NOAA Technical Memorandum NOS NGS 30

DETERMINATION OF PLUMB LINE CURVATURE BY ASTRONOMICAL AND GRAVIMETRIC METHODS

Rockville, Md.
February 1981

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DETERMINATION OF PLUMB LINE CURVATURES BY ASTRONOMICAL AND GRAVIMETRIC METHODS

Erwin Groten

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DETERMINATION OF PLUMB LINE CURVATURES BY ASTRONOMICAL AND GRAVIMETRIC METHODS

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ABSTRACT. The actual plumb line curvatures for several selected stations of the U.S. triangulation network in the Rocky Mountain area are estimated using two formulas by H. Bodemüller. By taking terrain inclination data from topographic maps, as well as elevations and conservative estimates of horizontal gravity gradients, it is shown that the determination of actual plumb line corrections is practically impossible. By varying the data to within reasonable limits totally different estimates for the constituents of the plumb line correction are found. Taking into account the 30" by 30" estimates for mean elevations in the United States, it seems that the only fairly reliable estimate of actual plumb line corrections is obtainable from topographic models. Also normal plumb line curvature is considered.

INTRODUCTION

Errors in astronomical observations as well as actual plumb line corrections appear randomly. A typical exception, however, is the boundary between the Great Plains and the Rocky Mountain chains in Colorado. The distortions caused by the plumb line curvature can be averaged out under favorable circumstances almost everywhere else.

The normal plumb line correction, however, is of a purely systematic quality; west of longitude $\lambda \doteq 102°$ it can cause errors on the order of a few meters for extended north-south astronomical levelings. In general, its effect can be compared with the accumulation of small systematic errors in leveling. Although it can be precisely evaluated, the plumb line correction is almost always neglected in triangulation work. Some numerical estimates are given for stations in high mountainous areas.

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It is concluded that the best way to avoid difficulties associated with both types of plumb line correction is to calculate and compare all related geodetic quantities at the Earth's surface. For gravimetric height anomalies and plumb line deflections, this is readily done by using Bjerhammar's or Molodensky's solution for astrogravimetric or astronomical leveling, or by collocation using surface data. Even though the geoid cannot be totally dismissed as a global reference surface, it should no longer be used with the aforementioned problem. If adjustments are involved, they should be done in a three-dimensional fashion.

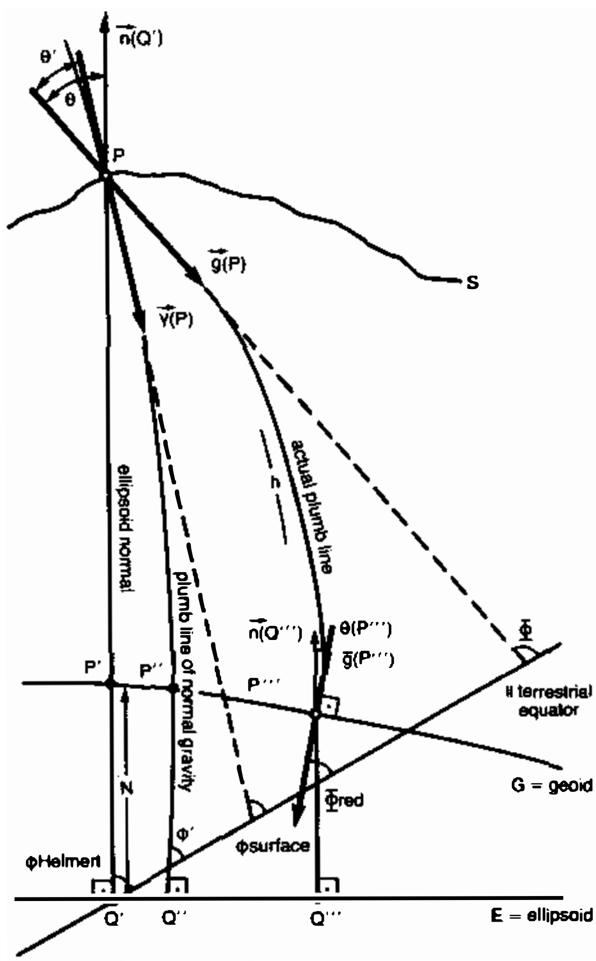

Figure 1.--Normal and actual plumb line curvatures at the geoid and at the Earth's surface.

