V. Theoretical Fundamentals of Inertial Gravimetry

- Basic gravimetry equation
- Essential IMU data processing for gravimetry
- Kalman filter approaches
- Rudimentary error analysis – spectral window
- Instrumentation

Inertial Gravimetry

- **Vector gravimetry, instead of scalar gravimetry**
  - **determine three components of gravity in the *n*-frame**
- **Use precision accelerometer triad, instead of gravimeter**
  - **OTF (off-the-shelf) units designed for inertial navigation rather than gravimetry**
- **Usually consider strapdown mechanization, instead of stabilized platform**
- **Need precision gyroscopes to minimize effect of orientation error on horizontal components**
- **Two documented approaches of data processing**
  - **either integrate accelerometer data or differentiate GPS data**

Direct Method

- **Recall strapdown mechanization**

Gravitational Vector in *n*-frame ← $\boldsymbol{g}^n = \mathbf{C}_i^n \left(\ddot{\boldsymbol{x}}^i - \mathbf{C}_b^i \boldsymbol{a}^b \right)$

accelerometers

transformation from inertial frame to *n*-frame ← **GPS**

transformation from body frame to inertial frame ← **gyros**

$\ddot{\boldsymbol{x}}^i$ – **kinematic accelerations obtained from GPS-derived positions, *x*, in *i*-frame.**

$\boldsymbol{a}^b$ – **inertial accelerations measured by accelerometers in body frame**

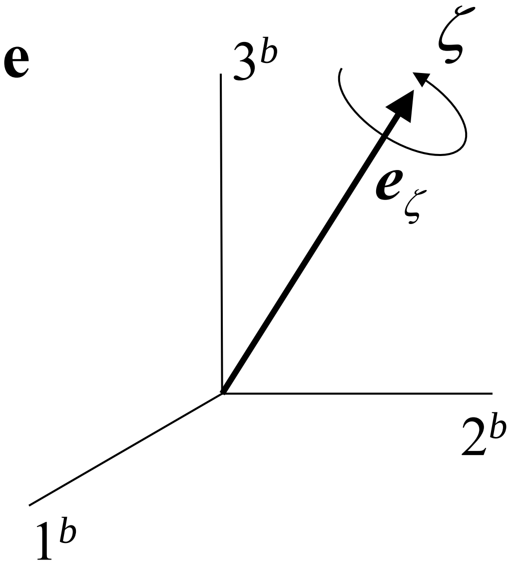
– **accelerometer data typically are** $\delta \boldsymbol{v}_k = \int_{t_{k-1}}^{t_k} \boldsymbol{a}^b(t)\,dt \quad \Rightarrow \quad \boldsymbol{a}_k^b \approx \dfrac{\delta \boldsymbol{v}_k}{\delta t}$ **(e.g.)**

better approximations can be formulated

◦ **where** $\delta t = t_k - t_{k-1}$ **is the data interval, e.g.,** $\delta t = 1/50$ s

Determination of $\mathbf{C}_b^i$ (1)

- **Let $\boldsymbol{e}_\zeta$ be the unit vector about which a rotation by the angle, ζ, rotates the *b*-frame to the *i*-frame**



$$\boldsymbol{e}_\zeta = \begin{pmatrix} b \\ c \\ d \end{pmatrix} \rightarrow \textbf{quaternion vector:} \quad \boldsymbol{q} = \begin{pmatrix} \cos(\zeta/2) \\ b\sin(\zeta/2) \\ c\sin(\zeta/2) \\ d\sin(\zeta/2) \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$\mathbf{C}_b^i = \begin{pmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2(q_2q_3+q_1q_4) & 2(q_2q_4-q_1q_3) \\ 2(q_2q_3-q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2(q_3q_4+q_1q_2) \\ 2(q_2q_4+q_1q_3) & 2(q_3q_4-q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{pmatrix}$$

- **Quaternions satisfy the differential equation:** $\frac{d}{dt}\boldsymbol{q} = \frac{1}{2}\mathbf{A}\boldsymbol{q}$

