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Mathematical Basis of Analytic Aerotriangulation

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Technical Bulletin No. 8 presents the mathematical basis of the solution of analytic aerotriangulation that is currently being programmed for electronic computation in the Coast and Geodetic Survey.

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Mathematical Basis of Analytic Aerotriangulation

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ANALYTIC aerotriangulation is a method for accurately determining the ground positions of objects throughout a strip or block of overlapping aerial photographs, using a relatively few known ground positions, by means of digital calculations based on coordinate measurements of pertinent image positions on each photograph. This method differs from the usual instrumental method that is based on measurements of a stereoscopic model which is perfected or solved by means of an analogue device (stereoscopic plotter). The analytic method offers certain worthwhile advantages accruing from automation, digital accuracy, systematic adjustment, and freedom from the discrepancies contributed by the plotting instrument.

The method discussed here is none other than that of Dr. Hellmut H. Schmid[1 2] of the Ballistic Research Laboratories. It is presented in a simpler manner for practical operation in the Coast and Geodetic Survey.

The general subject has received considerable attention in recent years but computational difficulties prevented any practical application until the advent of the electronic computer.

Von Gruber[3] presented a remarkable treatment of the subject in 1932 concluding that it had "no practical significance for the current work of map making." It is noteworthy that the Schmid approach is essentially identical to that of von Gruber.

Church[4] utilized an analytical approach which simulated ground survey practices for teaching purposes at Syracuse University, reporting his work in a series of bulletins between 1930 and 1950.

Herget[5] of Ohio State University announced another solution for the problem in 1954 which has since been developed in detail at Cornell University[6] (1956-58) under contract with the Engineer Research and Development Laboratories. The approach simulates the solution used in plotting instruments. This method is currently being programmed by the U. S. Geological Survey and is reported by Dodge.[7]

In England a different form of analytical method has been in practical use since about 1945 and is reported by Arthur[8] in 1955.

In summary, the general solution is currently being programmed for computation in one form or another at Ballistic Research Laboratories, Air Force Missile Test Center, Engineer Research and Development Laboratories, Army Map Service, Ohio State University, Cornell University, Aeronautical Chart and Information Center, U. S. Geological Survey, as well as the Coast and Geodetic Survey.

THE GEOMETRIC BASIS

An object A on the ground (fig. 1), an aerial camera station O (perspective center) and a photographic image a of the object are considered to lie on a line. Rectangular coordinates of the three points are considered to be expressed in related systems: $A\ (X, Y, Z)$; $O\ (X_o, Y_o, Z_o)$; $a(x^*, y^*, z^*)$. The derivation is entirely independent from the fact that the coordinate system may be geodetic or geocentric.

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[1] An analytical treatment of the orientation of a photogrammetric camera by Hellmut H. Schmid, *Ballistic Research Laboratories*, Aberdeen Proving Ground, Md., Report No. 880, 1953.

[2] An analytical treatment of the problem of triangulation by stereophotogrammetry by Hellmut H. Schmid, *Ballistic Research Laboratories*, Aberdeen Proving Ground, Md., Report No. 961, 1955.

[3] Photogrammetry, collected lectures and essays by O. von Gruber, *Chapman and Hall*, pp. 13, 38, 1932.

[4] Revised geometry of the aerial photograph by Earl Church, *Syracuse University*, Syracuse, N. Y., Bulletin 15, 1945, and others.

[5] Reduction of aerial photographs on electronic computers by Paul Herget, *Photogrammetric Engineering*, Vol. XX, No. 5, p. 842, 1954.

[6] A solution of the general analytical aerotriangulation problem by Cornell University, *Engineer Research and Development Laboratories*, Final Technical Report, June 1956 to May 1958.

[7] Sequential presentation of analytic aerotriangulation by the direct geodetic restraint method by Hugh F. Dodge, U. S. Geological Survey, October 1957 (typescript).

[8] A stereocomparator technique for aerial triangulation by D. W. G. Arthur, *Ordnance Survey*, London, Professional Papers, New Series No. 20, 1955.