at P: $\Theta_{astronomic}\,[\vec{g}(P),\ \vec{n}(Q')]$

$\Theta'_{gravimetric}\,[\vec{g}(P),\ \vec{\gamma}(P)]$

at P''': $\Theta_{astronomic}\,[\vec{g}(P'''),\ \vec{n}\ (Q''')]$

$\Theta_{gravimetric}\,[\vec{g}(P'''),\ \vec{\gamma}(P''') \uparrow\!\downarrow \vec{n}(Q''')]$

$\uparrow\!\downarrow$ means nearly antiparallel.

We use the following notations:

1. The "terrestrial equator" is the equatorial plane associated with the CIO pole.

2. $\vec{n}$ is the surface normal to the ellipsoid.

3. $\Theta_P(\xi,\eta)$ is the plumb line deflection at point P according to Helmert (1884).

4. S = Earth's surface, G = geoid, E = ellipsoid.

5. h = orthometric height; H = h + N = ellipsoidal height; N = geoid height. h is rigorously measured along the <u>actual</u> plumb line; however, it is often approximated by $\overline{PP'}$, thus $\overline{PP'''} \rightarrow \overline{PP'}$.

6. Φ is the astronomical latitude; (Φ,Λ) are the astronomical coordinates.

7. Geodetic coordinates (ϕ,λ) refer to $\vec{n}$ in <u>Helmert's</u> system,

8. $\vec{g}(P)$ is gravity at (P); $\vec{g}(P''')$ is gravity on the geoid at P''' located on the actual plumb line of P.

9. $\Phi_{red.} \equiv \Phi$ (reduced) is the (observed) astronomical latitude corrected for actual plumb line curvature.

10. $\Theta'_P(\xi',\eta')$ is the plumb line deflection referred to $\vec{\gamma}(P)$ instead of $\vec{n}$.

11. $\Theta(P''')$ is the deflection corrected for the curvature of the actual plumb line and referred to the ellipsoid normal at Q'''.

When astronomical deflections of the vertical are observed at the Earth's surface we measure (Φ,Λ) and get (ϕ,λ) from triangulation. In most cases geodetic coordinates (ϕ,λ) are computed according to Helmert's (1884) projection. Consequently, deflections of the vertical at P, i.e., at the Earth's surface, are obtained in the form

$$\begin{aligned} \xi &= \Phi - \phi_{Helmert}\,(Q') \\ \eta &= [\Lambda - \lambda_{Helmert}\,(Q')] \cos\phi \text{ at } P. \end{aligned} \tag{1}$$

If the triangulation adjustment has been done on the ellipsoid and if (Φ,Λ) has been corrected for the curvature of the <u>actual</u> plumb line, then we get

$$\begin{aligned} \xi &= \Phi_{red} - \phi(Q''') \\ \eta &= [\Lambda_{red} - \lambda(Q''')] \cos\phi \text{ at } P'''. \end{aligned} \tag{2}$$

Groten (1979) refers to these data (which are related to $\vec{n}\,(Q''')$ and <u>not</u> to $\vec{\gamma}(P'''))$ as "Pizzetti-projection" results. In eq. (2) $\phi \equiv \phi(Q''')$, whereas in the case of $\Theta(P)$ we have $\cos\phi \equiv \cos\phi\,(Q')$.

We can now compute gravimetric (ξ,η) using Stokes' classical approach from Δg reduced down to the geoid. The vectors $\vec{n}(Q''')$ and $\vec{\gamma}(P''')$ are almost parallel to each other. Therefore, the $\Theta'(\xi',\eta')$ -values obtained at P''' can be directly compared with the astrogeodetic data at P''' if the reference ellipsoid parameters of the normal gravity ellipsoid and the triangulation reference ellipsoid (a = semimajor axis, f=flattening and $\vec{r}_o$= location of the center of the ellipsoid) do not differ.