  – **where** $\mathbf{A} = \begin{pmatrix} 0 & \omega_1 & \omega_2 & \omega_3 \\ -\omega_1 & 0 & \omega_3 & -\omega_2 \\ -\omega_2 & -\omega_3 & 0 & \omega_1 \\ -\omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix}$ **and** $\boldsymbol{\omega}_{ib}^b = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} =$ gyro data

  – **this is a linear D.E. with no singularities**

Determination of $\mathbf{C}_b^i$ (2)

- **Solution to D.E., if A is assumed constant,**

$$\hat{\boldsymbol{q}}_k = \left(I + \frac{1}{2}\boldsymbol{\Theta}_k + \frac{1}{8}\boldsymbol{\Theta}_k^2 + \frac{1}{48}\boldsymbol{\Theta}_k^3 + \cdots \right)\hat{\boldsymbol{q}}_{k-1}\,, \quad k = 1, 2, \cdots$$

  - **using gyro data, typically given as** $\delta\boldsymbol{\theta}_k = \int_{t_{k-1}}^{t_k} \boldsymbol{\omega}_{ib}^b dt$
  - **where** $\boldsymbol{\Theta}_k = \int_{t_{k-1}}^{t_k} \mathbf{A} dt$
  - **note: A is assumed constant only in the solution to the D.E., not in using the gyro data (model error, not data error in $\boldsymbol{\Theta}$)**
  - $\hat{\boldsymbol{q}}_0$ **is given by an initialization procedure**
  - **solution is a second-order algorithm, neglecting terms of order δt^3**
  - **higher-order algorithms are easily developed (Jekeli 2000)**

Determination of δg^n (1)

- **Kalman filter approach to minimizing estimation errors**
  - estimate IMU systematic errors and gravity disturbance vector
  - formulate in *i*-frame and assume negligible error in $\mathbf{C}_i^n$
  - system state updates (observations) are differences in accelerations

$$
\begin{aligned}
\boldsymbol{y} &= \left(\tilde{\boldsymbol{a}}^i + \tilde{\boldsymbol{g}}^i\right) - \tilde{\ddot{\boldsymbol{x}}}^i \\
&= \left(\boldsymbol{a}^i + \delta\boldsymbol{a}^i\right) + \left(\boldsymbol{g}^i - \delta\overline{\boldsymbol{g}}^i\right) - \left(\ddot{\boldsymbol{x}}^i + \delta\ddot{\boldsymbol{x}}^i\right) \qquad \tilde{\boldsymbol{g}}^i = \boldsymbol{\gamma}^i + \boldsymbol{\Omega}_{ie}^i\boldsymbol{\Omega}_{ie}^i\boldsymbol{x}^i \\
&= \underbrace{\mathbf{C}_b^i\delta\boldsymbol{a}^b - \left[\boldsymbol{\psi}^i\times\right]\mathbf{C}_b^i\boldsymbol{a}^b}_{\delta\boldsymbol{a}^i} - \delta\ddot{\boldsymbol{x}}^i - \delta\overline{\boldsymbol{g}}^i
\end{aligned}
$$

  - δa^i includes accelerometer errors and orientation errors
  - over-script, ~, denotes indicated (measured) quantity

Determination of δg^n (2)

- **System state vector, $\boldsymbol{\varepsilon}_D$**
  - **orientation errors, ψ^i;** $\qquad \dfrac{d}{dt}\boldsymbol{\psi}^i = -\mathbf{C}_b^i \delta\boldsymbol{\omega}_{ib}^b$
  - **IMU biases, scale-factor errors, assumed as random constants**
  - **gravity disturbance components, modeled, e.g., as second-order Gauss-Markov processes in *n*-frame, with** $\delta\overline{\boldsymbol{g}}^i = \mathbf{C}_n^i \delta\overline{\boldsymbol{g}}^n$

$$\frac{d^2}{dt^2}\delta\overline{\boldsymbol{g}}^n = -2\begin{pmatrix} \beta_N & 0 & 0 \\ 0 & \beta_E & 0 \\ 0 & 0 & \beta_D \end{pmatrix}\frac{d}{dt}\delta\overline{\boldsymbol{g}}^n - \begin{pmatrix} \beta_N^2 & 0 & 0 \\ 0 & \beta_E^2 & 0 \\ 0 & 0 & \beta_D^2 \end{pmatrix}\delta\overline{\boldsymbol{g}}^n + \boldsymbol{w}_{\delta\overline{g}}$$