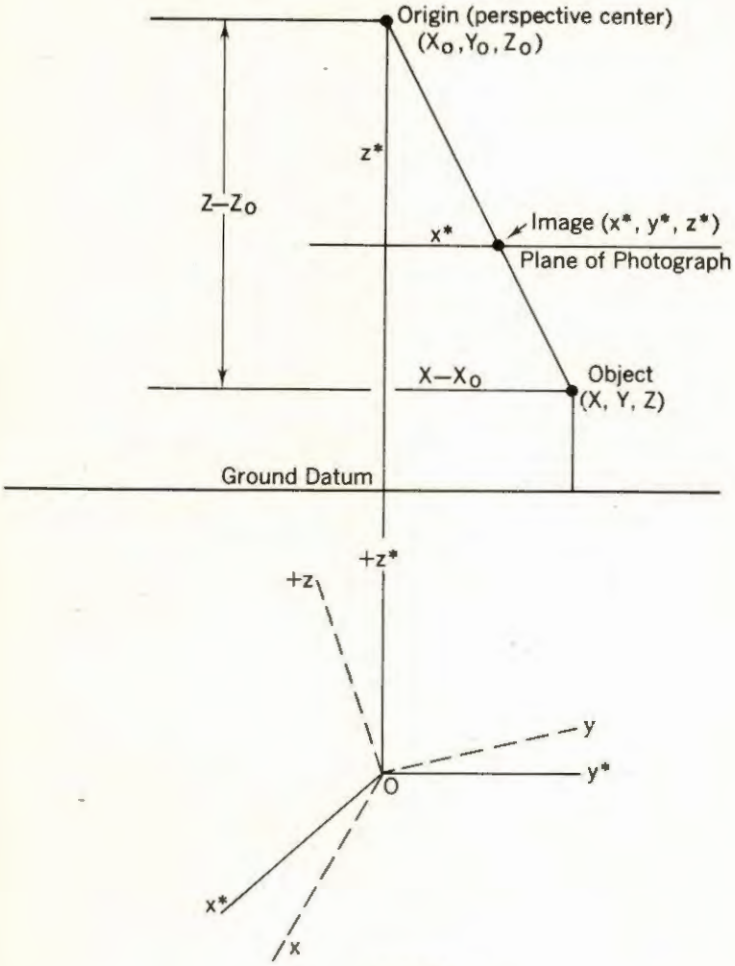

FIGURE 1.

In this $x^*y^*z^*$-coordinate system the image plane is considered to be parallel to the XY-plane: the photograph is not tilted. In the presence of tilt, x^*, y^*, z^* are evaluated by means of a transformation (analytic rectification). An aerial photograph is always considered to be tilted. The *observed* rectangular coordinates of the image a on a tilted photograph are called x, y, z. The x, y, z and the x^*, y^*, z^* systems of coordinates have the point O as their common origin, and they differ only in angular orientation. (z is the focal length of the aerial camera.) By the usual analytic geometry formulas for rotation of the axes in three dimensions, one set of coordinates can be expressed in terms of the other set:

$$x = x^*A_1 + y^*A_2 + z^*A_3$$

$$y = x^*B_1 + y^*B_2 + z^*B_3 \qquad (1), (30)$$

$$z = x^*C_1 + y^*C_2 + z^*C_3 \ .$$

A method for the evaluation of the cosines $A_1 \ldots C_3$ is given below in Algebraic Derivation of the Rotation Formulas.

The following ratio is evident from the similar triangles in the figure:

$$(X\text{-}X_o)/(Z\text{-}Z_o) = x^*/z^*. \qquad (2)$$

Letting $(X) = (X\text{-}X_o)$, etc., and solving for x^*

$$x^* = (X)\, z^*/(Z) \ . \qquad (3)$$

Similarly, by visualizing the YZ-plane,

$$y^* = (Y)\, z^*/(Z) \ . \qquad (4)$$

By algebraic identity

$$z^* = (Z)\, z^*/(Z) \ . \qquad (5)$$

If (3), (4), and (5) are substituted into (1),

$$x = A_1(X)\, z^*/(Z) + A_2(Y)\, z^*/(Z) + A_3(Z)\, z^*/(Z),$$

or, recognizing the common factor $z^*/(Z)$,

$$\begin{aligned} x &= [(X)A_1 + (Y)A_2 + (Z)A_3]\, z^*/(Z) \\ y &= [(X)B_1 + (Y)B_2 + (Z)B_3]\, z^*/(Z) \qquad (6) \\ z &= [(X)C_1 + (Y)C_2 + (Z)C_3]\, z^*/(Z) \ . \end{aligned}$$

The expressions for x and y are each divided by that for z:

$$\begin{aligned} \frac{x}{z} &= \frac{(X)A_1 + (Y)A_2 + (Z)A_3}{(X)C_1 + (Y)C_2 + (Z)C_3} \\ \frac{y}{z} &= \frac{(X)B_1 + (Y)B_2 + (Z)B_3}{(X)C_1 + (Y)C_2 + (Z)C_3} \end{aligned} \qquad (7)$$

Expressions (7) constitute the equations for a line in space on which lie the three points—the ground object, the camera lens point, and the photographic image. The equations appear in many texts on photogrammetry[9] and are also familiar to students of projective geometry. The equations are said to

[9] See footnote 3 above.

represent a projective transformation. They are also regarded as the conditions for a three-point collineation. They are the fundamental condition equations of the Schmid approach to analytic aerotriangulation.