When Bjerhammar's method is applied by using, for example, formulas by Heiskanen and Moritz (1967, p. 320), we have

$$\begin{Bmatrix} \xi' \\ \eta' \end{Bmatrix}_P = \frac{1}{4\pi\gamma} \iint_s \Delta g^* \; \frac{\partial S(r,\psi)}{\partial \psi} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \alpha \; ds \tag{3}$$

(with Δg^* being gravity on the Bjerhammar sphere, i.e., on an exact sphere, $S(r,\psi)$=Stokes-Pizzetti function). Then (ξ',η') is evaluated at $P(\vec{r})$ on the Earth's physical surface, γ = normal gravity, α = azimuth, s = unit sphere, ds is an element of s and $\vec{r}$ = geocentric radius vector of P. Molodensky's method yields the same quantities (ξ',η') at $P(\vec{r})$ related to $\vec{g}(P)$ and $\vec{\gamma}(P)$.

CURVATURE OF THE NORMAL GRAVITY FIELD (γ)

The gravimetric deflection of the vertical is the angle defined by $\vec{g}(P)$ and $\vec{\gamma}(P)$ where $\vec{g}$ and $\vec{\gamma}$ are the gravity vector and the normal gravity vector, respectively. In the classical theory of physical geodesy, P is a point on the geoid; whereas in modern theory, according to Bjerhammar, Molodensky, and others, P is situated at the Earth's surface where geodetic and astronomical measurements are made.

As a result of the curvature of the normal gravity field the direction of $\vec{\gamma}(Q'')$ at a point situated at the elevation H above the ellipsoid differs from $\vec{\gamma}(E)$ situated on the ellipsoid at the same normal gravity plumb line by Heiskanen and Moritz (1967, p. 196)

$$\Delta\phi = -0''.17\ H \sin 2\phi \tag{4}$$

where ϕ is again the latitude, H is the elevation above the ellipsoid expressed in kilometers and $0''.17$ is the numerical value of f^*/R referred to the International Ellipsoid: f^* is the gravitational flattening $(\gamma_a - \gamma_b)/\gamma_a$ (see, e.g., Heiskanen and Moritz 1967, p. 74) and R is the mean radius of the Earth. The value $0''.17$ can safely be used with the normal gravity formulas of the 1971 International Ellipsoid, adopted by the General Assembly of the International Union of Geodesy and Geophysics (held in Moscow), and the 1980 International Ellipsoid (adopted at IUGG General Assembly in Canberra).[2]

$\Delta\phi$ is defined in the sense [$\phi_{surface} \equiv$ geodetic latitude of $\vec{\gamma}(P)$]

$$\phi' - \phi_{surface} = \Delta\phi$$

where $\phi \doteq \phi_{Helmert}$ is the geodetic latitude of $\vec{n}(Q')$; ϕ' is shown in figure 1.

Since it is permissible to neglect the difference between ϕ' and ϕ in Helmert's projection, i.e., with $\phi_{Helmert}$ we have (Heiskanen and Moritz, 1967, p. 315)

$$\phi_{Helmert} - \phi_{surface} \doteq \Delta\phi \tag{5}$$

or

$$\xi_{Helmert} \doteq \xi_{surface} - \Delta\phi. \tag{6}$$

Astrogeodetic deflections of the plumb line

$$\xi = (\Phi - \phi) \text{ and } \eta = (\Lambda - \lambda) \cos \phi \tag{7}$$

are usually obtained according to Helmert's method where (Φ,Λ) are the astronomical quantities and (ϕ,λ) are geodetic coordinates obtained from

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[2] $f^* = 0.005258$ for the normal gravity formula of the International Ellipsoid of 1932; $f^* \doteq 0.005316$ for the normal gravity formula of 1980.

triangulation. Whenever the flattening, semimajor axis, and origin of the reference ellipsoid used in the computation of the triangulation coincide with the corresponding quantities of the aforementioned normal gravity field that were used to evaluate (ξ', η') from gravity anomalies Δg, eqs. (5) and (6) may be used for a direct comparison. It should be noted that η is not affected because the normal gravity field corresponds to an ellipsoid of rotation. Consequently, the orthogonal trajectories of the normal gravity field are plane curves lying on the geodetic meridian planes.

Assuming an accuracy of $\pm 0\overset{''}{.}5$ for ξ and $\pm 0\overset{''}{.}7$ for η we get corrections $\Delta\phi$ for elevations less than 3000 m, which may always be neglected. However, these corrections affect only the geodetic coordinates.