  - ◦ $w_{\delta g}$ **is a white noise vector process with appropriately selected variances**
  - ◦ $\beta_N, \beta_E, \beta_D$ **are parameters appropriately selected to model the correlation time of the processes** $= 2.146/\beta_{N,E,D}$

- **System dynamics equation**

$$\frac{d}{dt}\boldsymbol{\varepsilon}_D = \mathbf{F}_D\boldsymbol{\varepsilon}_D + \mathbf{G}_D\boldsymbol{w}_D$$

where w_D is a vector of white noise processes

Determination of δg^n (3)

- **Alternatively, omit gravity disturbance model**
  - observations assume normal gravitation is correct

$$
\begin{aligned}
\boldsymbol{y} &= \left(\tilde{\boldsymbol{a}}^i + \boldsymbol{g}^i\right) - \tilde{\ddot{\boldsymbol{x}}}^i \\
&= \left(\boldsymbol{a}^i + \delta\boldsymbol{a}^i\right) + \boldsymbol{g}^i - \left(\ddot{\boldsymbol{x}}^i + \delta\ddot{\boldsymbol{x}}^i\right) \\
&= \underbrace{\mathbf{C}_b^i \delta\boldsymbol{a}^b - \left[\boldsymbol{\psi}^i \times\right]\mathbf{C}_b^i \boldsymbol{a}^b}_{\delta\boldsymbol{a}^i} - \delta\ddot{\boldsymbol{x}}^i
\end{aligned}
$$

$$
\boldsymbol{g}^i = \boldsymbol{\gamma}^i + \boldsymbol{\Omega}_{ie}^i \boldsymbol{\Omega}_{ie}^i \boldsymbol{x}^i
$$

  - optimal estimates of IMU systematic errors by Kalman filter yield $\hat{\boldsymbol{y}}$
  - gravity disturbance estimates: $\delta\overline{\boldsymbol{g}}^i \approx \hat{\boldsymbol{y}} - \boldsymbol{y}$
    - assumes residual IMU systematic errors are small and white noise can be filtered
  - successfully applied technique (Kwon and Jekeli, 2001)

Indirect Method

- **Recall inertial navigation equations in *n*-frame**

$$\frac{d\boldsymbol{v}^n}{dt} = \mathbf{C}_b^n \boldsymbol{a}^b - \left(\boldsymbol{\Omega}_{ie}^n + \boldsymbol{\Omega}_{in}^n\right)\boldsymbol{v}^n + \bar{\boldsymbol{g}}^n$$

$$\frac{d\boldsymbol{x}^e}{dt} = \mathbf{C}_n^e \boldsymbol{v}^n$$

  - **integrate (i.e., get IMU navigation solution) and solve for $\bar{g}^n$ using a model and GNSS tracking data**
  - **analogous to traditional satellite tracking methods to determine global gravitational field, except gravity model is linear stochastic process, not spherical harmonic model**
  - **navigation solution from OTF INS should not be integrated with GNSS!**
    - **IMU and GNSS must be treated as separate sensors, just like in scalar gravimetry**
    - **use raw accelerometer and gyro data to obtain free-inertial navigation solution**
      - **one could pre-process IMU/GNSS data to solve for IMU systematic errors, neglecting gravity**

Determination of δg^n (1)