Fractions are cleared for convenience:

$$x\,[(X)C_1 + (Y)C_2 + (Z)C_3] - z\,[(X)A_1 + (Y)A_2 + (Z)A_3] = 0$$
$$y\,[(X)C_1 + (Y)C_2 + (Z)C_3] - z\,[(X)B_1 + (Y)B_2 + (Z)B_3] = 0 \qquad (8)$$

All nine cosines $A_1 \ldots C_3$ can be expressed in terms of three angles as shown by Equations (33) below. Angles a, b, c have been adopted; they are associated respectively with the x, y, z axes. A sequential, rather than simultaneous application of these angles has been selected because of certain simplifications introduced thereby. If Equations (33) are substituted into (8),

$$\begin{aligned} x\,[(X\text{-}X_o)\sin b + (Y\text{-}Y_o)(-\sin a\cos b) + (Z\text{-}Z_o)\cos a\cos b] - z[(X\text{-}X_o)\cos b\cos c \\ + (Y\text{-}Y_o)(\cos a\sin c + \sin a\sin b\cos c) \\ + (Z\text{-}Z_o)(\sin a\sin c - \cos a\sin b\cos c)] = 0 \\ y\,[(X\text{-}X_o)\sin b + (Y\text{-}Y_o)(-\sin a\cos b) + (Z\text{-}Z_o)\cos a\cos b] - z\,[(X\text{-}X_o)(-\cos b\sin c) \\ + (Y\text{-}Y_o)(\cos a\cos c - \sin a\sin b\sin c) \\ + (Z\text{-}Z_o)(\sin a\cos c + \cos a\sin b\sin c)] = 0 \end{aligned} \qquad (9)$$

An inverse form of the equations is also needed during the process to express ground coordinates in terms of photographic image coordinates on two overlapping photographs. If Equation (2) is solved for X,

$$X = X_o' + (Z\text{-}Z_o')x^{*\prime}/z^{*\prime}$$
$$X = X_o'' + (Z\text{-}Z_o'')x^{*\prime\prime}/z^{*\prime\prime}$$

where a single prime mark (') refers to the first of the two photographs and a double prime ('') refers to the second. By equating the two expressions and solving for Z,

$$Z = \frac{(X_o'' - X_o')z^{*\prime}z^{*\prime\prime} + Z_o'x^{*\prime}z^{*\prime\prime} - Z_o''x^{*\prime\prime}z^{*\prime}}{x^{*\prime}z^{*\prime\prime} - x^{*\prime\prime}z^{*\prime}}\,. \qquad (10)$$

This equation permits one to compute Z for any image in terms of the six orientation and position parameters of each of two overlapping photographs in the event X, Y, or Z are not known from control data. The denominator will never approach zero.

Once Z is determined, either from (10) or from control data, X and Y are explicitly defined:

$$X = X_o + (Z\text{-}Z_o)x^*/z^*$$
$$Y = Y_o + (Z\text{-}Z_o)y^*/z^* \qquad (11)$$

However, if X and Y only are given for a control point, then Z can be computed from one photograph using the same relations in a different form:

$$Z = Z_o + (X\text{-}X_o)z^*/x^* \qquad (12)$$

The terms x^*, y^*, z^* are defined in Equations (31).

DERIVATION OF THE ITERATIVE EQUATIONS

Inasmuch as the geometric condition equations (8) or (9) are too involved to be solved in a direct manner for each of the six parameters for each photograph, special equations are more suitable for an iterative type of solution. These special iteration or observation equations need to be augmented to include a least square adjustment of the observed image coordinates, which are subject to random types of discrepancies from several sources, and where the number of images observed is considerably larger than the number of unknowns.

The iterative solution is based on initially selected logical approximations for the six unknown parameters a, b, c, X_o, Y_o, Z_o, and a solution for corrections to them, Δa, Δb, Δc, ΔX_o, ΔY_o, ΔZ_o, which, when applied, will make the corrected approximations conform more nearly to the geometry

implied by the Equations (9). Thus the original equations (9) are replaced by a different set of correction (observation) equations which are linear, whose unknowns are the corrections $\Delta a \ldots \Delta Z_o$, and which derive from (9) by partial differentiation. The purpose of this section is to show the derivation of the observation equations and to indicate how the coefficients in the observation equations can be computed from the observed and estimated data.