PLUMB LINE CURVATURE OF THE GRAVITY FIELD (g)

The plumb line curvature of the actual gravity field is much more important but extremely difficult to determine. It may amount up to >1" in high mountains.

The basic difference in comparison to the normal gravity field is due to the fact that ξ as well as η are affected by the plumb line curvature of the actual gravity field. All the computational formulas are based on the rigorous relations (Heiskanen and Moritz 1967, p. 194)

$$\delta\Phi = - \int \frac{1}{g} \frac{\partial g}{\partial x} \, dh \tag{8}$$

$$\delta\Lambda = - \frac{-1}{\cos\Phi} \int \frac{1}{g} \frac{\partial g}{\partial y} \, dh \tag{9}$$

where the integrals are taken along the actual plumb line from the geoid up to the Earth's surface; x is positive towards the north, and dh is the height increment. Heiskanen and Moritz (1967, p. 195) cite one set of the various computational approximation formulas (see Bodemüller 1957)

$$\delta\Phi = - \frac{h}{\bar{g}} \frac{\partial \bar{g}}{\partial x} + \frac{g - \bar{g}}{\bar{g}} \tan \beta_1 \tag{10}$$

$$\delta\Lambda = \frac{-h}{\bar{g}\cos\phi}\frac{\partial\bar{g}}{\partial y} + \frac{g-\bar{g}}{\bar{g}\cos\phi}\tan\beta_2 \qquad (11)$$

where h is orthometric height and $\bar{g}$ is mean gravity along the plumb line between the Earth's physical surface and the geoid. The gradients can be found from torsion balance data after applying some doubtful terrain reductions, or by using some well-known integration formulas. The gradients may also be deduced by numerical integration processes.

Even though horizontal gradients of gravity are less sensitive to small density variations than vertical gravity gradients, they are difficult to evaluate precisely. It is well known that, in general, second derivatives of the potential are discontinuous where density changes discontinuously. β_1 and β_2 are the inclination angles of the terrain in the north-south and east-west directions with respect to the local horizon.

In eqs. (10) and (11) mean gravity values $\bar{g}$ are used. However, the argument that those equations are therefore easier to handle than the original integral formulas is only true if density varies randomly between the Earth's surface and the geoid. However, this is seldom the case.

Let us briefly inspect eqs. (10) and (11): The second term on the right side is dominated by the inclination of the terrain for steep topographic slopes. For $\beta<30°$ the term increases for constant $(g/\bar{g}-1)$ almost linearly with β. For irregular terrain where β may be of the order of 45° (note that $\tan\beta=1$ for $\beta=45°$) or even greater the second term may dominate the curvature. The first term is mainly affected by the surrounding topography to a distance of < 50 km; in most cases the influence of the topography in the nearest neighborhood (distances < 10 km) will be dominant. The first and the second terms can, of course, cancel each other to some extent. This happens where the inclination of the terrain at the point of interest produces an effect which is opposite to the effect of the surrounding areas. A good estimate on horizontal gradients, their variations, and order of magnitude is found in a study by Groten et al. (1979). These data are based on several thousand torsion balance observations in the Rhinegraben.

However, they do not <u>fully</u> reflect the gradient variation because they are corrected for local terrain effects.

A well-known formula of Heiskanen and Moritz (1967, p. 167) gives

$$\bar{g} = g + 0.0424\ h \tag{12}$$

which is based on the density value of $g = 2.67$ cgs, where g is in gals and h is in kilometers. Consequently,

$$\frac{g-\bar{g}}{\bar{g}} = \frac{g}{\bar{g}} - 1 \doteq \frac{g}{g+0.0424\ h} - 1 = \frac{1}{1+0.0424\,\frac{h}{g}} - 1. \tag{13}$$

For $h = 1$ km and $g = 10^3$ gal we obtain

$$\frac{g-\bar{g}}{\bar{g}} = 0.999958 - 1 \doteq -4\cdot 10^{-5}. \tag{14}$$

Note that $4.8 \cdot 10^{-5}$ corresponds to an angle of about 10".

It is well known that we can safely assume that density in the upper layer of the crust (between the Earth's surface and the geoid) may vary by 20 to 30 percent. Deviations from that estimate are known. For example, in the Rhinegraben stronger deviations exist. In general, however, this is a good estimate. Therefore, instead of a constant density of $\rho=2.67$ g/cm^3 we should assume variations of ±20 percent.

