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H. Arnold Karo, Director

Use of Near-Earth Satellite Orbits for Geodetic Information

PAUL D. THOMAS

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Technical Bulletin Series

This series of Technical Bulletins was inaugurated to present to the personnel of the Coast and Geodetic Survey and to others the results of research and development in the various fields of the Bureau's activities. Since many of the bulletins deal with new practices and new techniques, the views expressed are those of the authors and do not necessarily represent final Bureau policy.

Technical Bulletin No. 11 summarizes some of the preliminary background material and planning for the use of a geodetic near-earth satellite which is expected to be placed in orbit during the latter part of 1961. It covers the principal geodetic uses, the type of satellite most useful, the recommended program, and tracking methods. An appendix discusses energy requirements, accuracy of velocity, injection angles required to produce a near-earth satellite, and use of the satellite trajectory to determine geodetic data.

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Use of Near-Earth Satellite Orbits for Geodetic Information

PAUL D. THOMAS, Mathematician

U. S. Coast and Geodetic Survey

INTRODUCTION

EDWARD EVERETT HALE, better known as the author of "The Man Without a Country," wrote a fiction story (under the pseudonym Frederic Ingham) called "The Brick Moon," which was published by the *Atlantic Monthly* in installments from October 1869 to February 1870.

Hale proposed construction of an artificial moon for reasons which are surprisingly modern. One finds on page 30 of the October 1869 issue:

"If from the surface of the earth, by a gigantic peashooter, you could shoot a pea upward from Greenwich, aimed northward as well as upward; if you drove it so fast and far that when its power of ascent was exhausted, and it began to fall, it should clear the earth, and pass outside the North Pole; if you had given it sufficient power to get it half round the earth without touching, that pea would clear the earth forever. It would continue to rotate above the North Pole, . . . above the South Pole and Greenwich, forever, with the impulse with which it had first cleared our atmosphere and attraction. If only we could see that pea as it revolved in that convenient orbit, then we could measure the longitude from that, as soon as we knew how high the orbit was, . . . we must make it a very large pea . . . it must stand fire well . . . it must be brick . . . the pea shooter, of course, is only an illustration . . . we would build two gigantic flywheels . . . they should revolve, their edges nearly touching, in opposite directions, for years, . . . to accumulate power, driven by some water fall . . . one should be a little heavier than the other . . . the brick moon . . . should be gently rolled down a gigantic groove . . . till it lighted on the edge of both wheels at the same instant . . . it would be snapped upwards as a drop of water from a grindstone." It is interesting to note that the Navy Department has a current project for utilizing a satellite for navigational purposes, called "Transit." Azimuth and elevation of the radio beam from the satellite will be measured by surface ships and aircraft as to a star; line of position will be calculated by means of prepared tables of the satellite orbit.

The invention of rockets by the Chinese, in the 12th Century, and subsequent developments made possible the advent of artificial satellites. In the 1920's, the work of Hermann Oberth and his group in Germany was significant. Oberth, who is called the German father of modern rocketry, was the first to seriously consider the possibility of establishing an earth satellite, and in 1923 he proposed a manned space station to serve as an observatory. At the same time, in the United States, the experiments of Robert H. Goddard with the first liquid-fueled rockets were monumental. The German V-2 ballistic rocket, unleashed near the end of World War II, had a range of 118 miles and reached a height of 50 miles in its trajectory.

Thus, it was the development of large rockets and telemetering techniques during World War II which made feasible the practical consideration of artificial satellites, since it would be possible, if a satellite orbit could be achieved, to collect information in space without human observers and return it to earth by radio transmitter.

On February 24, 1949, at White Sands Proving Ground, N. Mex., the U. S. Army fired a two-stage rocket—combining the V-2 and the WAC Corporal—to a height of 250 miles, and that same year, Willy Ley, a former member of Oberth's group, proposed the firing into orbit of a 200-pound satellite by means of a 220,000-pound rocket. In 1953, S. Fred Singer, University of Maryland physicist, suggested an instrumented satellite which he called the "Mouse." The Office of Naval Research actually set up a satellite project in 1954 called the "Orbiter" which was to use a military rocket as the first stage. In connection with U. S. participation in the International Geophysical Year (IGY) program, Government satellite rocket research was separated in 1955 from the military missiles program and the well known satellite project "Vanguard" established.

In 1956, a three-stage rocket consisting of the Army Redstone and two smaller stages went to a height of 680 miles. With a directed booster as final stage, ***this rocket could have achieved a satellite orbit.*** Early in 1957, a test three-stage rocket fired by the Lockheed Aircraft Corp. ***reached 1,000 miles*** because of a firing malfunction. The last stages of the rocket were