The observation or correction equations are formed from Equations (8) or (9) by partial differentiation of calculus. Equation (8), demonstrates the orderliness and relationships of the terms, and (9) is the identical equation expressed in terms of the fundamental variables. Most comments about either form apply to both forms. Equation (8) is sometimes more convenient to discuss because of its relative brevity.

For purposes of further brevity in this discussion, Equation (8) is expressed as

$$\begin{aligned} wx - uz &= 0 \\ wy - vz &= 0 \ . \end{aligned} \tag{13}$$

These relations state the condition that the sums are zero for any image-lens-object collineation. But if incorrectly estimated values for the six unknown variables are substituted into the equations, it is obvious that the sums will probably not be zero. Let F_x and F_y denote the sums using the estimated values:

$$\begin{aligned} w'x - u'z &= F_x \\ w'y - v'z &= F_y \ . \end{aligned} \tag{14}$$

F_x and F_y are also the total changes in the values of the functions caused by all the discrepancies in the estimations of the variables, and may be written ΔF_x and ΔF_y in the sense of calculus. They are normally relatively small. These functions may be regarded as measures of incorrectness of the estimates.

We seek relatively small changes $\Delta a \ldots \Delta Z_o$ in the estimated variables, and minimum residual corrections Δx and Δy (v_x, v_y) to the observed quantities, such that ΔF_x and ΔF_y will be zero simultaneously for all the image-lens-object collineations under consideration (18 to 25 on each of an unlimited number of photographs). Partial differential calculus is used to derive expressions involving the various changes:

$$\begin{aligned} \frac{\partial F_x}{\partial a}\Delta a + \frac{\partial F_x}{\partial b}\Delta b + \ldots + \frac{\partial F_x}{\partial Z_o}\Delta Z_o + \Delta F_x &= 0 \\ \frac{\partial F_y}{\partial a}\Delta a + \frac{\partial F_y}{\partial b}\Delta b + \ldots + \frac{\partial F_y}{\partial Z_o}\Delta Z_o + \Delta F_y &= 0 \end{aligned} \tag{15}$$

The terms like $\partial F_x/\partial a$ represent numerical coefficients which can be determined with sufficient accuracy from the approximated parameters by means of formulas derived presently. Terms like Δa are the corrections, unknowns, which are to be determined by a simultaneous solution of a set of equations like (15), which are linear.

An interpretation of Equation (15) is that a discrepancy in F, say ΔF_x, is equal to the sum of the individual discrepancies caused by a change $\partial F_x/\partial a$, etc., of one unit in each variable multiplied by the total number of units Δa of change in the corresponding variable. It is noteworthy that, inasmuch as the correct values of the variables cause ΔF to be zero, ΔF is equal to the inequality of Equation (9) when the estimated values are substituted.

In Equation (14) F_x and F_y are treated as though they are *errors* caused by using the estimated values. As a result, the values of the variables in the solution will indicate the errors in the estimations. But it is more convenient to consider these values as *corrections* which can be *added* to the estimates. Whereupon it is expedient to reverse the algebraic signs of ΔF_x and ΔF_y, and also to transpose them to the other sides of the equal signs, which is done in Equation (15).

The complete expansion of the partial derivatives of Equation (9) are as follows:

$$\begin{aligned} &[(X\text{-}X_o)\sin b + (Y\text{-}Y_o)(\text{-}\sin a \cos b) + (Z\text{-}Z_o)\cos a \cos b]\,\Delta x \\ &+ \{x[(Y\text{-}Y_o)(\text{-}\cos a \cos b) + (Z\text{-}Z_o)(\text{-}\sin a \cos b)] \\ &\quad -z[(Y\text{-}Y_o)(\text{-}\sin a \sin c + \cos a \sin b \cos c) \\ &\qquad + (Z\text{-}Z_o)(\cos a \sin c + \sin a \sin b \cos c)]\}\,\Delta a \end{aligned}$$

