QUADRATURE.
            DO 476 IB = 1, 4
            DO 476 NI = 1, INTACT
            INTNI = INTGTY - NI
            TFCN(IB) = TFCN(IB) + ALPHA(NI,INTGT)*(FDF(NI,IB) + FDF(INTNI,IB))
           1  *FAKTOR
        476 CONTINUE
            GO TO 999
      C
      C...  THE TAYLOR SERIES METHOD.
      C
        601 CONTINUE
            DR2FC = G11A - DRSDFI**2
            DR2FC2 = G22A - DRSDLM**2
            DRS2DF = RST1*DR2FC                                          (1.16)
            DRS2DL = RST1*DR2FC2                                         (1.18)
            DELPHI = D1(I)
            DLDP = D21(I)
            DPHI2 = D11(I)
            ZWOLF = 12.0 + (0.1 - 0.0238095238T*DPHI2)*DPHI2**2
            DLDP3 = DLDP/3.0
      C...   COMPUTE F .
      C               0
            FZERO = RST1*FB
            DO 490 IJK = 2, 4
            ICY = IJK - 1
            FZERO(IJK) = RST3*RSTAR(ICY)
            PL(IJK,1,1) = FZERO(IJK)                                     (1.37A)
            PL(IJK,1,2) = FZERO(IJK)                                     (1.37B)
            FZERO(IJK) = FB*FZERO(IJK)                                   (1.26)
        490 CONTINUE
      C...   COMPUTE SECOND DERIVATIVES OF F .
      C                                    0
            ALBT(1,1) = COSPHI*(RHX+RHX)*(    R1**2 + ARE* R2    ) - FB  (1.35A)
           1      -FOUR*SINPHI*RHX*R1
            FPHIB = TWOR*R1*RHX
            ALBT21 = FC - COSPHI*FPHIB
            ALBT(2,1) = ALBT21 + ALBT21                                  (1.35B)
            ALBT(3,1) = -FB                                              (1.35C)
            ALBT(4,1) = FB + FB                                          (1.35D)
      C     ALBT(1,2) = 0.0                                              (1.36A)
      C     ALBT(2,2) = 0.0                                              (1.36B)
            ALBT(3,2) = -FB                                              (1.36C)
            ALBT(4,2) = ALBT(4,1)                                        (1.36D)
      C
            DRDPSQ = DRSDFI**2
            DRDLSQ = DRSDLM**2
            R2RF   = RST2*DRSDFI
            R2RL   = RST2*DRSDLM
            R2R2F = RST2*DRS2DF
            R2R2L = RST2*DRS2DL
            DO 440 IVV =1, 3
            DSRP(IVV) =  -RST1*DRDFI(IVV) + R2RF*RSTAR(IVV)
            DSRL(IVV) =  -RST1*DRDLM(IVV) + R2RL*RSTAR(IVV)
            DSR2P(IVV) = R2R2F*RSTAR(IVV) - RST1*(D2RDF2(IVV)
```