In spite of the fact that we have to introduce a hypothetical density in the case of terrain models, the estimation of curvatures using terrain models seems to be more appropriate than the aforementioned approach where an extremely dense gravity net is needed for reasonable accuracy. Recent investigations have proven that such computations are efficient and easy to perform. From such computations, as well as from estimates that rely on the interrelation of plumb line curvature and orthometric correction K of leveling (Heiskanen and Moritz, 1967, p. 195)[3], i.e.,

$$\delta\Phi = \partial K/\partial x; \quad \delta\Lambda = \partial K/(\cos\phi\ \partial y), \tag{15}$$

we know that plumb line curvature corrections will remain below ±0.5" in hilly areas. However, in high regions such as the Rocky Mountains the corrections may be as high as a few seconds of arc. Such corrections show a random behavior in irregular topography. But in areas such as Colorado where the mountain chains border the Great Plains the corrections tend to be systematic. Consequently, in those areas a systematic error is anticipated, whereas errors resulting from plumb line curvature corrections (or the errors caused by totally omitting those corrections) might be averaged out within other large areas.

When formulas (10) and (11) are replaced by more general, well-known formulas, such as (where C = geopotential number)

$$\delta\Phi = \frac{\partial}{\partial x}\left(h - \frac{C}{g}\right) \tag{16}$$

$$\delta\Lambda = \frac{\partial}{\cos\phi\partial y}\left(h - \frac{C}{g}\right) \quad \left(\text{with } h = \frac{C}{g}\right), \tag{17}$$

the whole problem of plumb line deflections becomes obvious. With maps of mean elevations for small blocks, such as 30" by 30", the topographic contribution to the plumb line deflection is more accurately estimated than from conventional formulas where g and elevation enter. Moreover, horizontal gradients of g and h, or C/g, are the relevant quantities instead of the quantities themselves. However, the whole problem is avoided if gravimetric (ξ',η') are calculated at the Earth's surface (instead of at the geoid). For additional details, see the appendix.

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[3]These formulas are found in slightly different form in conventional geodetic textbooks (e.g., see Heiskanen and Moritz, 1967, p. 195) using C = W^o - W, with W^o being the value of the potential W on the geoid; dC = -dW.

CONCLUSIONS

In principle, at elevations less than 3000 m the curvature of the normal plumb line can always be neglected. At those few stations where the elevation is higher, it can formally be neglected unless a greater number of stations are clustered together so that a systematic distortion might occur. On the other hand, the normal plumb line curvature is a one-sided effect which can accumulate in very long astronomic leveling profiles along meridians. Even though those results are no longer of actual interest, the accumulation of small errors can imply significant distortion as in the case of spirit leveling.

The curvature of the actual plumb line is of a totally different quality and character: It affects ξ and η in a more or less random way. It has a strong systematic influence only in areas of systematic topographical features such as mountain chains. In the interior of extended mountain areas it again behaves randomly, in general. Equations (10) and (11) give a general idea of the problem involved in determining the actual plumb line deflections. The inclination β of the station neighborhood can cancel the effect of topographic areas that are 10 to 30 km apart or it can magnify that influence. Since mean gravity along the plumb line is needed we have to introduce hypothetical quantities of doubtful value. Consequently, many arguments exist against the application of eqs. (10) and (11). Although the superiority of eqs. (10) and (11) over eqs. (8) and (9) seems apparent, in principle the determination of mean values in this case is not any easier than the determination of the quantities in eqs. (8) and (9). The first terms on the right side of eqs. (10) and (11), as well as the right sides of (8) and (9), are equally doubtful as is seen from the measurements of gravity gradients or from the computations of mean horizontal gravity gradients.

Consequently, it is best to compare surface values of (ξ,η) obtained from astronomical observations with results calculated either by applying Molodensky's or Bjerhammar's method, or collocation. If Bjerhammar's method is rigorously applied it needs no ellipsoidal correction; however, the "downward continuation" down to an exact sphere will always be problematic.