$$
\begin{aligned}
&+ \{x[(X-X_o)\cos b + (Y-Y_o)\sin a \sin b + (Z-Z_o)(-\cos a \sin b)] \\
&\quad -z[(X-X_o)(-\sin b \cos c) + (Y-Y_o)\sin a \cos b \cos c \\
&\qquad + (Z-Z_o)(-\cos a \cos b \cos c)]\} \Delta b \\
&-z[(X-X_o)(-\cos b \sin c) + (Y-Y_o)(\cos a \cos c - \sin a \sin b \sin c) \\
&\quad + (Z-Z_o)(\sin a \cos c + \cos a \sin b \sin c)] \Delta c \\
&- (x \sin b - z \cos b \cos c) \Delta X_o \\
&- [-x \sin a \cos b - z(\cos a \sin c + \sin a \sin b \cos c)] \Delta Y_o \\
&- [x \cos a \cos b - z(\sin a \sin c - \cos a \sin b \cos c)] \Delta Z_o \\
&+ (x \sin b - z \cos b \cos c) \Delta X \\
&+ [-x \sin a \cos b - z(\cos a \sin c + \sin a \sin b \cos c)] \Delta Y \\
&+ [x \cos a \cos b - z(\sin a \sin c - \cos a \sin b \cos c)] \Delta Z \\
&+ \{x[(X-X_o)\sin b + (Y-Y_o)(-\sin a \cos b) + (Z-Z_o)\cos a \cos b] \\
&\quad -z[(X-X_o)\cos b \cos c + (Y-Y_o)(\cos a \sin c + \sin a \sin b \cos c) \\
&\qquad + (Z-Z_o)(\sin a \sin c - \cos a \sin b \cos c)]\} = 0 ;
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&[(X-X_o)\sin b + (Y-Y_o)(-\sin a \cos b) + (Z-Z_o)\cos a \cos b] \Delta y \\
&+ \{y[(Y-Y_o)(-\cos a \cos b) + (Z-Z_o)(-\sin a \cos b)] \\
&\quad -z[(Y-Y_o)(-\sin a \cos c - \cos a \sin b \sin c) \\
&\qquad + (Z-Z_o)(\cos a \cos c - \sin a \sin b \sin c)]\} \Delta a \\
&+ \{y[(X-X_o)\cos b + (Y-Y_o)\sin a \sin b + (Z-Z_o)(-\cos a \sin b)] \\
&\quad -z[(X-X_o)\sin b \sin c + (Y-Y_o)(-\sin a \cos b \sin c) \\
&\qquad + (Z-Z_o)\cos a \cos b \sin c]\} \Delta b \\
&-z\,[(X-X_o)(-\cos b \cos c) + (Y-Y_o)(-\cos a \sin c - \sin a \sin b \cos c) \\
&\quad + (Z-Z_o)(-\sin a \sin c + \cos a \sin b \cos c)] \Delta c \\
&- [y \sin b - z(-\cos b \sin c)] \Delta X_o \\
&- [-y \sin a \cos b - z(\cos a \cos c - \sin a \sin b \sin c)] \Delta Y_o \\
&- [y \cos a \cos b - z(\sin a \cos c + \cos a \sin b \sin c)] \Delta Z_o
\end{aligned}
$$

$$+ [y \sin b - z(-\cos b \sin c)] \Delta X$$

$$+ [-y \sin a \cos b - z(\cos a \cos c - \sin a \sin b \sin c)] \Delta Y$$

$$+ [y \cos a \cos b - z(\sin a \cos c + \cos a \sin b \sin c)] \Delta Z$$

$$+ \{y [(X-X_o)\sin b + (Y-Y_o)(-\sin a \cos b) + (Z-Z_o)\cos a \cos b]$$

$$-z [(X-X_o)(-\cos b \sin c)$$

$$+ (Y-Y_o)(\cos a \cos c - \sin a \sin b \sin c)$$

$$+ (Z-Z_o)(\sin a \cos c + \cos a \sin b \sin c)]\} = 0 \ .$$

The next steps constitute a simplification of the appearance of Equations (16) for the purpose of aiding in their utilization. Inasmuch as the derivation is complete, the simplification adds no photogrammetric nor mathematical principles. First the trigonometric terms are grouped as they are in Equation (8), with the introduction of terms like $A_1{}^*$, meaning the partial derivative of A_1 with respect to the angle b. At the same time, Δx and Δy (v_x, v_y) are solved for:

$$v_x = \frac{\begin{array}{l} \{x[-C_3(Y) + C_2(Z)] - z[-A_3(Y) + A_2(Z)]\} \Delta a \\ + \{x[C_1{}^*(X) + C_2{}^*(Y) + C_3{}^*(Z)] - z[A_1{}^*(X) + A_2{}^*(Y) + A_3{}^*(Z)]\} \Delta b \\ -z[B_1(X) + B_2(Y) + B_3(Z)] \Delta c \\ -(C_1x - zA_1)\Delta X_o - (C_2x - zA_2)\Delta Y_o - (C_3x - zA_3)\Delta Z_o \\ + (C_1x - A_1z)\Delta X + (C_2x - A_2z)\Delta Y + (C_3x - A_3z)\Delta Z \\ + \{x[C_1(X) + C_2(Y) + C_3(Z)] - z[A_1(X) + A_2(Y) + A_3(Z)]\} \end{array}}{C_1(X) + C_2(Y) + C_3(Z)} ;$$