- **Solve for gravity disturbance treated as an error state (among many others) in the linear perturbation of navigation equations**
  - **typical error states collected in state vector, ε_I, include:**
    - **position errors, velocity errors, orientation errors**
    - **IMU systematic errors (biases, etc.)**
    - **gravity disturbance components**

$$\frac{d}{dt}\boldsymbol{\varepsilon}_I = \mathbf{F}_I\boldsymbol{\varepsilon}_I + \mathbf{G}_I\boldsymbol{w}_I$$ **where $\boldsymbol{w}_I$ is a vector of white noise processes**

  - **integration is done numerically (e.g., using linear finite differences)**

$$\boldsymbol{\varepsilon}_I\left(t_k\right) = \boldsymbol{\Phi}_I\left(t_k, t_{k-1}\right)\boldsymbol{\varepsilon}_I\left(t_{k-1}\right) + \mathbf{G}_I\left(t_k\right)\overline{\boldsymbol{w}}_I\left(t_k\right)$$ **where Φ = state transition matrix**

  - **observations are differences, IMU-indicated minus GNSS positions, treated as updates to the corresponding system states**

$$\boldsymbol{y}\left(t_k\right) = \mathbf{H}\left(t_k\right)\boldsymbol{\varepsilon}_I\left(t_k\right) + \boldsymbol{v}\left(t_k\right)$$ **where $\boldsymbol{v}$ is a vector of discrete white noise processes**

Determination of δg^n (2)

- **Gravity disturbance model**
  - stochastic process; e.g., second-order Gauss-Markov process,

$$\frac{d^2}{dt^2}\delta \boldsymbol{g}^n = -2\begin{pmatrix} \beta_N & 0 & 0 \\ 0 & \beta_E & 0 \\ 0 & 0 & \beta_D \end{pmatrix}\frac{d}{dt}\delta \boldsymbol{g}^n - \begin{pmatrix} \beta_N^2 & 0 & 0 \\ 0 & \beta_E^2 & 0 \\ 0 & 0 & \beta_D^2 \end{pmatrix}\delta \boldsymbol{g}^n + \boldsymbol{w}_{\delta g}$$

    - $\boldsymbol{w}_{\delta g}$ is a white noise vector process with appropriately selected variances
    - β_N, β_E, β_D are parameters appropriately selected to model the correlation time of the processes $= 2.146/\beta_{N,E,D}$

- **Kalman filter/smoother estimate, is optimal in the sense of minimum mean square error**
  - theoretically the gravity model is an approximation since the gravity field is not a linear, finite-dimensional, set of independent along-track signals as required/modeled in the system state formalism
  - successful estimation depends on stochastic separability of gravity disturbance from accelerometer errors and coupled gyro errors

Rudimentary Error analysis

- Assume that ***n***- and ***i***-frames coincide (approx. valid for < 1 hour)

negligible

$$\boldsymbol{g}^n \approx \ddot{\boldsymbol{x}}^n - \boldsymbol{a}^n \Rightarrow \qquad \delta \boldsymbol{g}^n = \delta \ddot{\boldsymbol{x}}^n + \boldsymbol{\Psi}^n \mathbf{C}_b^n \boldsymbol{a}^b - \mathbf{C}_b^n \delta \boldsymbol{a}^b - \boldsymbol{\Gamma}^n \delta \boldsymbol{p}^n$$