We therefore conclude that the use of the geoid should be totally avoided in this connection whenever possible. In spite of the relatively low accuracy of astronomical observations which leads to errors in (ξ, η) of the order of $\pm 0''.5$ to $0''.8$ and even higher, we can neglect plumb line curvatures at high-elevation stations only in "exceptional" cases, e.g., in wide areas where only one or two isolated high-elevation stations exist. However, it is much simpler to solve the problem of astronomical (=astrogeodetic) leveling as well as the geodetic boundary value problems (including the determination of ξ,η and N) at the physical surface of the Earth. The comparison of geoidal quantities with quantities measured at the Earth's surface obviously involves difficulties which cannot be overcome or handled with sufficient accuracy according to modern standards.

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APPENDIX.--ERROR ESTIMATES FOR THE DEFLECTION OF THE VERTICAL

The following numerical details supplement the general statements and remarks given in this report. The continental United States extends in the north-south direction to about 2000 km. In general, the elevation of the eastern United States is less than 2000 m. By neglecting the curvature of the normal plumb line an error in the ξ-component of the deflection of the vertical amounting to $<0\overset{\prime\prime}{.}34$ is committed. This error is less than the standard deviation for the plumb line deflection field of the first-order horizontal networks in the United States. For an astrogeodetic north-south leveling profile of the eastern part of the United States, the error of geoid undulations caused by neglecting normal plumb line curvature is always less than 3 m. Actually, it is much less because the elevations are, in general, less than 2000 m even over the full length of 2000 km. (See table 1.)

Table 1 includes a sample of a dozen first-order triangulation stations in Colorado. Even in high mountainous areas the first-order stations, in general, have elevations less than 3000 m. If we use $h \leq 2500$, we still end up with errors of $<0\overset{\prime\prime}{.}43$ in ξ. This corresponds to a systematic distortion of <4.5 m for a north-south astrogeodetic leveling profile across the entire country. This means that relative geoid undulations obtained from an astrogeodetic leveling will always be affected by a systematic error of less than 5 meters. In reality it will be less than 4 m if we consider the elevations in more detail.

When we inspect the topography surrounding triangulation stations in high mountains, as given in table 1, we discover that the areas where the gravity field must be known with highest accuracy (in order to take account of relatively strong horizontal gravity gradients and associated curvatures) are such that it is practically impossible to perform precise measurements in a dense gravity station net. Moreover, the elevations are difficult to determine at those gravity stations. Consequently, in the neighborhood of a trigonometric station where gravity is most needed we cannot get it with sufficient accuracy. In smooth topography where gravity can be determined it is not needed.

Table 1.--A sampling of geodetic and astrogeodetic data from the Rocky Mountain area

	Station	ϕ	λ	h (meter)	ξ (arc sec)	η (arc sec)
1	EL PASO E BASE 1879	38° 57' 21".99	104° 27' 41".92	1994	-5.07	-7.32
2	EL PASO W BASE 1879	38° 58' 42".84	104° 35' 19".26	2166	-5.00	-8.32
3	GOLF 1935	37° 09' 06".84	104° 30' 46".08	1922	7.95	-0.96
4	HOUSIER 1953	39° 29' 06".66	106° 57' 14".74	3510	2.28	9.63
5	ROMEO N BASE 1935	37° 16' 07".11	105° 57' 42".74	2317	2.14	0.78
6	ROMEO S BASE 1935	37° 10' 47".96	105° 59' 07".13	2353	2.80	0.99
7	SAN LUIS 1935	39° 05' 35".11	105° 07' 03".99	2366	-9.80	1.91
8	STEAMBOAT N BASE 1939	40° 25' 26".16	106° 49' 33".93	2088	-2.00	12.36
9	STEAMBOAT S BASE 1939	40° 21' 23".29	106° 49' 41".02	2088	0.43	15.03
10	UTE NW BASE 1952	37° 48' 14".23	104° 43' 08".85	1779	0.50	-5.95
11	UTE SE BASE 1952	37° 44' 03".40	104° 38' 39.32	1835	-1.69	-6.95
12	WOLF 1936	37° 38' 12".49	106° 30' 57".30	2798	8.01	-3.98

In comparison to the computation of (N,ξ,η) from Δg we need a substantially denser gravity field (station distance of the order of 100 m and less) in the immediate triangulation station neighborhood (of radius $d < 30$ km) for computing the <u>actual</u> plumb line curvature components.