```
     1                  +2.0*DRSDFI*DSRP(IVY))
      DSR2L(IVY) = R2R2L*RSTAR(IVY) - RST1*(D2RDL2(IVY)
     1                  +2.0*DRSDLM*DSRL(IVY))
  440 CONTINUE
C...  OTHER PL(1,-,-) IN EQUIVALENCE STATEMENTS.
      PL(1,1,1) = RST1                                               (1.31A)
      PL(1,4,1) = RST3*DRDPSQ                                        (1.31D)
      PL(1,1,2) = RST1                                               (1.32A)
      PL(1,4,2) = RST3*DRDLSQ                                        (1.32D)
C
      TWORS3 = RST3 + RST3
      FPHI(1) = -RST2*DRSDFI*FA + RST1*FPHIB
C
      DO 460  IVY = 1, 3
      IVORY = IVY+1
      PLRST = (RST1+RST1)*PL(IVORY,1,1)
      PL23 = RST23*PL(IVORY,1,1)
      PL(IVORY,2,1) = PLRST*DRSDFI + RST2*DSRP(IVY)                  (1.37B)
      FPHI(IVORY) = -PL(IVORY,2,1)*FA + PL(IVORY,1,1)*FPHIB
      PL(IVORY,2,2) = PLRST*DRSDLM + RST2* DSRL(IVY)                 (1.38B)
      PL(IVORY,3,1) = PLRST*DRS2DF + RST2*DSR2P(IVY)                 (1.37C)
      PL(IVORY,3,2) = PLRST*DRS2DL + RST2*DSR2L(IVY)                 (1.38C)
      PL(IVORY,4,1) = PL23*DRDPSQ +TWORS3*DRSDFI*DSRP(IVY)           (1.37D)
      PL(IVORY,4,2) = PL23*DRDLSQ +TWORS3*DRSDLM*DSRL(IVY)           (1.38D)
  460 CONTINUE
C
      DO 410 JJ = 1, 2
      DO 410 II = 1, 4
      FCOMP(II,JJ) = 0.0
      DO 410 KK = 1, 4
      FCOMP(II,JJ) =                                                 (1.33)
     1              FCOMP(II,JJ) + PL(II,KK,JJ)*ALBT(KK,JJ)          (1.34)
  410 CONTINUE
C
      SD2 = 0.2*SINPHI*DPHI2
      DO 470 IVORY = 1, 4
      FCOMP(IVORY,1) =                                               (1.48)
     1                 FCOMP(IVORY,1) + SD2 * FPHI(IVORY)            (1.50)
  470 CONTINUE
C
      DO 510 N = 1, 4
      TFCN(N) = TFCN(N) +(ZWOLF*FZERO(N) + 2.0*(FCOMP(N,1)*DPHI2 +   (1.23)
     1          FCOMP(N,2)*DLAM2))*DLDP3                             (1.25A)
  510 CONTINUE
C
      GO TO 999
C
C...  THE ONE-POINT NUMERICAL CUBATURE.
C
  701 CONTINUE
      RHXQ = WTK(I)*RHX
      TFCN(1) = TFCN(1) + RST1*RHXQ                                  (3.3)
      RHXQ3 = RST3*RHXQ
      DO 799 N = 2, 4
      TFCN(N) = TFCN(N) + RHXQ3*RSTAR(N-1)                           (3.6)
```

```
  799 CONTINUE
C
  999 CONTINUE
C
  475 CONTINUE
C...  ROTATE ATTRACTION VECTOR INTO INERTIAL SYSTEM.
      TFCN(1) = FACTOR*TFCN(1)
      TFX = FACTOR*(COSTH*TFCN(2) - SINTH*TFCN(3))
      TFY = FACTOR*(SINTH*TFCN(2) + COSTH*TFCN(3))
      TFCN(4) = FACTOR*TFCN(4)
      TFCN(2) = TFX
      TFCN(3) = TFY
      RETURN
      END
```

```
    R          PHI        LAMBDA            J                  UX                  UY                  UZ                  G

1.047000    0.000000  282.000000   .9555855701E+00  -.1899349292E+00   .8935121503E+00   .1574606630E-04   .9134764585E+00
                                   .9551098376E+00  -.1896642801E+00   .8923002828E+00 -0.                 .9122348019E+00
                                  -.2757324670E-03   .2706490710E-03  -.1211867455E-02  -.1574606650E-04  -.1241656575E-02
                              ABS(DG) =    .12418E-02      DEFLECTION =    .001271757 DEG.     TIME =    .4630 SEC.

1.047000    1.000000  282.000000   .9555824950E+00  -.1898629909E+00   .8931430484E+00  -.1605287805E-01   .9132415645E+00
                                   .9551098376E+00  -.1896353953E+00   .8921643812E+00  -.1592069253E-01   .9122348019E+00
                                  -.2726573662E-03   .2275976440E-03  -.9786672751E-05   .1321855156E-03  -.1006762601E-02
                              ABS(DG) =    .10134E-02      DEFLECTION =    .007291569 DEG.     TIME =    .4620 SEC.

1.047000    3.000000  282.000000   .9553829155E+00  -.1896645347E+00   .8922345418E+00  -.4896076307E-01   .9134357943E+00
                                   .9551098576E+00  -.1894043518E+00   .8910774163E+00  -.4774268068E-01   .9122348019E+00
                                  -.2750778637E-03   .2601828953E-03  -.1157125552E-02   .3180825963E-03  -.1200992374E-02
                              ABS(DG) =    .12279E-02      DEFLECTION =    .016055118 DEG.     TIME =    .4610 SEC.

1.047000    5.000000  282.000000   .9553799941E+00  -.1892415566E+00   .8901938100E+00  -.7976767724E-01   .9135755365E+00
                                   .9551098376E+00  -.1889425502E+00   .8889048109E+00  -.7950650172E-01   .9122348019E+00
                                  -.2701564790E-03   .2989863185E-03  -.1288999190E-02   .2611755169E-03  -.1340734585E-02
                              ABS(DG) =    .13487E-02      DEFLECTION =    .009214519 DEG.     TIME =    .4710 SEC.