TABLE OF ARTIFICIAL EARTH SATELLITES

Name	Launch date, life-time and descent date	Shape and weight	Size	Date	Orbital inclination (deg.)	Orbital period (min.)	Perigee height (n. miles)	Apogee height (n. miles)	Orbital eccentricity	Argument of perigee (deg.)
*Sputnik 1 1957α_2 instrumented sphere	1957 Oct. 4.8 ? 92 days ?1958 Jan. 4	Sphere 184 lb.	23 in. dia.	1957 Oct. 4.9 1957 Oct. 25.8 1957 Dec. 25.1	65.1 65.1 65.0	96.2 95.4 91.0	122 120 110	513 471 254	0.052 0.047 0.020	58 49 23
*Sputnik 1 1957α_1 rocket	1957 Oct. 4.8 57 days 1957 Dec. 1	Cylinder? ------	------	1957 Oct. 4.9 1957 Nov. 4.0 1957 Nov. 24.3	65.1 65.1 65.0	96.2 94.0 91.0	122 119 110	513 403 254	0.052 0.038 0.020	58 45 36
*Sputnik 2 1957β	1957 Nov. 3.1 162.0 days 1958 Apr. 14.08	Payload 1,120 lb.	?70 ft. long.	1957 Nov. 4.00 1958 Jan. 4.00 1958 Feb. 21.00 1958 Mar. 25.00 1958 Apr. 9.00	65.33 65.29 65.26 65.23 65.21	103.760 100.505 97.105 93.785 90.780	122 119 114 107 97	902 739 570 402 253	0.0987 0.0802 0.0605 0.0400 0.0215	59 35 14 359 352
Explorer 1 1958α	1958 Feb. 1.16 4 years	Cylinder 30.8 lb.	80 in. long. 6 in. dia.	1958 Feb. 1.15 1959 Jan. 3.05 1959 Mar. 21.06	33.2 33.2 33.1	114.8 111.7 111.2	191 189 189	1,379 1,228 1,203	0.141 0.125 0.122	121 139 296
Vanguard 1 1958β_2 instrumented sphere	1958 Mar. 17.5 ?100 years	Sphere 3½ lb.	6.4 in. dia.	1958 Mar. 17.5 1959 Mar. 12.5	34.3 34.3	134.18 133.94	353 353	2,140 2,129	0.191 0.190	129 274
Vanguard 1 1958β_1 rocket	1958 Mar. 17.5 ?100 years	Cylinder 50 lb.	4 ft. long. 20 in. dia.	1958 Mar. 17.5	34.3	134.18	353	2,140	0.191	129
*Explorer 3 1958γ	1958 Mar. 26.73 93 days 1958 June 28	Cylinder 31 lb.	80 in. long. 6 in. dia.	1958 Mar. 26.73 1958 May 15.0 1958 June 14.13	33.3 33.3 33.3	115.7 104.8 96.8	101 97 93	1,511 970 565	0.166 0.110 0.063	 326
Sputnik 3 1958δ_2 instrumented cone	1958 May 15.3 19 months	Cone 2,926 lb.	12.3 ft. long. 68 in. dia.	1958 May 15.3 1958 Nov. 17.00 1959 Mar. 2.9	65.19 65.16 65.13	105.97 103.000 100.750	122 120 118	1,015 867 754	0.111 0.095 0.082	58 351 310
*Sputnik 3 1958δ_1 rocket	1958 May 15.3 202.4 days 1958 Dec. 3.7	Cylinder? ------	------ ------	1958 May 15.3 1958 Aug. 15.10 1958 Oct. 11.25 1958 Nov. 13.66 1958 Nov. 30.59	65.19 65.14 65.09 65.04 65.00	105.9 102.000 98.000 94.000 90.000	122 119 115 108 95	1,011 816 615 413 217	0.111 0.089 0.066 0.041 0.017	58 26 5 347 339
Explorer 4 1958ϵ	1958 July 26.63 13 months	Cylinder 38.5 lb.	80 in. long. 6 in. dia.	1958 July 26.63 1958 Oct. 25.06 1959 Mar. 21.02	50.3 50.3 50.2	110.18 107.73 102.37	142 140 138	1,193 1,073 805	0.128 0.115 0.085	50 317 60
*Atlas 1958ζ	1958 Dec. 18.96 33.6 days 1959 Jan. 21.6	Cylinder 8,700 lb.	80 ft. long. 10 ft. dia.	1958 Dec. 18.96 1959 Jan. 1.78 1959 Jan. 17.02	32.4 32.4 32.3	101.47 98.12 92.67	104 95 88	796 627 355	0.089 0.070 0.037	130 249 37
Vanguard 2 1959α_1 instrumented sphere	1959 Feb. 17.67 ?50 years	Sphere 21.5 lb.	20 in. dia.	1959 Feb. 17.67	32.9	125.7	301	1,790	0.166	135
Vanguard 2 1959α_2 rocket	1959 Feb. 17.67 ?50 years	Cylinder 50 lb.	4 ft. long. 20 in. dia.	1959 Feb. 17.67	32.9	130.0	302	1,992	0.184	135
*Discoverer 2 1959γ	1959 Apr. 13.89 13 days 1959 Apr. 26.5?	1,300 lb.		1959 Apr. 13.9 1959 Apr. 24.6	90 90	90.4 88.9	130 103	190 132	0.008 0.004	160 96
Explorer 6 1959β	1959 Aug. 7 1 year	142 lb.	29 in. long. 26 in. dia. 4 Solar Vanes 18 X 18 in.		46.9	756.9	135	22,680		
*Discoverer 5 1959ϵ	1959 Aug. 13 47 days 1959 Sept. 28	1700 lb.	Agena rocket and reentry capsule.		90	94.0	118	391		
*Discoverer 6 1959δ	1959 Aug. 19 63 days 1959 Oct. 20	1700 lb.	Same as Discoverer 5		90		121	466		
Vanguard 3 (SLV-7) 1959η	1959 Sept. 18 30-40 years	Sphere with tube attached 100 lb.	20 in. sphere 26 in. attached tapered tube			130.0	275	2019		
Explorer 7 1959ι_1	1959 Oct. 13 20 years	2 cones joined at bases 91.5 lbs.	Cone base 30 in. dia. - Overall length 60 in.		50.3	101.2	300	584		
Explorer 7 Rocket Body 1959ι_2	1959 Oct. 13 20 years					101.2	298	585		
*Discoverer 7 1959κ	1959 Nov. 7 19 days 1959 Nov. 26	1700 lbs.	Same as Discoverer 5		90					
Discoverer 8 1959λ	1959 Nov. 20 3 months	1700 lbs.	Same as Discoverer 5		90	101.5	101	793		

NOTES: Dates are given in days and decimals of a day U.T. 1 nautical mile = 6,080 ft. = 1.853 km. Perigee and apogee heights for Sputniks are over an earth of radius 3,435 n.m. and for U.S. satellites over an earth of 3,442 n.m. Argument of perigee is the angle from northward crossing of the equator to perigee, measured along the orbit. The orbit values for the Sputniks are from British observations and theory. Those for the U.S. satellites have been compiled from a variety of sources. It is not known whether Discoverer 1, launched February 28, 1959, achieved orbit: it was, nevertheless, designated 1959θ. * Indicates satellites no longer in orbit.

FIG. 1.—World score of near-earth satellites as of December 1959.

supposed to fire the missile downward to test the heating effect of its high-speed re-entry into the atmosphere, but they fired it upward instead.

The U.S.S.R. had been quietly pursuing rocketry studies and experiments for many years. In 1925, the Soviet Government published a voluminous report by the Rocket Subsection of the Committee for the Exploration of the Stratosphere. October 4, 1957, saw the successful launching of the first artificial earth satellite by the Russians (the Sputnik). The resulting chain of "explosive" events culminated in vast organizational changes in Government and in expenditures in the race for the conquest of space. Since that memorable day, there have been many attempts and some successes in creating artificial satellites. The first American satellite, Explorer 1, was launched by the Army from Cape Canaveral, Fla., on January 31, 1958, using the Jupiter C rocket. The world score as of December 1959 is shown in figure 1. The space probe shots (Lunik 3 is of course an earth satellite but not a near-earth satellite) have not been included, nor the unsuccessful firings. We are all aware that the Soviet Coat of Arms was reportedly buried in the moon by Lunik 2 on September 13, 1959, and that the Russians claim to have photographed the back side of the moon with Lunik 3 on October 7, 1959. The artificial earth satellites still in orbit as of December 15, 1959, were Explorer 1 (1958 α); Vanguard 1 and its rocket (1958 β_1 and β_2); Sputnik 3 (1958 δ_2); Vanguard 2 and its rocket (1959 α_1 and α_2); Explorer 6 (1959 δ); Vanguard 3 (1959 η); Explorer 7 and its rocket (1959 ι_1 and ι_2); Discoverer 8 (1959 λ); and Lunik 3 (1959 θ)—a dozen artificial earth satellites of which two are Russian. (See p. 36.)

THE MOTION OF AN ARTIFICIAL SATELLITE

The basic orbit of an artificial earth satellite is an ellipse, with the center of the earth at one focus, and to reference it to the earth we have the following terms and defintions: ***orbital inclination***—the angle between the orbital plane and the earth's equator; ***orbital period***—the time for the satellite to make one complete transit of its orbit; ***perigee height***—the distance from the earth to the satellite at its nearest approach; ***apogee height***—its greatest distance from the earth (apogee and perigee are the end points of the major axis of the elliptic orbit—also called the line of apsides); ***eccentricity***—the eccentricity of the elliptic orbit; ***argument of perigee***—the angle from the northward crossing of the earth's equator to perigee, measured along the elliptic orbit. These terms are shown in figure 1.