Station HOUSIER does not have the highest elevation in the first-order network; the corresponding <u>normal</u> plumb line correction amounts to 0".58 or less. The ξ-components are generally supposed to have accuracies of about ±0".5. Moreover, first-order stations are located at elevations as high as 4419 m (Whitney, Southern California) or 4399 m (Mount Elbert, Colorado). These are, to some extent, "isolated points" where the normal plumb line curvature correction amounts to about 0".75. We need not consider Alaska in this case where even higher elevations occur; the low accuracy of astronomic deflection of the vertical in Alaska does not necessitate consideration of errors less than ±1". In the case of isolated points we face an "outlier" effect. However, since all normal plumb line curvature corrections have the same sign they are systematic in general.

It is, therefore, safe to avoid the whole normal plumb line curvature problem by determining gravimetric (ξ,η) or N at the Earth's surface whenever astrogeodetic deflections of the vertical are compared with corresponding gravimetric quantities. The astrogeodetic (=astronomical) leveling by which geoid undulation N is determined from (ξ,η) can readily be done in terms of surface quantities using Molodensky's method. If least squares collocation is applied to surface data the normal plumb line curvature problem is again dodged. The question itself comes up only for ξ (η is not affected at all), and stations in the United States west of $\lambda < 102°$ are of concern where station elevations $h > 2$ km occur.

The curvature of the actual plumb line is more difficult to discuss in detail. Inclination angles of terrain around stations such as MOUNT ELBERT amount to as much as $\beta = 45°$. Consequently, horizontal derivatives of the elevations mentioned previously can become as strong as

$$\frac{\delta h}{\delta x} \rightarrow \frac{\partial h}{\partial x} \cong \pm 1 \tag{1A}$$

and

$$\frac{\delta h}{\delta y} \rightarrow \frac{\partial h}{\partial y} \cong \pm 1. \tag{2A}$$

Analogously, the inclination angles can be as high as 45° implying

$$|\tan \beta_i| = 1 \ (i = 1,2).$$

Assuming horizontal gravity gradients of the order of 10 mgal/10 km we end up with correction terms on the order of

$$\frac{h}{\bar{g}} \frac{\partial \bar{g}}{\partial s} \leq 3.10^{-6} \cong 0.''6 \tag{3A}$$

where s takes the place of x or y and h = 3 km. However, 10 mgal/10 km is a very conservative estimate. This is readily explained using the following example: By taking a Bouguer coefficient of 0.1 mgal/m we get 100 mgal/km for inclination angles $\beta=45°$. It is seen that we arrive easily at several seconds of arc for the actual plumb line correction if we consider only the contribution of the part inherent in eq. (3A), for example. Moreover, for the term (i=1,2) in eqs. (10) and (11) (see eq. 14)

$$\frac{g-\bar{g}}{\bar{g}} \tan\beta_i = \frac{g-\bar{g}}{\bar{g}} = 4.10^{-5} \tag{4A}$$

(for $\tan\beta_i=1$) we get a reasonable limit for actual plumb line corrections to η in areas such as the Rocky Mountain chain. But even for lower elevations, the actual plumb line corrections can be prominent even if it may be averaged out over long distances so that the accumulation of "errors" is not as serious as in the normal plumb line curvature effect.

Because this pertubation is completely avoided whenever astrogeodetic as well as gravimetric results (ξ',η',N) are considered at the Earth's surface, the necessity to perform geodetic calculations at the Earth's surface is stressed. It seems that we have "reduced" data too often to the geoid and to the ellipsoid, respectively.

A more specific consideration of the topography makes it plausible that the actual plumb line curvature correction within the Colorado area shown in table 1 (which is typical for such mountain area) may indeed behave randomly, at least to some extent. Therefore, the neglect of actual plumb line curvatures could lead to random errors for nonsystematic terrain in adjustments of large systems.

The irregularities of topography encountered in the Rocky Mountains indicate that smoothing is necessary in any approach. The smoothing inherent in collocation is very appropriate if collocation, applied to downward continuation, is part of Bjerhammar's solution. In general, the smoothing in all these approaches will cause smaller absolute values of computational results in comparison to the observed data, i.e., astrogeodetic deflections, etc. Even though the data corrected for actual plumb line curvature are supposed to be slightly smoother than the corresponding original data at the Earth's surface, the latter should be preferred in the applications.

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