1.047000    7.000000  282.000000   .9553705226E+00  -.1885097708E+00   .8867737083E+00  -.1113487487E+00   .9154013834E+00
                                   .9551098376E+00  -.1882505513E+00   .8856492118E+00  -.1111734563E+00   .9122348019E+00
                                  -.2606849782E-03   .2592194402E-03  -.1124496449E-02   .1752923876E-03  -.1166581461E-02
                              ABS(DG) =    .11672E-02      DEFLECTION =    .002432695 DEG.     TIME =    .4630 SEC.

1.047000   10.000000  282.000000   .9553612344E+00  -.1870989664E+00   .8800540748E+00  -.1589166314E+00   .9136496558E+00
                                   .9551098376E+00  -.1867828535E+00   .8787442365E+00  -.1584079110E+00   .9122348019E+00
                                  -.2515967829E-03   .5161128670E-03  -.1509838253E-02   .5087204862E-03  -.1414853851E-02
                              ABS(DG) =    .14403E-02      DEFLECTION =    .016908847 DEG.     TIME =    .4760 SEC.

1.047000   15.000000  282.000000   .9553356206E+00  -.1850291595E+00   .8704011291E+00  -.2059924226E+00   .9133820633E+00
                                   .9551098376E+00  -.1848031969E+00   .8694306844E+00  -.2052081805E+00   .9122348019E+00
                                  -.2257829270E-05   .2259623984E-03  -.9704447528E-03   .7842420937E-03  -.1147261357E-02
                              ABS(DG) =    .12680E-02      DEFLECTION =    .033897945 DEG.     TIME =    .4700 SEC.

1.047000   20.000000  282.000000   .9552760925E+00  -.1785044555E+00   .8596098783E+00  -.3130607728E+00   .9136824590E+00
                                   .9551098376E+00  -.1782261244E+00   .8384879913E+00  -.3120026777E+00   .9122348019E+00
                                  -.1662548208E-03   .2783310858E-03  -.1121887011E-02   .1058095090E-02  -.1447657061E-02
                              ABS(DG) =    .15671E-02      DEFLECTION =    .037652564 DEG.     TIME =    .4850 SEC.

1.047000   25.000000  282.000000   .9552188084E+00  -.1721154338E+00   .8095428369E+00  -.3868174942E+00   .9135705233E+00
                                   .9551098376E+00  -.1718942140E+00   .8086986947E+00  -.3855270863E+00   .9122348019E+00
                                  -.1089707988E-03   .2212197870E-03  -.8441421629E-03   .1290407889E-02  -.1335721369E-02
                              ABS(DG) =    .15578E-02      DEFLECTION =    .050308338 DEG.     TIME =    .5000 SEC.

1.047000   30.000000  282.000000   .9551582216E+00  -.1643755672E+00   .7734608546E+00  -.4574909097E+00   .9135419822E+00
                                   .9551098576E+00  -.1642540848E+00   .7727547127E+00  -.4561174010E+00   .9122348019E+00
                                  -.4838399800E-04   .1214824892E-03  -.7061418949E-03   .1373508704E-02  -.1307180224E-02
                              ABS(DG) =    .15492E-02      DEFLECTION =    .052180359 DEG.     TIME =    .5370 SEC.