If the earth were spherical and had no atmosphere, the motion of a satellite would be relatively simple. It would be a fixed ellipse in a fixed plane with the center of the earth as one focus, the direct consequence of Kepler's first law of planetary motion: The orbit of a planet (a satellite) around the sun (the earth) is an ellipse, the sun (the earth) being situated at a focus. But the attraction of the noncentral mass of the earth's so-called equatorial bulge has a disturbing influence on the elliptic satellite orbit, and the resistance of the air "perturbs" its motion.

The effect of the noncentral gravitational field does not materially alter the general shape of the elliptic orbit, but changes continuously its orientation in space. Air drag distorts the ellipse with little influence on its orientation in space. Fortunately, the diurnal motion of the earth (its rotation) plays no part in the progress of the satellite itself but of course is important from the standpoint of the particular area of the earth from which the satellite is visible at any given time. The disturbing effects of the attraction of the sun, moon, and other celestial bodies are considered negligible for near-earth satellites.

Figure 2 portrays a portion of a satellite orbit. The effect of the attraction of the equatorial mass of the earth is to pull the satellite toward the equator, thus decreasing the inclination. But like a gyroscope the actual effect is to cause the plane of the orbit to precess westward for an eastward

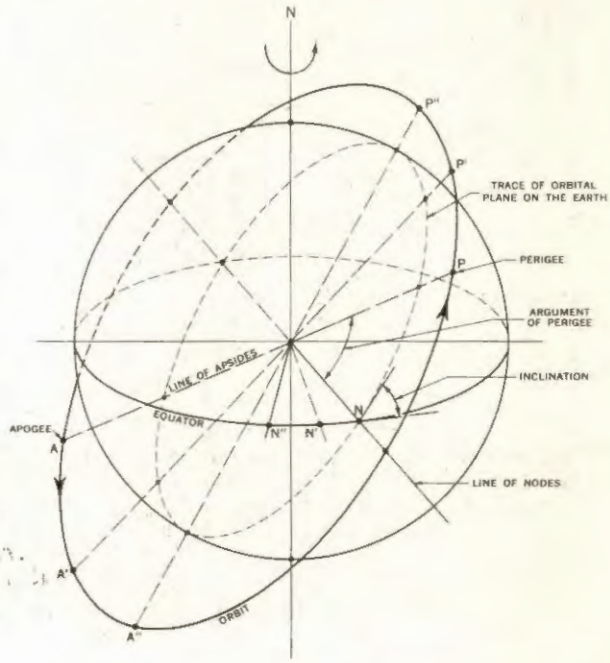

FIG. 2.—Graphic representation of a portion of the trajectory of a near-earth satellite.

launch about the focus (the center of the earth), maintaining the same inclination so that on a second revolution the line of nodes would be at N', on a third revolution at N", etc. Simultaneously, the perigee will have advanced correspondingly to the points P', P", etc. (The perigee moves in the direction of the satellite motion where the inclination is less than 63°.4, and in the opposite direction for inclination angles larger than 63°.4.) Thus, the "rigid" elliptic orbit rotates also about the focus (the center of the earth) in the plane of the orbit. It is the ability to determine the rates or periods of these nodal and perigee motions from observational data which make the artificial satellite useful for determining the shape of the earth. (See app., p. 32.)

Paradoxically, air resistance speeds up a satellite by forcing it to fall into a smaller orbit with a decreased major axis, decreased eccentricity, and corresponding decreased period. With each revolution, the apogee height decreases much more rapidly than the perigee height until the orbit is nearly circular. Thereafter perigee and apogee descend at nearly equal rates and the end of the satellite's celestial career is near as the satellite enters the lower denser atmosphere. While this is going on, the plane of the elliptic orbit is still revolving westward and the major axis of the ellipse is revolving in its plane, so viewing the satellite motion as a whole, the near-earth satellite actually traverses a continuous space trajectory around the rotating earth. It oscillates between a maximum north and south latitude corresponding to the inclination until it finally burns in the atmosphere. (See fig. 3.) Hence, if a satellite is to have long life, it is necessary that perigee be several hundred miles, probably not much less than 1,000 miles, since the early satellites showed that the effects of air drag were greater with altitude than had been indicated by the preflight empirical estimates based on exponential decay with altitude.

PRINCIPAL GEODETIC USES OF ARTIFICIAL SATELLITES

The principal geodetic uses of artificial satellites may be grouped as follows:

(a) For better geodetic positioning in areas of the earth not connected to continental networks of horizontal control, such as islands and intercontinental connections of geodetic datums.

(b) To deduce a more refined value for the flattening of the spheroidal representation of the earth (from the nodal motion of the satellite orbit).

(c) To obtain data on regional gravity anomalies from a long-term program using many satellites.

Better Geodetic Positioning

In connecting islands to continental geodetic datums or in intercontinental ties of geodetic

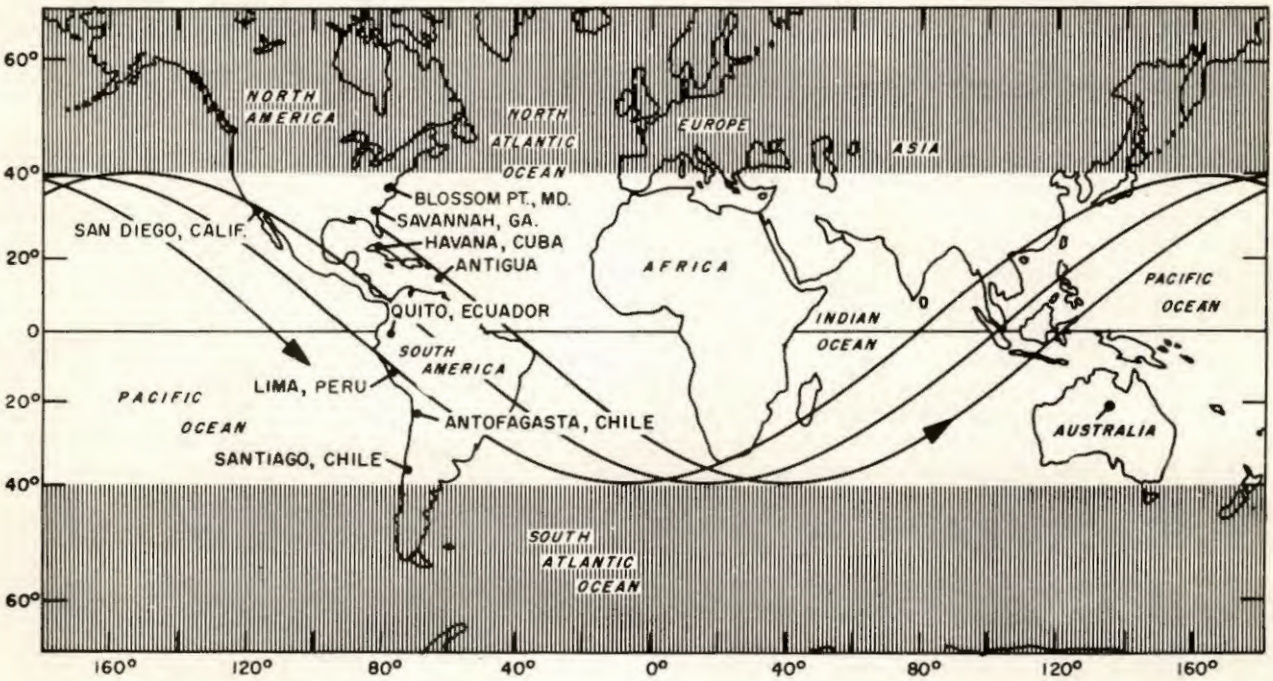

FIG. 3.—A partial trace over the earth of the trajectory of the first American artificial satellite showing its oscillatory motion in latitude and regression of the line of nodes of its orbit.