gravitation errors

errors in kinematic acceleration

errors in sensor orientation

errors in inertial acceleration

gravitational gradients

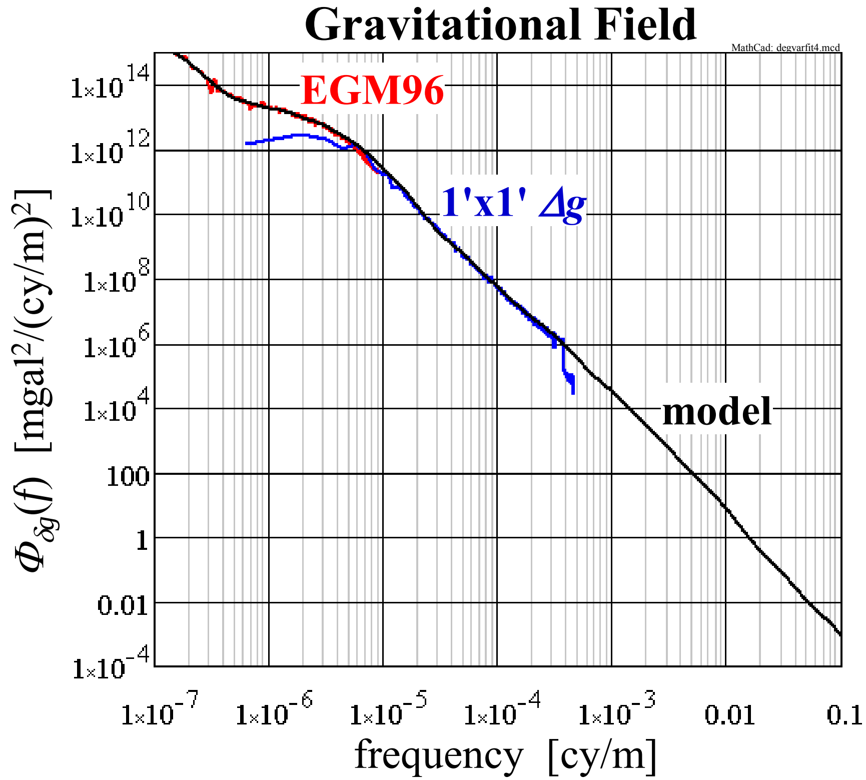
position errors

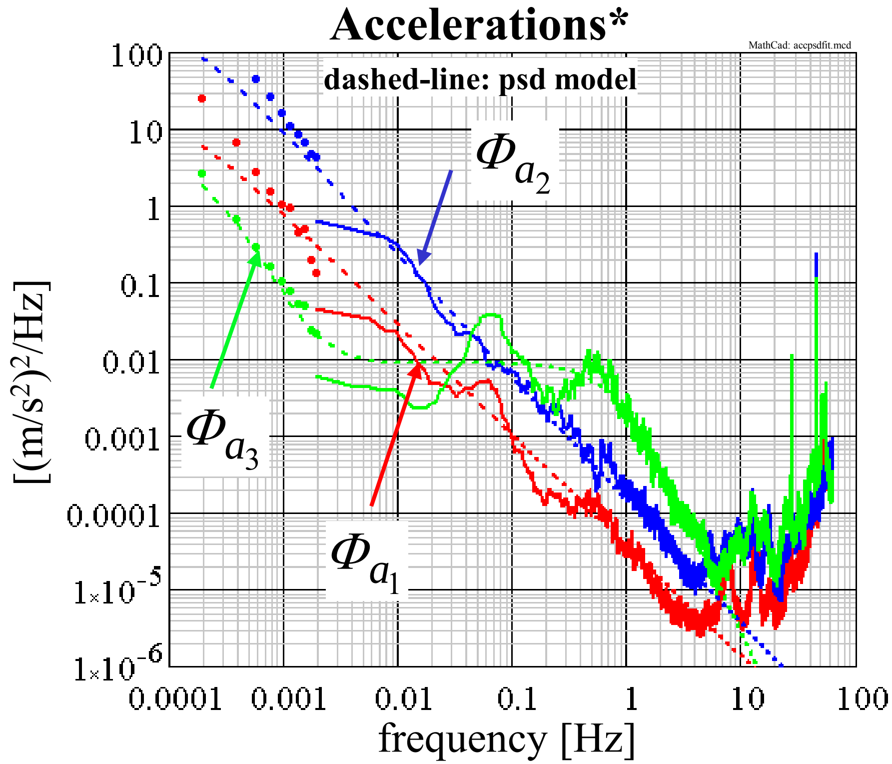
- δg **PSD** is obtained from models of IMU and position error PSDs and PSD of vehicle acceleration

$$\Phi_{\delta g_1} = \Phi_{\delta \ddot{x}_1} + \Phi_{\psi_3 a_2} + \Phi_{\psi_2 a_3} + \Phi_{\delta a_1}$$

$$\Phi_{\delta g_3} = \Phi_{\delta \ddot{x}_3} + \Phi_{\psi_2 a_1} + \Phi_{\psi_1 a_2} + \Phi_{\delta a_3}$$