1.047000   35.000000  282.000000   .9550595506E+00  -.1554306789E+00   .7312474790E+00  -.5247049054E+00   .9133437520E+00
                                   .9551098376E+00  -.1553638828E+00   .7309296008E+00  -.5232363868E+00   .9122348019E+00
                                   .5028700539E-04   .6679616983E-04  -.3178782245E-03   .1468518590E-02  -.1108950039E-02
                              ABS(DG) =    .15040E-02      DEFLECTION =    .063775226 DEG.     TIME =    .5910 SEC.
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| R | PHI | LAMBDA | U | UX | UY | UZ | G |
|---|---|---|---|---|---|---|---|
| 1.047000 | 40.000000 | 282.000000 | .9549420197E+00 | -.1451996431E+00 | .68308551.2E+00 | -.5872671690E+00 | .9124535524E+00 |
| | | | .9551098376E+00 | -.1452912678E+00 | .6835416733E+00 | -.5863732278E+00 | .9122548019E+00 |
| | | | .1678179317E-03 | -.9162476483E-04 | .4561580442E-03 | .8939412123E-03 | -.2187504356E-03 |
| | | | ABS(DG) = .10078E-02 | DEFLECTION = | .061779804 DEG. | TIME = | .6170 SEC. |
| 1.047000 | 42.500000 | 282.000000 | .9548897742E+00 | -.1397081448E+00 | .6572624837E+00 | -.6169935404E+00 | .9122461950E+00 |
| | | | .9551098376E+00 | -.1398351753E+00 | .6578727762E+00 | -.6162968992E+00 | .9122548019E+00 |
| | | | .2200634500E-03 | -.1270305147E-03 | .6102924832E-03 | .6964411372E-03 | -.1139302720E-04 |
| | | | ABS(DG) = .93468E-03 | DEFLECTION = | .058700702 DEG. | TIME = | .6290 SEC. |
| 1.047000 | 50.000000 | 282.000000 | .9547385079E+00 | -.1217693489E+00 | .5728787256E+00 | -.6993569949E+00 | .9122050290E+00 |
| | | | .9551098376E+00 | -.1219138492E+00 | .5735595659E+00 | -.6988124008E+00 | .9122548019E+00 |
| | | | .3713297776E-03 | -.1445002973E-03 | .6808403438E-03 | .5445940496E-03 | .2977297521E-04 |
| | | | ABS(DG) = .88375E-03 | DEFLECTION = | .055475835 DEG. | TIME = | .6710 SEC. |
| 1.047000 | 55.000000 | 282.000000 | .9546314808E+00 | -.1085360928E+00 | .5106032033E+00 | -.7474644917E+00 | .9117010919E+00 |
| | | | .9551098376E+00 | -.1087869619E+00 | .5118024164E+00 | -.7472590029E+00 | .9122548019E+00 |
| | | | .4783568742E-03 | -.2508690416E-03 | .1199213124E-02 | .2054888364E-03 | .5337100253E-03 |
| | | | ABS(DG) = .12423E-02 | DEFLECTION = | .070478566 DEG. | TIME = | .6920 SEC. |
| 1.047000 | 60.000000 | 282.000000 | .9545382730E+00 | -.9459744085E-01 | .4450599566E+00 | -.7897769920E+00 | .9114684514E+00 |
| | | | .9551098376E+00 | -.9483214005E-01 | .4461501414E+00 | -.7900185127E+00 | .9122548019E+00 |
| | | | .5715646044E-03 | -.2346991941E-03 | .1090184776E-02 | -.2415206949E-03 | .7663505326E-03 |