(17)

$$v_y = \frac{\begin{array}{l} \{y[-C_3(Y) + C_2(Z)] - z[-B_3(Y) + B_2(Z)]\} \Delta a \\ + \{y[C_1{}^*(X) + C_2{}^*(Y) + C_3{}^*(Z)] - z[B_1{}^*(X) + B_2{}^*(Y) + B_3{}^*(Z)]\} \Delta b \\ + z[A_1(X) + A_2(Y) + A_3(Z)] \Delta c \\ -(yC_1 - zB_1)\Delta X_o - (yC_2 - zB_2)\Delta Y_o - (yC_3 - zB_3)\Delta Z_o \\ + (C_1y - B_1z)\Delta X + (C_2y - B_2z)\Delta Y + (C_3y - B_3z)\Delta Z \\ + \{y[C_1(X) + C_2(Y) + C_3(Z)] - z[B_1(X) + B_2(Y) + B_3(Z)]\} \end{array}}{C_1(X) + C_2(Y) + C_3(Z)} .$$

It seems desirable to express (17) using still simpler looking coefficients: (compare with (15))

$$v_x = p_1\Delta a + p_2\Delta b + p_3\Delta c + p_4\Delta X_o + p_5\Delta Y_o + p_6\Delta Z_o - p_4\Delta X - p_5\Delta Y - p_6\Delta Z + p_7$$
$$v_y = q_1\Delta a + q_2\Delta b + q_3\Delta c + q_4\Delta X_o + q_5\Delta Y_o + q_6\Delta Z_o - q_4\Delta X - q_5\Delta Y - q_6\Delta Z + q_7 \quad . \tag{18}$$

If a chronological set of new letters $A, B, C \ldots T$ (which bear no significance to the derivation) is introduced for common groups of trigonometric terms, a computational routine is fairly obvious. Also, it is convenient to let the letters a, b, c represent the sines of the three angles instead of the angles themselves, which provides a significant curtailment of computation inasmuch as the angles themselves are not needed.

A large part of the program consists of computing the numerical values of the 14 coefficients of the observation equations (18), or $p_1 \ldots q_7$. The given data consist of the estimated values for the six terms a, b, c, X_o, Y_o, Z_o. The following equations are useful in programming.

For each photograph:

$$\begin{aligned}
a' &= (1-a^2)^{1/2} & b' &= (1-b^2)^{1/2} & c' &= (1-c^2)^{1/2} \\
A &= b'c' & F &= bc & K &= cC \\
B &= -b'c & G &= ab & L &= cD \\
C &= -ab' & H &= -a'b & M &= a'c + c'G \\
D &= a'b' & I &= -c'C & N &= ac + c'H \\
E &= -bc' & J &= -c'D & P &= a'c' - cG \\
 & & & & Q &= ac' - cH
\end{aligned} \tag{19}$$

For each image:

$$\begin{aligned}
x^* &= Ax + By + bz \\
y^* &= Mx + Py + Cz \\
z^* &= Nx + Qy + Dz
\end{aligned} \tag{31}$$

$$Z = \frac{(X_o''-X_o')z^{*\prime}z^{*\prime\prime} + Z_o'x^{*\prime}z^{*\prime\prime} - Z_o''x^{*\prime\prime}z^{*\prime}}{x^{*\prime}z^{*\prime\prime} - x^{*\prime\prime}z^{*\prime}} \tag{10}$$

$$\begin{aligned}
X &= X_o + x^* (Z-Z_o)/z^* \\
Y &= Y_o + y^* (Z-Z_o)/z^*
\end{aligned} \tag{11}$$

$$Z = Z_o + z^* (X-X_o)/x^* \tag{12}$$

$$R = \frac{1}{b(X-X_o) + C(Y-Y_o) + D(Z-Z_o)} \tag{20}$$

$$S = C(Z-Z_o) - D(Y-Y_o) \tag{21}$$

$$T = b'(X-X_o) + G(Y-Y_o) + H(Z-Z_o) \quad . \tag{22}$$

Then the 14 coefficients are:

$$\begin{aligned}
p_1 &= R\{Sx - z[M(Z-Z_o) - N(Y-Y_o)]\} \\
p_2 &= R\{Tx - z[E(X-X_o) + I(Y-Y_o) + J(Z-Z_o)]\} \\
p_3 &= -Rz[B(X-X_o) + P(Y-Y_o) + Q(Z-Z_o)] \\
p_4 &= -R(bx - Az) \\
p_5 &= -R(Cx - Mz) \\
p_6 &= -R(Dx - Nz) \\
p_7 &= x - Rz[A(X-X_o) + M(Y-Y_o) + N(Z-Z_o)] \;;
\end{aligned} \tag{23}$$