PSD Models





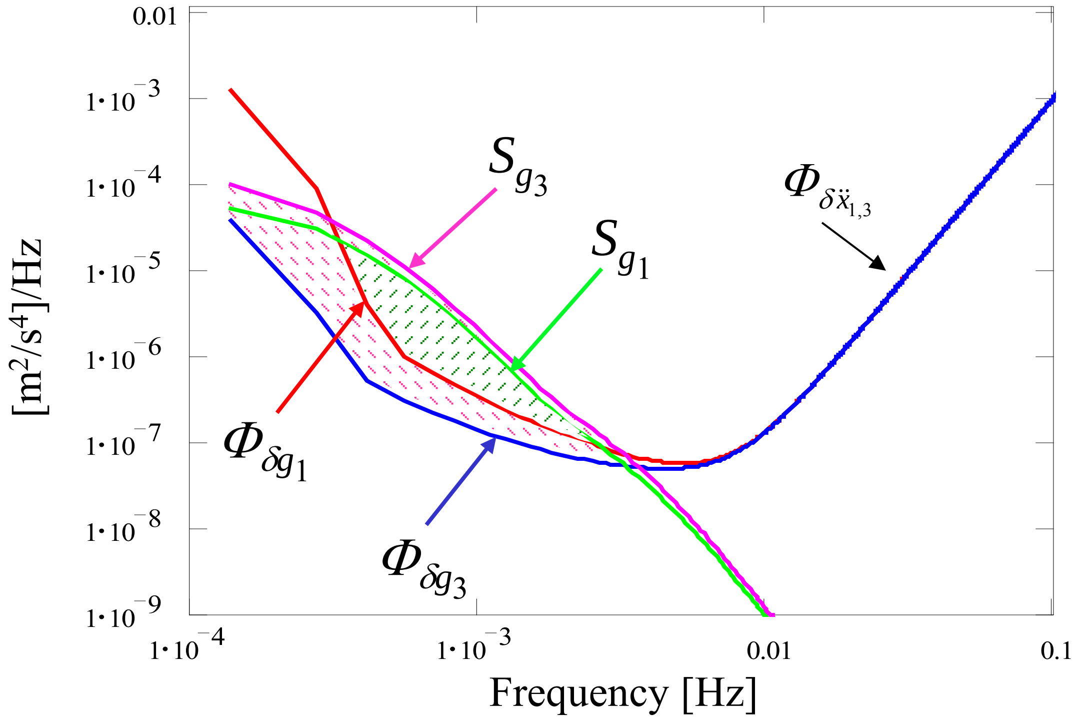
* data from Twin Otter aircraft

- **Accelerometers: white noise (±25 mGal)**
- **Gyros: rate bias (± 0.003°/hr) plus white noise (±0.06 °/hr/√Hz)**
- **Orientation: initial bias (0.005°)**
- **Position: white noise (±0.1 m)**

approximated by simple PSD models

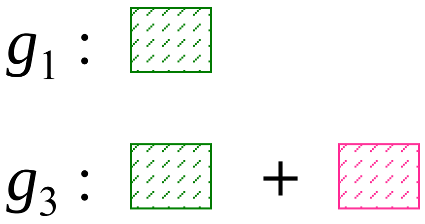
- **Aircraft speed = 250 km/hr; altitude = 1000 m**

PSDs of Gravity Errors, δg_1, δg_3, Versus Along-Track PSDs of Signals, g_1, g_3



Spectral Window:

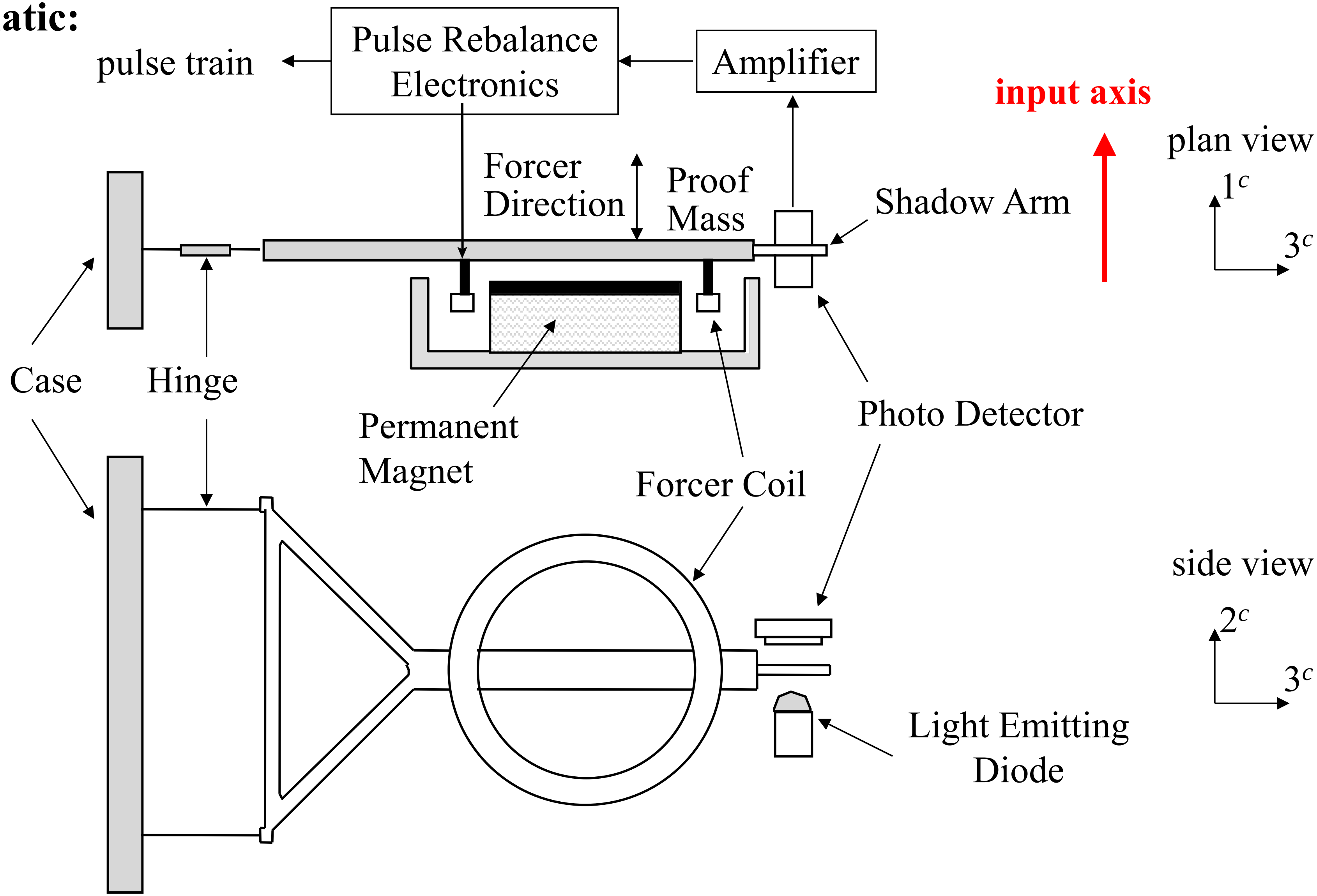
signal-to-noise ratio > 1



- **Signal-to-noise ratio > 1 over larger bandwidth for g_3**
- **Significant error source is orientation bias, especially for g_1, g_2**
- **Signal PSD's move to right *and* down with increased velocity**

Pendulous, Force-Rebalance Accelerometer (e.g.)