| | | | ABS(DG) = .11410E-02 | DEFLECTION = | .053117523 DEG. | TIME = | .6960 SEC. |
| 1.047000 | 65.000000 | 282.000000 | .9544440205E+00 | -.7984358748E-01 | .3756431044E+00 | -.8259975776E+00 | .9109087432E+00 |
| | | | .9551098376E+00 | -.8015558837E-01 | .3771023945E+00 | -.8267655046E+00 | .9122548019E+00 |
| | | | .6658171646E-03 | -.3120008880E-03 | .1459290063E-02 | -.7679270500E-03 | .1326058716E-02 |
| | | | ABS(DG) = .16783E-02 | DEFLECTION = | .064655576 DEG. | TIME = | .7340 SEC. |
| 1.047000 | 70.000000 | 282.000000 | .9543680277E+00 | -.6459879024E-01 | .3038961625E+00 | -.8558917993E+00 | .9105364646E+00 |
| | | | .9551098376E+00 | -.6486900426E-01 | .3051846706E+00 | -.8572203118E+00 | .9122548019E+00 |
| | | | .7418098938E-03 | -.2702140260E-03 | .1288508125E-02 | -.1328512526E-02 | .1698337346E-02 |
| | | | ABS(DG) = .18704E-02 | DEFLECTION = | .0[illegible]7255752 DEG. | TIME = | .7750 SEC. |
| 1.047000 | 75.000000 | 282.000000 | .9543053191E+00 | -.4886218349E-01 | .2298648795E+00 | -.8793487517E+00 | .9102085298E+00 |
| | | | .9551098376E+00 | -.4908872786E-01 | .2309443071E+00 | -.8811511548E+00 | .9122548019E+00 |
| | | | .8045185603E-03 | -.2265443782E-03 | .1079427642E-02 | -.1802423169E-02 | .2026272160E-02 |
| | | | ABS(DG) = .21131E-02 | DEFLECTION = | .037697291 DEG. | TIME = | .8690 SEC. |
| 1.047000 | 80.000000 | 282.000000 | .9542591384E+00 | -.3275407560E-01 | .1540847951E+00 | -.8961330819E+00 | .9098733176E+00 |
| | | | .9551098376E+00 | -.3293485661E-01 | .1549463180E+00 | -.8983759055E+00 | .9122548019E+00 |
| | | | .8506992514E-03 | -.1807810112E-03 | .8615229370E-03 | -.2242823592E-02 | .2361484344E-02 |
| | | | ABS(DG) = .24094E-02 | DEFLECTION = | .030065763 DEG. | TIME = | 1.0420 SEC. |
| 1.047000 | 85.000000 | 282.000000 | .9542352356E+00 | -.1641113147E-01 | .7718605489E-01 | -.9067004925E+00 | .9101279006E+00 |
| | | | .9551098376E+00 | -.1653033120E-01 | .7776909390E-01 | -.9087634731E+00 | .9122548019E+00 |
| | | | .8746020554E-03 | -.1191997375E-03 | .5830390167E-03 | -.2062980598E-02 | .2106901356E-02 |
| | | | ABS(DG) = .21471E-02 | DEFLECTION = | .026002464 DEG. | TIME = | 1.1050 SEC. |
| 1.047000 | 89.500000 | 282.000000 | .9542215221E+00 | -.1662218806E-02 | .7799724483E-02 | -.9097753062E+00 | .9098102585E+00 |
| | | | .9551098376E+00 | -.1655112073E-02 | .7786690095E-02 | -.9122000668E+00 | .9122548019E+00 |
| | | | .8883155598E-03 | .7106732864E-05 | -.1303438949E-04 | -.2424760585E-02 | .2424543406E-02 |
| | | | ABS(DG) = .24248E-02 | DEFLECTION = | .002244425 DEG. | TIME = | 1.1020 SEC. |