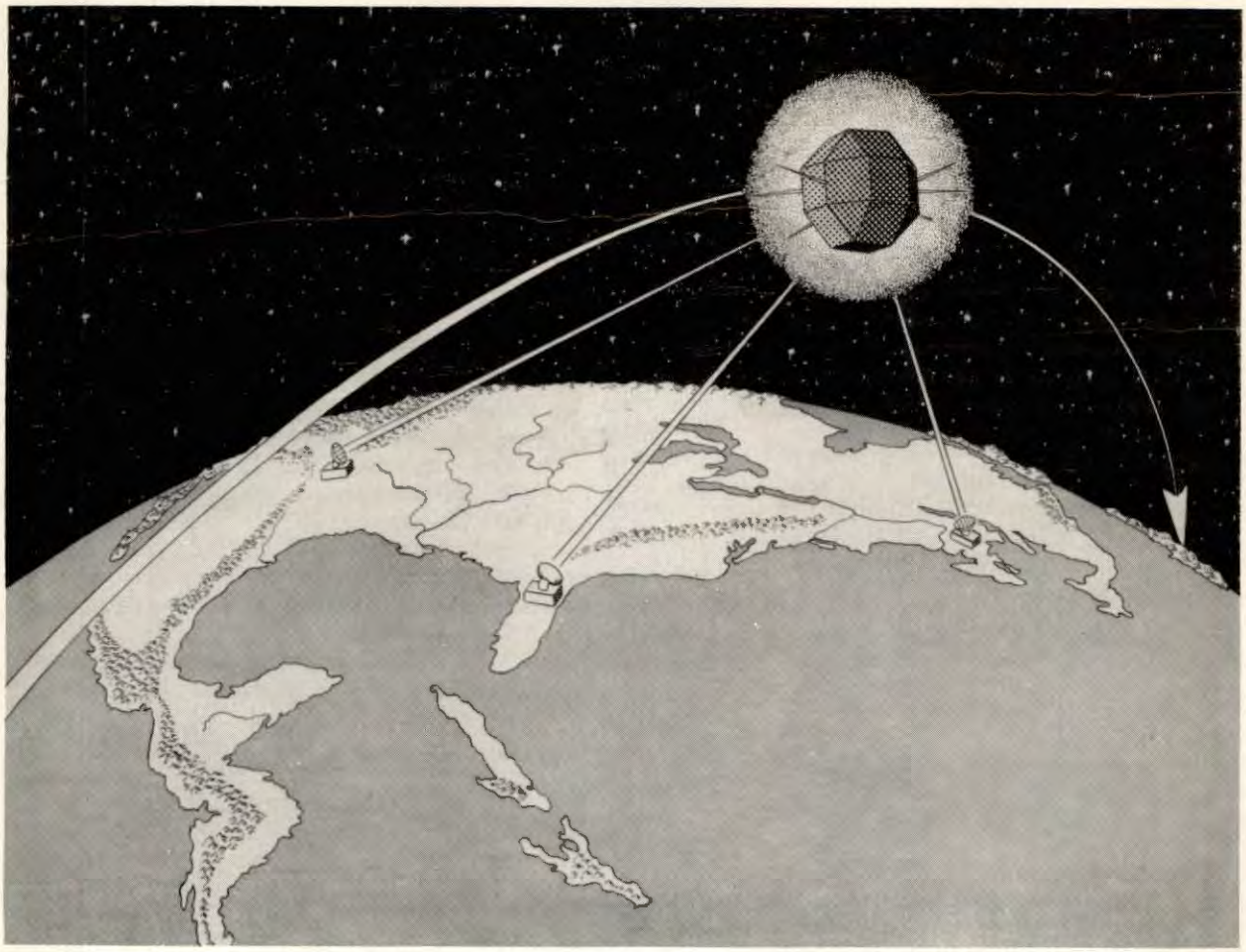

FIG. 4.--Simultaneous observations on a satellite to effect Geodetic datum ties.

datums separated by oceans, deserts, etc., there are at least two general methods using mobile or fixed optical tracking instrumentation: (1) Simultaneous continuous observations from stations in the separate nets, the satellite being used much as the flare in the known flare triangulation technique (fig. 4). This method imposes tremendous operational and coordinating efforts and is generally considered undesirable unless there is no alternative. (2) Continuous time-correlated observations of a satellite whose orbital characteristics are predictable to the accuracy required for geodetic purposes (and consistent with the inherent accuracy of the observing instrumentation), so that a usable ephemeris can be constructed of the satellite motion. The position as seen at one station can then be transferred to the equivalent time and position at another station, the calculated orbit providing a means of effective triangulation.

More Refined Value for Flattening of Earth

Newton deduced by use of his law of gravitation that the attraction of a uniform sphere, or of a sphere in which the density at any point is a function of its distance from the center of the sphere, is the same as that of a particle whose mass is that of the sphere but situated at the sphere's center. But the nonuniform nature of the earth's mass distribution results in a gravitational field which is not inversely proportional to the square of the radial distance. However, the external gravitational potential can be expressed as a series of spherical harmonics (and hence in a series of Legendre polynomials), the coefficients of each harmonic depending on the shape and mass distribution of the earth. If the coefficients can be determined, it is possible to deduce the figure of the earth. (See app., pp. 25-30.)

Several mathematical methods have been devised to evaluate the coefficients and were used with observational data from some of the past and existing satellites. In one method the second-order differential equations for the satellite motion are made integrable by including all of the zero-order term in the potential and just enough of the second-order term (there is no first-order term) to account for the constant (secular) nodal and perigee motions. The resulting equations can be integrated and the results are expressions for the nodal and perigee motions as series in the inclination of the orbit and the ellipticity or flattening of the earth. Various values have been published for the flattening as obtained from satellite data ranging from 1/297.9 to 1/298.32 although some of these admittedly were obtained from data in which the effects of air drag and asymmetry in the earth's gravitational field had not or could not have been accounted for. It is significant that most results tend to 1/298, as compared to 1/297 for the International Ellipsoid.

Regional Gravity Anomalies

If perigee of a near-earth satellite is high enough so that the perturbations due to air drag are negligible, and effects of the attraction of the sun and moon are negligible, then the chief source of irregularity in its motion will be the asymmetry of the earth's gravitational field. At the height of the satellite there is a certain averaging effect on the surface gravity anomalies and the motion of the satellite itself is so rapid that it tends to average the effects of many anomalies, so that the important quantity is the average value of the anomalies over rather large areas. Now, motion of the nodes in an artificial satellite depends on both the second harmonic (flattening) and on the fourth harmonic of the external gravitational potential and the two cannot be separated for any one satellite, but for two satellites with different inclinations and distances from the earth they can be separated. (See app., p. 32.) Hence, for adequate determination of the average value of the gravity anomalies over large areas, several satellites at different perigee heights and inclinations would be desirable and with all perigee heights great enough for air drag effects to be negligible, or accountable.