$$q_1 = R\{Sy - z[P(Z-Z_o) - Q(Y-Y_o)]\}$$

$$q_2 = R\{Ty - z[F(X-X_o) + K(Y-Y_o) + L(Z-Z_o)]\}$$

$$q_3 = x - p_7$$

$$q_4 = -R(by - Bz) \qquad (24)$$

$$q_5 = -R(Cy - Pz)$$

$$q_6 = -R(Dy - Qz)$$

$$q_7 = y + p_3 \ .$$

ALGEBRAIC DERIVATION OF ROTATION FORMULAS

An xyz-coordinate system, based on the photo fiducial axes, is used to identify the position of an image on a photograph. An auxiliary $x^*y^*z^*$-system is introduced, being comparable to the xyz-system, having the same origin, and differing in angular orientation only. The x^*, y^*, z^* axes are selected parallel, respectively, to the X, Y, Z axes of the local geocentric or other rectangular ground coordinate system in which $+Z$ is upward, and X and Y form a right handed sequence, such that if $+X$ is eastward, then $+Y$ is northward.

A sequence of rotations is considered (fig. 2) through which can be expressed $x^*y^*z^*$-coordinates in terms of the x, y, z-values, and vice versa. The first rotation a (fig. 2a), like ω or x-tilt, is about the x^*-axis considered counterclockwise positive as viewed from the plus end of the x^*-axis, which corresponds with textbook notation in plane analytic geometry. (Rosenfield[10] considers the rotation *clockwise* as viewed from the *origin*, which is physically identical.) The second rotation b, like ϕ or y-tilt, is about the inclined $y^{*\prime}$-axis, and the third rotation c like κ or swing, is about the twice rotated z^*-axis, which is the photograph perpendicular (lens axis), or the z-axis.

By plane analytic geometry[11] the *new* $x^{*\prime}$, $y^{*\prime}$, $z^{*\prime}$ coordinates of an image point P are expressed

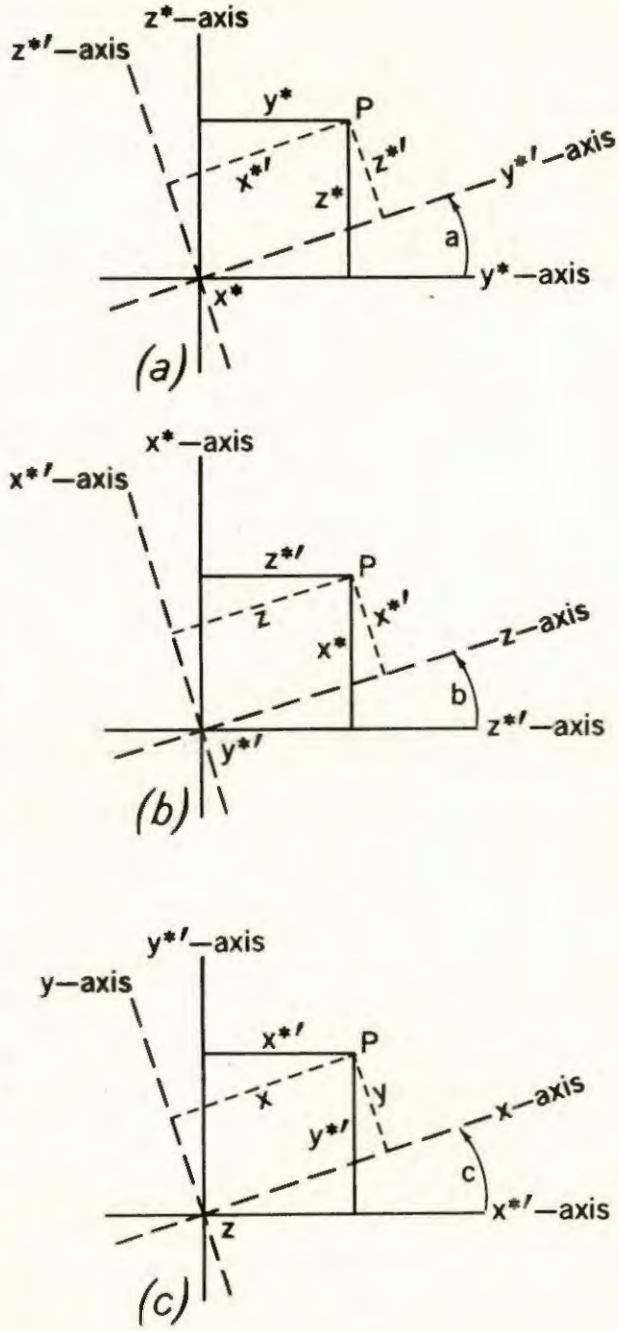

FIGURE 2.

in terms of the *original* x^*, y^*, z^* coordinates for the first a rotation:

$$y^{*\prime} = y^* \cos a + z^* \sin a$$
$$z^{*\prime} = -y^* \sin a + z^* \cos a \ . \qquad (25)$$

The x^* coordinates are unchanged. The second rotation b is strictly related to this once-rotated

[10] The problem of exterior orientation in photogrammetry by George H. Rosenfield, *RCA Service Co.*, Patrick Air Force Base, Fla., preliminary publication, January 1958.