| R | PHI | LAMBDA | U | UX | UY | UZ | G |
|---|---|---|---|---|---|---|---|
| 1.047000 | 90.000000 | 282.000000 | .9542214621E+00 | -.1215291363E-13 | -.2068808914E-04 | -.9099154999E+00 | .9099155001E+00 |
| | | | .9551098376E+00 | -.6412930969E-14 | .3017046813E-13 | -.9122348019E+00 | .9122348019E+00 |
| | | | .8883755077E-03 | .5739982662E-14 | .2068808917E-04 | -.2319302046E-02 | .2319301811E-02 |
| | | | ABS(DG) = .23194E-02 | DEFLECTION = | .001302693 DEG. | TIME = | .8580 SEC. |
| 1.300000 | 90.000000 | 282.000000 | .7687186783E+00 | -.6038053189E-14 | -.1583409184E-05 | -.5906188548E+00 | .5906188548E+00 |
| | | | .7692307692E+00 | -.4159711624E-14 | .1956990455E-13 | -.5917159763E+00 | .5917159763E+00 |
| | | | .5120909246E-03 | .1878341564E-14 | .1583409204E-05 | -.1097121523E-02 | .1097121521E-02 |
| | | | ABS(DG) = .10971E-02 | DEFLECTION = | .000153606 DEG. | TIME = | .5450 SEC. |
| 2.100000 | 90.000000 | 282.000000 | .4761396001E+00 | -.2008311306E-14 | .4913608507E-14 | -.2266959797E+00 | .2266959797E+00 |
| | | | .4761904762E+00 | -.1594084500E-14 | .7499577936E-14 | -.2267573696E+00 | .2267573696E+00 |
| | | | .5087604768E-04 | .4142268063E-15 | .2585969428E-14 | -.6138990277E-04 | .6138990277E-04 |
| | | | ABS(DG) = .61390E-04 | DEFLECTION = | .000000000 DEG. | TIME = | .1760 SEC. |
| 1.047000 | -5.000000 | 282.000000 | .9553735285E+00 | -.1891920585E+00 | .8899675267E+00 | .7977411629E-01 | .9133453581E+00 |
| | | | .9551098376E+00 | -.1889425502E+00 | .8889048109E+00 | .7950650172E-01 | .9122348019E+00 |
| | | | -.2636908899E-03 | .2495082468E-03 | -.1062715862E-02 | -.2676145671E-03 | -.1110556200E-02 |
| | | | ABS(DG) = .11239E-02 | DEFLECTION = | .010854275 DEG. | TIME = | .4740 SEC. |
| 1.047000 | -15.000000 | 282.000000 | .9553065394E+00 | -.1833806908E+00 | .8625417741E+00 | .2367588020E+00 | .9130506668E+00 |
| | | | .9551098376E+00 | -.1832016265E+00 | .8618958880E+00 | .2361037403E+00 | .9122348019E+00 |
| | | | -.1967017515E-03 | .1790643409E-03 | -.6458860952E-03 | -.6550616524E-03 | -.8158648457E-03 |
| | | | ABS(DG) = .93720E-03 | DEFLECTION = | .028954022 DEG. | TIME = | .4830 SEC. |
| 1.047000 | -35.000000 | 282.000000 | .9550313708E+00 | -.1552911152E+00 | .7306276860E+00 | .5239962491E+00 | .9124166894E+00 |
| | | | .9551098376E+00 | -.1553638828E+00 | .7309296008E+00 | .5232363868E+00 | .9122348019E+00 |
| | | | .7846679773E-04 | -.7276756043E-04 | .3019147822E-03 | -.7598623188E-03 | -.1818874807E-03 |
| | | | ABS(DG) = .82088E-03 | DEFLECTION = | .050271152 DEG. | TIME = | .6070 SEC. |
| 1.047000 | -60.000000 | 282.000000 | .9545172423E+00 | -.9453919930E-01 | .4448277443E+00 | .7892317325E+00 | .9108765615E+00 |
| | | | .9551098376E+00 | -.9483214005E-01 | .4461501414E+00 | .7900185127E+00 | .9122348019E+00 |
| | | | .5925953065E-03 | -.2929407507E-03 | .1322397103E-02 | .7867802144E-03 | .1358240672E-02 |
| | | | ABS(DG) = .15664E-02 | DEFLECTION = | .049041122 DEG. | TIME = | .7010 SEC. |
| 1.047000 | -75.000000 | 282.000000 | .9542991670E+00 | -.4887031010E-01 | .2299256970E+00 | .8792846522E+00 | .9101624212E+00 |
| | | | .9551098376E+00 | -.4908872786E-01 | .2309443071E+00 | .8811511548E+00 | .9122348019E+00 |
| | | | .8106706502E-03 | -.2184177653E-03 | .1018610193E-02 | .1866502623E-02 | .2072380753E-02 |
| | | | ABS(DG) = .21375E-02 | DEFLECTION = | .032934941 DEG. | TIME = | .8890 SEC. |
| 1.047000 | -90.000000 | 282.000000 | .9542214621E+00 | -.1020711293E-13 | .2068808899E-04 | .9099154999E+00 | .9099155001E+00 |
| | | | .9551098376E+00 | -.6412930969E-14 | .3017046813E-13 | .9122348019E+00 | .9122348019E+00 |
| | | | .8883755077E-03 | .3794181963E-14 | -.2068808896E-04 | .2319302044E-02 | .2319301808E-02 |
| | | | ABS(DG) = .23194E-02 | DEFLECTION = | .001302693 DEG. | TIME = | .8770 SEC. |
| 1.047000 | 60.000000 | 267.000000 | .9545383120E+00 | .2381718256E-01 | .4543644167E+00 | -.7897632181E+00 | .9114495169E+00 |
| | | | .9551098376E+00 | .2387134034E-01 | .4554923079E+00 | -.7900185127E+00 | .9122348019E+00 |
| | | | .5715256194E-03 | .5415777695E-04 | .1127891242E-02 | -.2552946414E-03 | .7852850031E-03 |
| | | | ABS(DG) = .11577E-02 | DEFLECTION = | .053449624 DEG. | TIME = | .7180 SEC. |
| 1.047000 | 60.000000 | 285.000000 | .9545385308E+00 | -.1177750464E+00 | .4394809962E+00 | -.7897606414E+00 | .9114474084E+00 |
| | | | .9551098376E+00 | -.1180518702E+00 | .4405755774E+00 | -.7900185127E+00 | .9122348019E+00 |
| | | | .5713067922E-03 | -.2768237699E-03 | .1094581196E-02 | -.2578712998E-03 | .7873935318E-03 |
| | | | ABS(DG) = .11581E-02 | DEFLECTION = | .053363703 DEG. | TIME = | .7200 SEC. |