TYPE OF SATELLITE MOST USEFUL FOR GEODETIC PURPOSES

The geodetic satellite experiment differs basically from other types of satellite experiments. Most of the scheduled experiments use the satellite as a convenient vehicle for carrying specific instrumentation into otherwise inaccessible environments. Orbit information is necessary only to the extent that the approximate location of the flight path can be established. But the geodetic satellite experiment, as has been demonstrated, depends basically upon the laws governing the motion of the satellite in its orbit. That is, time correlated position data must be accurate enough to deduce mathematically the parameters of the motion, the effects of air drag and earth's asymmetric gravitational field so that reliable ephemerides of the motion can be constructed for further geodetic (or navigational) use. Hence, a satellite for geodetic purposes (chiefly for connecting widely separated geodetic datums) or navigational purposes should preferably have the following characteristics:

(a) Be spherical in shape. Although aerodynamically a sphere has high drag (one of the reasons the cannonball was abandoned long ago), it has always the same presentation area in the direction of motion. (The drag is a function of the area presented to the resisting medium in the direction of the motion. A tumbling cylinder, for instance, would have a constantly changing presentation area.)

(b) Have a perigee of not much less than 1,000 miles and a high density to surface area ratio so that air drag can either be ignored or minimized.

(c) Have an orbit of maximum inclination and minimum ellipticity. Maximum inclination would make possible its observation from higher latitudes and minimum ellipticity would assure an almost circular orbit which would tend to be more uniform and the motion more amenable to construction of an ephemeris. (For deducing the flattening from the perigee and nodal motions, an orbit of large eccentricity and minimum air drag is more desirable, since the line of nodes and the line of apsides are then better determined from the orbital position data.)

(d) Have the mass of the instrumentation symmetrically distributed spherically within the spherical shell, if possible, to eliminate any libration effect.

(e) Have a flashing beacon of sufficient brightness—brilliance of a fifth-order magnitude star at an altitude of 800 to 1,000 miles—to be registered photographically by a tracking camera, the flash source to be controlled by a command receiver. This would allow the satellite to be tracked when it was on the dark side of the earth.

(f) Have a radio beacon and a transponder (300 mc. or higher) for the determination of radio fixes. (Possibly Hiran-type distance measurements could be made.)

(g) Have an accurate clock to broadcast time signals perhaps coordinated with the flash mechanism. This would eliminate the maintenance of accurate clocks by the mobile tracking stations and aid in simultaneous observations from several stations.

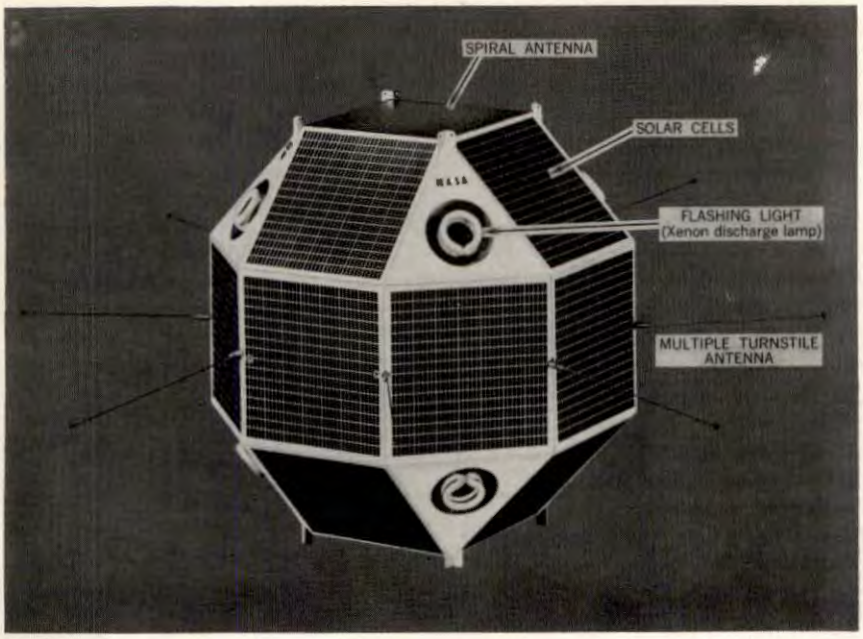

FIG. 5.—The geodetic satellite under development by NASA.

Consideration is being given to these features and others as they develop by the National Aeronautical and Space Administration (NASA), which has the responsibility for experimental satellites and plans to launch a geodetic satellite in 1961. Instrumentation availability, payload capabilities, mass-to-area ratio (this depends on latest atmospheric density information), requirements on surface area for control of internal temperature, or other technical considerations may compromise some of the desirable features, but the result should be the best that the state of the art permits. Figure 5 shows the geodetic satellite configuration as being developed by NASA. Present plans call for the Thor-Delta rocket configuration to put this satellite in orbit.

RECOMMENDED GEODETIC SATELLITE PROGRAM

The following is, in essence, the general plan presented by the Coast and Geodetic Survey, at the request of NASA, for observing the geodetic satellite on a worldwide basis.

Requirements

The geodetic satellite to be launched by NASA should provide a means for obtaining accurate geocentric coordinates of a network of points on the earth. Time-position observations on the satellite from more or less uniformly spaced points over the entire surface of the earth should give data for a precise determination of the size and shape of the earth. These data can also be employed in connecting existing geodetic datums into a unified system. Complete uniformity in coverage is impracticable, but should be approximated. It will be desirable to have carefully selected and more closely spaced points on large existing geodetic datums in order that these may be accurately positioned and oriented one to the other either by simultaneous observations on the satellite, where possible, or by use of the orbital theory, if this can be highly developed. For uniform spacing in all directions, roughly 100 points would provide for 1,500-mile spacing, 200 points for 1,000-mile spacing, and 400 points for 750-mile spacing.

The proposed program is international in scope and would require the joint efforts of many countries. The objectives are purely scientific, and not military, and the results should be freely available everywhere. Since existing knowledge of the size and shape of the earth and of the relationships between major geodetic datums is now, in general, adequate for military needs, it is emphasized that local military objectives, if any, should be in addition to and not part of the proposal herein outlined.

Considerations

Most of the details of the geodetic satellite, such as those related to acquisition both optically and electronically, the type of orbit, etc., have already been worked out by NASA.

The optical observing system recommended at this time is the BC-4 ballistic camera system or an equivalent. This system uses glass plates, not films.

The Baker-Nunn optical system and the Minitrack and Microlock electronic systems will be

operated independently from this program, under the direction of NASA. These systems will determine an approximate orbit needed in advance for the orientation of the ballistic cameras.

Operation of Program

To insure a better chance of success in the short time allotted for planning, cooperation will be necessary among geodesists and photogrammetrists, both national and international. Plans recommended by the Coast and Geodetic Survey to the National Aeronautics and Space Administration stressed the need of recognized competence in geodesy and photogrammetry, progressive activity in international organizations, and capacity to undertake the responsibility of directing and coordinating the observing program, the photogrammetric reduction of the observations, and the mathematical analysis of the results.