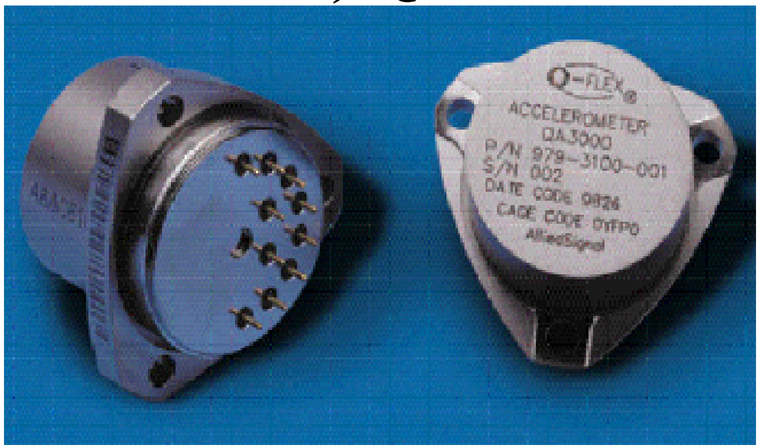
- **Schematic:**



- **Torque needed to keep proof mass in equilibrium is a measure of acceleration**

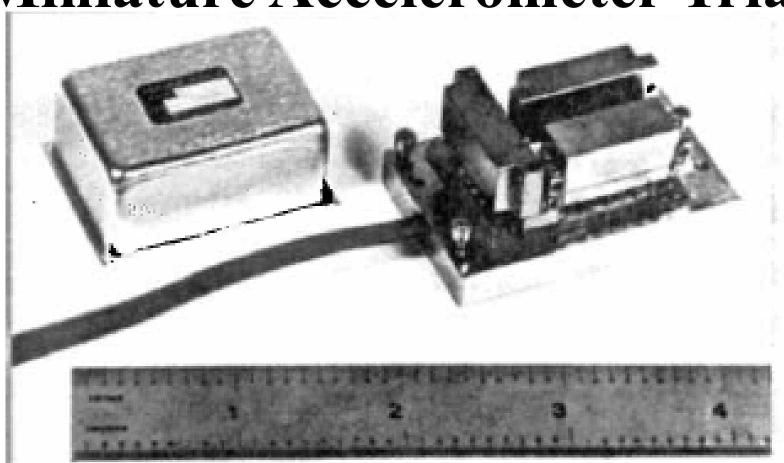
Pendulous, Force-Rebalance Accelerometer (e.g.)

- **Honeywell (Allied Signal, Sundstrandt) QA3000**



Performance	QA3000-030
Input Range [g]	±60
Bias [mg]	<4
One-year Composite repeatability [μg]	<40
Temperature Sensitivity [μg/°C]	<15
Scale Factor [mA/g]	1.20 to 1.46
One-year Composite Repeatability [ppm]	<80
Temperature Sensitivity [ppm/°C]	<120
Axis Misalignment [μrad]	<1000
One-year Composite Repeatability [μrad]	<70
Vibration Rectification [μg/g^{2}rms]	<10 (50-500 Hz) <35 (500-2000 Hz)
Intrinsic Noise [μg-rms]	<7 (0-10 Hz) <70 (10-500 Hz) <1500 (500-10,000 Hz)

- **Litton (Northrop Grumman) A-4 Miniature Accelerometer Triad**



PERFORMANCE	
Bias Repeatability	25 μg
Scale Factor	120 ppm
Alignment	2 sec

INS Used for Airborne Gravimetry

Honeywell Laseref III

University of Calgary

Honeywell H-770

Intermap Technologies

Litton LN100

Ohio State University

Strapdown Platforms

ITC-2 Inertial Survey System

Stabilized Platform

Bauman Moscow State Technical University

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