| R | PHI | LAMBDA | U | UX | UY | UZ | G |
|---|---|---|---|---|---|---|---|
| 1.047000 | 60.000000 | 288.000000 | .9545377853E+00 | -.1405954639E+00 | .4327274578E+00 | -.7897621706E+00 | .9114518218E+00 |
| | | | .9551098376E+00 | -.1409480283E+00 | .4337934264E+00 | -.7900185127E+00 | .9122348019E+00 |
| | | | .5720522923E-03 | -.3525644238E-03 | .1065968552E-02 | -.2563420592E-03 | .7829801712E-03 |
| | | ABS(DG) = | .11517E-02 | DEFLECTION = | .053066631 DEG. | TIME = | .7050 SEC. |
| 1.157000 | 90.000000 | 282.000000 | .8636207841E+00 | -.8326362671E-14 | -.5716817902E-05 | -.7454477874E+00 | .7454477875E+00 |
| | | | .8643042351E+00 | -.5251498074E-14 | .2470635596E-13 | -.7470218108E+00 | .7470218108E+00 |
| | | | .6834510325E-03 | .3074864597E-14 | .5716817927E-05 | -.1574023353E-02 | .1574023331E-02 |
| | | ABS(DG) = | .15740E-02 | DEFLECTION = | .000439400 DEG. | TIME = | .7660 SEC. |
| 1.157000 | 89.968750 | 282.000000 | .8636207878E+00 | -.8436749644E-04 | .3912010065E-03 | -.7454477132E+00 | .7454478206E+00 |
| | | | .8643042351E+00 | -.8471095155E-04 | .3985336932E-03 | -.7470216997E+00 | .7470218108E+00 |
| | | | .6834472533E-03 | -.3434551078E-06 | .7332686715E-05 | -.1573986518E-02 | .1573990206E-02 |
| | | ABS(DG) = | .15740E-02 | DEFLECTION = | .000499216 DEG. | TIME = | .8670 SEC. |
| 1.157000 | 89.937500 | 282.000000 | .8636207921E+00 | -.1687349917E-03 | .7881188805E-03 | -.7454474204E+00 | .7454478561E+00 |
| | | | .8643042331E+00 | -.1694218779E-03 | .7970672679E-03 | -.7470213664E+00 | .7470218108E+00 |
| | | | .6834429732E-03 | -.6868862235E-06 | .8948387397E-05 | -.1573945970E-02 | .1573954702E-02 |
| | | ABS(DG) = | .15740E-02 | DEFLECTION = | .000559276 DEG. | TIME = | .9510 SEC. |
| 1.157000 | 89.875000 | 282.000000 | .8636208022E+00 | -.3374698983E-03 | .1581954331E-02 | -.7454461793E+00 | .7454479342E+00 |
| | | | .8643042351E+00 | -.3388435542E-03 | .1594133587E-02 | -.7470200330E+00 | .7470218108E+00 |
| | | | .6834329097E-03 | -.1373655888E-05 | .1217925660E-04 | -.1573853739E-02 | .1573876555E-02 |
| | | ABS(DG) = | .15739E-02 | DEFLECTION = | .000679857 DEG. | TIME = | .9720 SEC. |
| 1.157000 | 86.000000 | 282.000000 | .8636256330E+00 | -.1079205159E-01 | .5076682985E-01 | -.7436813946E+00 | .7454902789E+00 |
| | | | .8643042351E+00 | -.1083419657E-01 | .5097088739E-01 | -.7452021032E+00 | .7470218108E+00 |
| | | | .6786021279E-03 | -.4214497773E-04 | .2040575316E-03 | -.1520708655E-02 | .1531531853E-02 |
| | | ABS(DG) = | .15349E-02 | DEFLECTION = | .007822539 DEG. | TIME = | .9300 SEC. |
| 1.157000 | 70.000000 | 282.000000 | .8637264227E+00 | -.5294024261E-01 | .2490608405E+00 | -.7008990323E+00 | .7457167190E+00 |
| | | | .8643042351E+00 | -.5312071073E-01 | .2499132951E+00 | -.7019708832E+00 | .7470218108E+00 |
| | | | .5778123667E-03 | -.1804681148E-03 | .8524546075E-03 | -.1071850904E-02 | .1305091795E-02 |
| | | ABS(DG) = | .13813E-02 | DEFLECTION = | .034744355 DEG. | TIME = | .6540 SEC. |
| 1.157000 | 55.000000 | 282.000000 | .8639184090E+00 | -.8887677206E-01 | .4181311024E+00 | -.6116367792E+00 | .7462119331E+00 |
| | | | .8643042351E+00 | -.8908477629E-01 | .4191109208E+00 | -.6119244634E+00 | .7470218108E+00 |
| | | | .3858260440E-03 | -.2080042318E-03 | .9798183986E-03 | -.2876642426E-03 | .8098776973E-03 |
| | | ABS(DG) = | .10421E-02 | DEFLECTION = | .050331429 DEG. | TIME = | .5990 SEC. |
| 1.157000 | 42.600000 | 282.000000 | .8641179683E+00 | -.1141835718E+00 | .5371878026E+00 | -.5059256242E+00 | .7467056720E+00 |
| | | | .8643042351E+00 | -.1143266009E+00 | .5378643690E+00 | -.5056411126E+00 | .7470218108E+00 |
| | | | .1862667965E-03 | -.1430290757E-03 | .6765663621E-03 | .2845116435E-03 | .3161388046E-03 |
| | | ABS(DG) = | .74776E-03 | DEFLECTION = | .051985640 DEG. | TIME = | .5520 SEC. |
| 1.157000 | 30.000000 | 282.000000 | .8642950633E+00 | -.1344751048E+00 | .6326488299E+00 | -.3740652856E+00 | .7471632578E+00 |
| | | | .8643042351E+00 | -.1345063613E+00 | .6328026771E+00 | -.3735109054E+00 | .7470218108E+00 |
| | | | .9171793643E-05 | -.3125642834E-04 | .1538471473E-03 | .5543802244E-03 | -.1414469711E-03 |
| | | ABS(DG) = | .57618E-03 | DEFLECTION = | .042836000 DEG. | TIME = | .4640 SEC. |
| 1.157000 | 15.000000 | 282.000000 | .8644221913E+00 | -.1501061488E+00 | .7062041149E+00 | -.1937929918E+00 | .7475371774E+00 |
| | | | .8643042351E+00 | -.1500223522E+00 | .7057996752E+00 | -.1933434717E+00 | .7470218108E+00 |
| | | | -.1179562159E-03 | .8379662201E-04 | -.4044396896E-03 | .4495200384E-03 | -.5153665618E-03 |
| | | ABS(DG) = | .61046E-03 | DEFLECTION = | .025087131 DEG. | TIME = | .4180 SEC. |