A major item for decision is the desired number of observing units and points from which observations are to be taken. As a start, a maximum of 100 observing units are suggested, each to occupy at least two points. If the satellite functions successfully over the period contemplated (400 days), many observing units could easily occupy more than two points. Central control is essential in directing the movement of the observing units in the field to insure maximum coverage compatible with adequate geometrical strength.

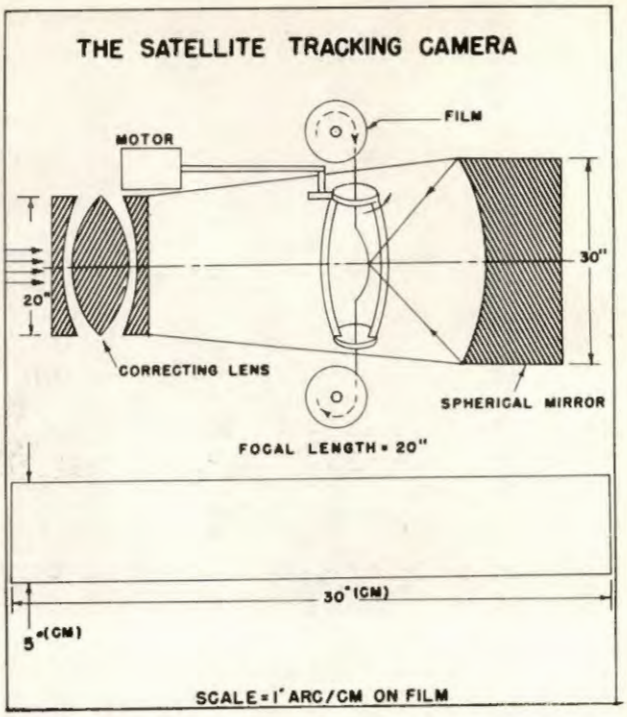

FIG. 8.—Schematic of lens system in the Baker-Nunn tracking camera. (Drawing not to scale.)

The United States military groups may wish to locate certain additional points for special purposes. Their program could be carried on independently and still have the benefit of the accurate worldwide network.

Figure 6 shows the known triangulation nets on the earth and a suggested net of observation points to effect a tie of the various geodetic datums.

FIG. 7.—The Baker-Nunn satellite tracking camera and installation.

SATELLITE TRACKING METHODS

For geodetic purposes, continuous optical and electronic tracking of the satellite is desirable. Positioning should be the order of 50-100 feet. As shown in figure 6, there should be enough mobile and permanent stations throughout the world so that complete connections of the various continental and other geodetic datums can be made.

Optical Techniques

Fixed Cameras.—The system used for the IGY optical tracking program (now under the cognizance of NASA) is the net of 12 Baker-Nunn satellite tracking cameras (5 additional units are being installed to obtain coverage in higher latitudes) and their associated Model III Norrman Crystal Clocks. These clocks provide absolute accuracy at all tracking stations to about one-thousandth of a second. (See fig. 7.)

WORLD
TRIANGULATION
CHART
PRINCIPAL ARCS OF TRIANGULATION
LINES OF HIRAN TRILATERATION
PROPOSED GEODETIC SATELLITE OBSERVATION SITES
ARCTIC OCEAN
ARCTIC
GREENLAND
GREENLAND SEA
BARENTS SEA
LAPTEV SEA
EAST SIBERIAN SEA
BAFFIN BAY
NORWEGIAN SEA
SIBERIA
UNION OF SOVIET SOCIALIST REPUBLICS
RSFSR
MONGOLIA
GOBI DESERT
CHINA
SEA OF JAPAN
NORTH
ATLANTIC
OCEAN
ALGERIA
LIBYA
SAUDI ARABIA
ARABIAN SEA
BAY OF BENGAL
PACIFIC
PHILIPPINES
CAROLINE ISLANDS
INDIAN
OCEAN
CORAL SEA
AUSTRALIA
TASMAN SEA
BRAZIL
SOUTH
ATLANTIC
OCEAN
DRAKE PASSAGE
SCOTIA SEA
FRENCH WEST AFRICA
GULF OF GUINEA
MOZAMBIQUE CHANNEL

The tracking camera is a modified F-1 Schmidt system with a 50-cm. apochromatic triple-element correcting system and an 80-cm. spherical mirror. This camera, when stationary, can photograph easily a sixth magnitude satellite moving at a rate of 1 degree per second. (Against a star background, the satellite position does not depend on the direction of the local vertical at the camera station.)

The use of strip film (55 mm.) and a third axis of rotation makes possible short exposure times during bright twilight conditions (fig. 8). Fifty cm. spheres can be photographed to more than 2,500 km. and 6-meter spheres to the moon's distance. This equipment provides an accuracy of about 5 seconds of arc along the motion of the satellite and at least 2 seconds in a direction transverse to the path. Since absolute times the world over are known to no greater precision than 0.001 second (8 meters in satellite motion), and a precision in tracking of 2 seconds of arc corresponds to a distance on earth of the order of 3 meters at a distance of 300 km., accuracy of a direction of the satellite from a tracking station is of the order of 10 to 15 meters. Figure 9 shows the location of the original 12 Baker-Nunn camera installations.

Mobile Cameras.—These are usually in the form of phototheodolites or kinetheodolites of which there are several types and makes. Probably most exploited at American test ranges are the Askania (originally from the German Askania plant). (See fig. 10.) This instrument is controlled by two operators, one tracking in azimuth and the other in elevation. An internal flash-camera periodically photographs the scales, thus recording azimuth and elevation angles; at the same time the satellite is photographed through a long-focus lens mounted on the kinetheodolite. The purpose of the latter operation is to allow for errors in tracking; the displacement of the satellite's image on the photograph enables a cor-

FIG. 9.--The original 12 Baker-Nunn camera sites.

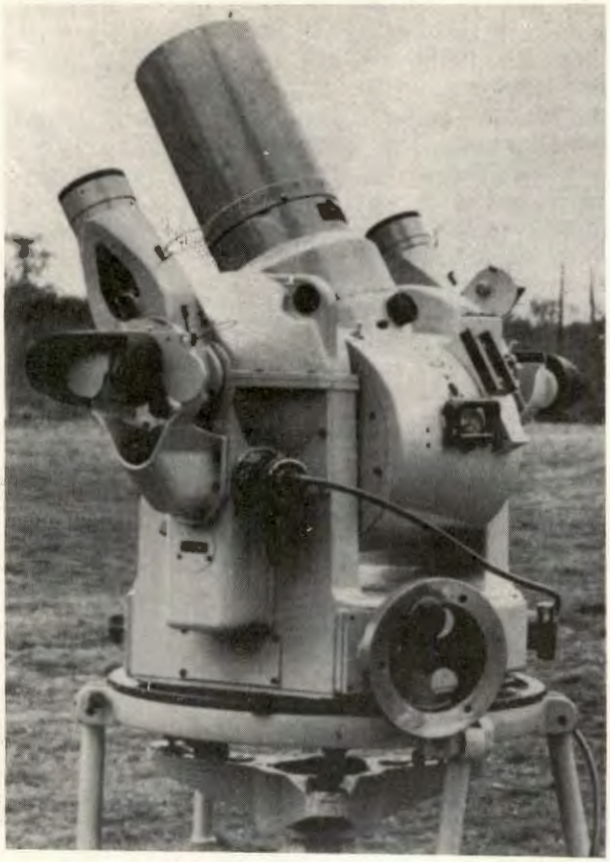

FIG. 10.—The Askania kinetheodolite.

rection to be made to the recorded azimuth and elevation angles. The exposures are controlled by accurate timing equipment at a rate of about 5 per second, so that during the course of a single transit of the satellite some hundreds of readings may be recorded. Under good operating conditions, an accuracy of 20 seconds of arc can be achieved with a timing error of about 20 milliseconds. Of course an accurate clock is also needed at such mobile stations unless the geodetic satellite is equipped with one. Kinetheodolites are not considered accurate enough for geodetic work but are useful for supplementary observations.