[11] Analytic geometry by Frederick H. Steen and Donald H. Ballou, *Ginn and Company*, p. 101, 1943.

$x^*y^{*\prime}z^{*\prime}$-system inasmuch as this rotation is about the $y^{*\prime}$ axis as shown in figure 2b. Applying the same rotational formulas to the $x^*z^{*\prime}$-system,

$$\begin{aligned} z &= z^{*\prime} \cos b + x^* \sin b \\ x^{*\prime} &= -z^{*\prime} \sin b + x^* \cos b \ . \end{aligned} \tag{26}$$

The y^* coordinates are not changed. The expression for $z^{*\prime}$ in Equation (25) is substituted into Equation (26):

$$\begin{aligned} z &= -y^* \sin a \cos b + z^* \cos a \cos b \\ &\quad + x^* \sin b \\ x^{*\prime} &= y^* \sin a \sin b - z^* \cos a \sin b \\ &\quad + x^* \cos b \ . \end{aligned} \tag{27}$$

The final c-rotation is about the z-axis as shown in figure 2c. Applying the same rotation formulas:

$$\begin{aligned} x &= x^{*\prime} \cos c + y^{*\prime} \sin c \\ y &= -x^{*\prime} \sin c + y^{*\prime} \cos c \ . \end{aligned} \tag{28}$$

The z^* coordinate is unchanged. The expression for $x^{*\prime}$ from Equation (27) and $y^{*\prime}$ from Equation (25) are substituted into Equation (28):

$$\begin{aligned} x &= x^* \cos c \cos b + y^* \sin a \sin b \cos c \\ &\quad -z^* \cos a \sin b \cos c + y^* \cos a \sin c \\ &\quad +z^* \sin a \sin c \\ y &= -x^* \cos b \sin c - y^* \sin a \sin b \sin c \\ &\quad +z^* \cos a \sin b \sin c + y^* \cos a \cos c \\ &\quad +z^* \sin a \cos c \ . \end{aligned}$$

Factoring,

$$\begin{aligned} x &= x^* \cos b \cos c + y^* (\cos a \sin c + \sin a \sin b \cos c) \\ &\quad +z^* (\sin a \sin c - \cos a \sin b \cos c) \\ y &= x^* (-\cos b \sin c) + y^* (\cos a \cos c - \sin a \sin b \sin c) \\ &\quad +z^* (\sin a \cos c + \cos a \sin b \sin c) \\ z &= x^* \sin b + y^* (-\sin a \cos b) + z^* (\cos a \cos b) \ . \end{aligned} \tag{29}$$

This completes the derivation.

Terms $A_1 \ldots C_3$ are introduced for convenience:

$$\begin{aligned} x &= A_1 x^* + A_2 y^* + A_3 z^* \\ y &= B_1 x^* + B_2 y^* + B_3 z^* \\ z &= C_1 x^* + C_2 y^* + C_3 z^* \end{aligned} \tag{30}$$

To be explicit, the inverse expression is:

$$\begin{aligned} x^* &= A_1 x + B_1 y + C_1 z \\ y^* &= A_2 x + B_2 y + C_2 z \\ z^* &= A_3 x + B_3 y + C_3 z \ . \end{aligned} \tag{31}$$

In matrix notation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} ; \tag{32}$$

and the coefficients of Equation (29) are displayed in tabular form:

$$\begin{bmatrix} A_1 A_2 A_3 \\ B_1 B_2 B_3 \\ C_1 C_2 C_3 \end{bmatrix} = \left[\begin{array}{l|l|r} \cos b \cos c & \cos a \sin c + \sin a \sin b \cos c & \sin a \sin c - \cos a \sin b \cos c \\ -\cos b \sin c & \cos a \cos c - \sin a \sin b \sin c & \sin a \cos c + \cos a \sin b \sin c \\ \sin b & -\sin a \cos b & \cos a \cos b \end{array}\right] \tag{33}$$

Equation (32) is identical to Rosenfield's Equation (2.7)[12].

[12] See footnote 10 above.

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