| R | PHI | LAMBDA | U | UX | UY | UZ | G |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 1.157000 | 5.000000 | 282.000000 | .8644664093E+00 | -.1548474989E+00 | .7285280310E+00 | -.6527216420E-01 | .7476572044E+00 |
| | | | .8643042351E+00 | -.1547235489E+00 | .7279170670E+00 | -.6510724077E-01 | .7470218108E+00 |
| | | | -.1621741881E-03 | .1239499465E-03 | -.6109640207E-03 | .1649234299E-03 | -.6353936041E-03 |
| | | | ABS(DG) = .64486E-03 | DEFLECTION = | .008438638 DEG. | TIME = | .4160 SEC. |
| 1.157000 | 0.000000 | 282.000000 | .8644713549E+00 | -.1554443950E+00 | .7313304747E+00 | .3153818584E-05 | .7476678558E+00 |
| | | | .8643042351E+00 | -.1553145678E+00 | .7306975919E+00 | -0. | .7470218108E+00 |
| | | | -.1671198113E-03 | .1298252273E-03 | -.6328827733E-03 | -.3153818584E-05 | -.6460449696E-03 |
| | | | ABS(DG) = .64607E-03 | DEFLECTION = | .000427121 DEG. | TIME = | .4180 SEC. |
| 1.157000 | -5.000000 | 282.000000 | .8644655987E+00 | -.1548450956E+00 | .7285174891E+00 | .6527759575E-01 | .7476469087E+00 |
| | | | .8643042351E+00 | -.1547235489E+00 | .7279170670E+00 | .6510724077E-01 | .7470218108E+00 |
| | | | -.1613635649E-03 | .1215466787E-03 | -.6004221316E-03 | -.1703549767E-03 | -.6250979062E-03 |
| | | | ABS(DG) = .63585E-03 | DEFLECTION = | .008925603 DEG. | TIME = | .4140 SEC. |
| .100000E+04 | 65.000000 | 282.000000 | .1000004413E-02 | -.8786766496E-07 | .4133848622E-06 | -.9063117864E-06 | .1000004412E-05 |
| | | | .1000000000E-02 | -.8786727737E-07 | .4133830387E-06 | -.9063077870E-06 | .1000000000E-05 |
| | | | -.4413349547E-08 | .3875885018E-12 | -.1823460699E-11 | .3999364557E-11 | -.4412499310E-11 |
| | | | ABS(DG) = .44125E-11 | DEFLECTION = | .000000038 DEG. | TIME = | .1790 SEC. |



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