A type of mobile camera, usually used for missile tracking work, is called the ballistic camera. Several types exist as made by Zeiss, Wild, and other companies. Figure 11 shows the BC-4 ballistic camera as made by Wild. It is a phototheodolite featuring the movements and the graduated horizontal and elevation circles of the mount of the T-4 theodolite. Photographic plates 18 cm. square are used. Although several lens systems may be used, the Astrotar lens, of 304 mm. focal length and 117 mm. aperture appears to be especially suited for observing self-luminous satellites. The cameras are equipped with rotary between-the-lens disk shutters, supplemented by a capping shutter. The rotary disk shutters provide for chopping of continuous self-illuminated trajectory trails at specified intervals, controlling the length of individual exposures and accurately synchronizing widely separated cameras. The capping shutter provides an extended range of sequence selection and is used to record time-coded star trails for either lens calibration or camera orientation problems.

These cameras are used in pairs (or in larger numbers), actually treating the tracking problem as one in photogrammetric aërial triangulation, each camera obtaining a time correlated photograph (or sequence of photographs) of the missile or satellite against a star background. The position of the satellite is then determined from the known positions (with respect to time) of the stars by a least squares adjustment procedure, as is done with the Baker-Nunn fixed cameras. In order to do this, the distances on the plate from reference marks on it must be measured to the trace of the satellite and to the stars. Several types of machines exist for making such "plate measurements." Figure 12 shows the Mann machine (also

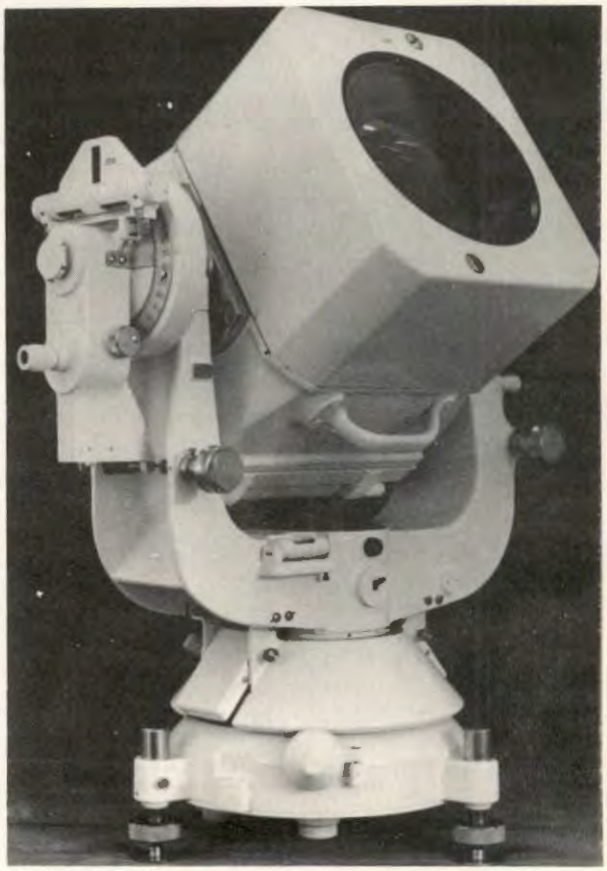

FIG. 11.—The Wild BC-4 ballistic camera.

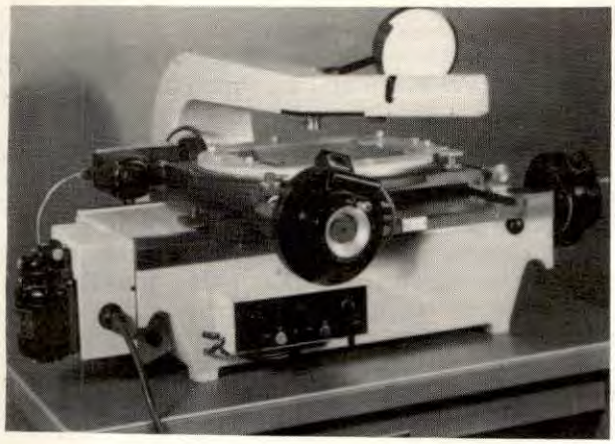

FIG. 12.--The Mann Comparator.

called a comparator), as owned by the Coast and Geodetic Survey. This machine has digitized counters and readout to an IBM punch-card machine. These cards may then be fed into a high speed computer which has had data from the star catalogs programed into it to aid in identifying the stars in the background of the satellite on the plate. Finally, through computations from these data an azimuth and elevation may be derived from each of the cameras to the satellite at a particular time and from the base lines between the cameras, the position at that time of the satellite can be computed by triangulation.

A mobile air-conditioned observing station (fig. 13) employing the Wild BC-4 camera is being built by the Instrument Corp. of Florida, which may also house plate development equipment; a plate measuring engine; and timing, recording, and communications equipment. The unit may be airlifted or transported by other means.

It is believed that mobile equipments, employing cameras of the Wild BC-4 type and properly operated with subsequent measuring and reduction procedures sufficiently controlled with respect to accuracy, will provide position data useful for geodetic purposes.

Investigation may show that use of longer focal lengths will make less critical the precision of the plate measurements making possible such measurements in the field and relieving to some extent the inevitable bottleneck in the data reduction procedures.

All optical equipment is sensitive to meteorological conditions, but for a long-lived geodetic satellite this should be a transient handicap. Optical observations have played the major role in establishing accurate orbits for those satellites which have been visible, although the data reduction from the photographic records has become a

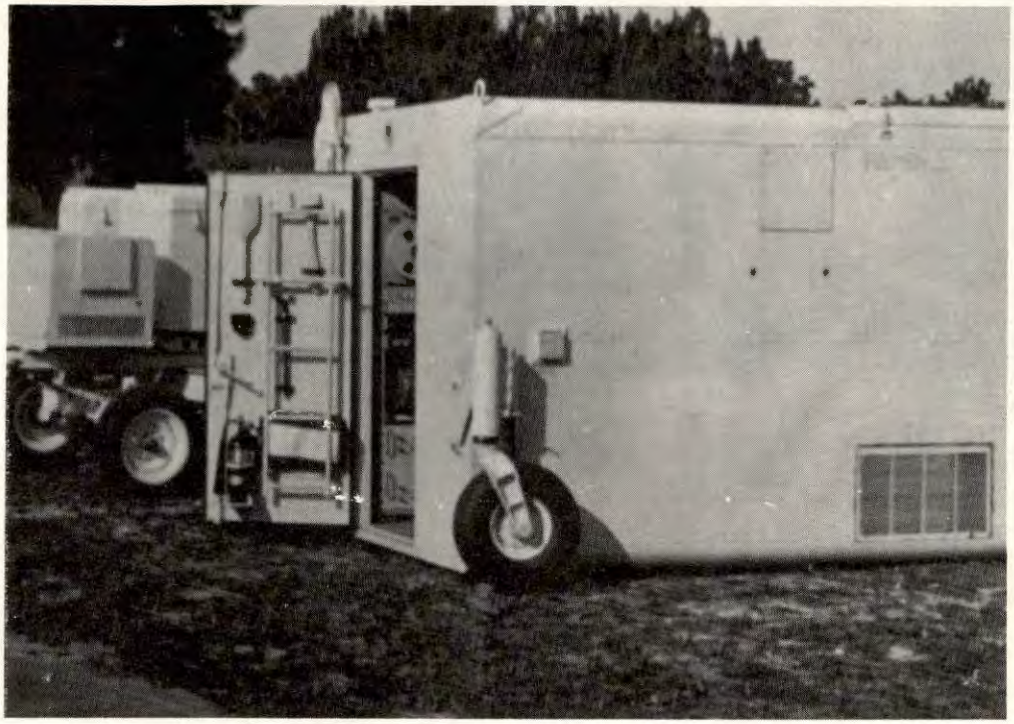

FIG. 13.--Mobile observing units as built by the Florida Instrument Co.

monumental bottleneck in spite of efforts to mechanize the process.

Electronic Techniques

During the early life of a space vehicle when little is known about its path, its transmitter advertises its position over a wide area, so that it can be located without previous knowledge of the approximate orbit. Orbital information can be obtained by means of the Doppler effect, interferometer techniques, and radar. Refraction of radio waves in the ionosphere can lead to considerable errors, particularly at low frequencies; hence, high frequencies are more desirable for tracking purposes (108 mc. or higher, as used in American satellites).

Doppler Effect.—If the satellite transmitter sends out a continuous, unmodulated wave at a fixed frequency, the signal received on the ground exhibits a change in frequency, due to the relative velocity of satellite and observing station. The receiver frequency is a function of the transmitted frequency, the velocity of radio propagation (the velocity of light), and the rate of change of the distance between satellite and observer (the radial velocity). If, then, the frequency is recorded as the satellite approaches and recedes, the radial velocity can be computed. And, since at the instant of closest approach the radial velocity will be zero (the relative velocity is then normal to the sight-line) it is possible to deduce the minimum distance and the relative velocity. (See app., p. 33.)

The orbital period can be found by observing successive transits at the same station. However, the period is not simply the difference between the times of closest approach; a correction must be made to allow for the fact that, in the intervening period, the rotation of the earth has altered the observer's position relative to the satellite orbit.

The accuracy with which the orbit can be determined from Doppler data alone depends, among other things, on the separation between the satellite and the observing stations, and between the stations themselves. For favorable triangulation these distances should be of the same order which requires a large number of operating ground stations. Over the distances involved, the curvatures of the orbit and of the earth are too large to be ignored, which complicates the analysis.

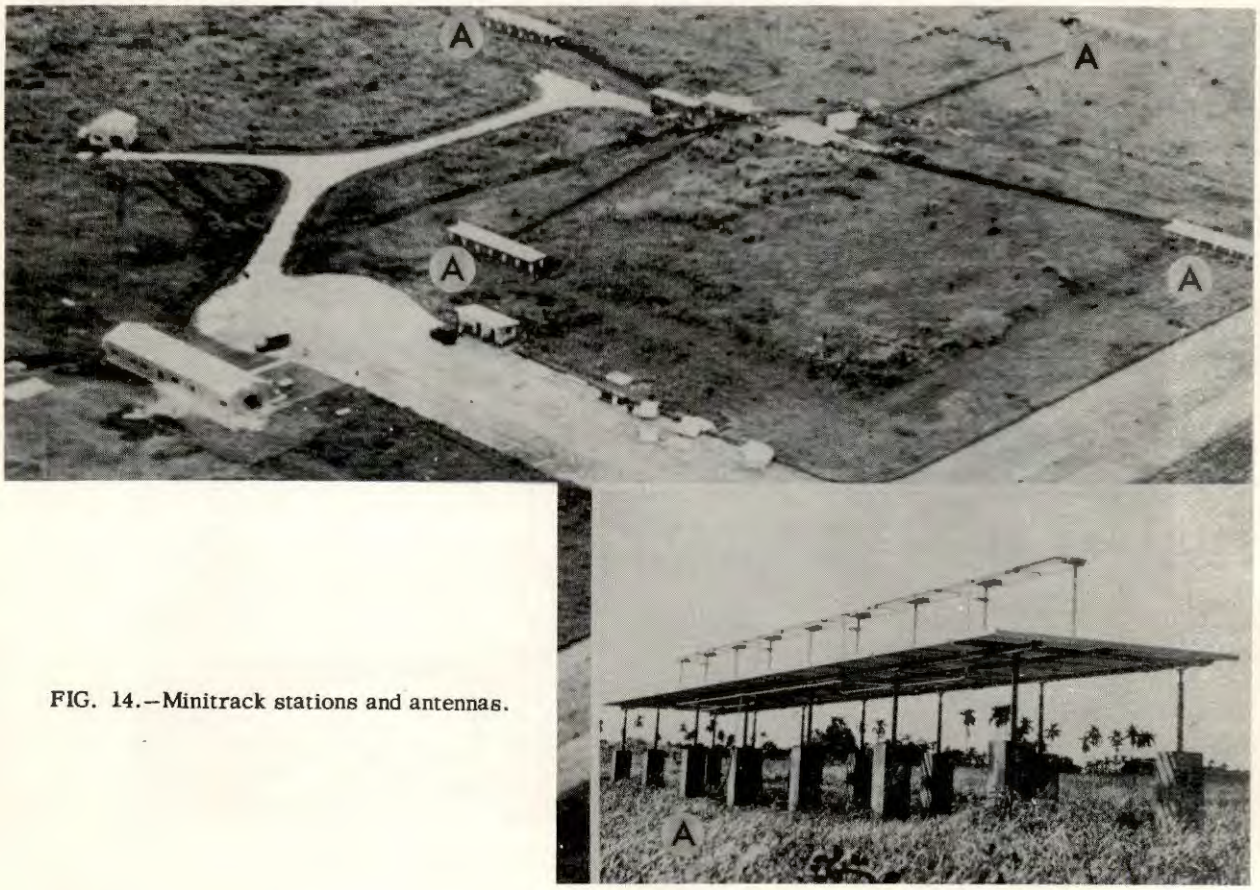

FIG. 14.—Minitrack stations and antennas.

Interferometer Principle.—Unlike the Doppler system, which measures radial velocity, the interferometer provides direction information so that the two methods are complementary. As its name implies, the interferometer makes use of the interference pattern between the signals received at a pair of antennas, a pattern determined by the difference in the distance between the satellite and each of the receiving antennas. The separation between the antennas, usually 50 to 100 meters, is small compared with the satellite's distance, so that the waves from the transmitter may be considered to traverse parallel paths. The measured phase difference of the two antennas gives a continuous indication of the cosine of the direction of the satellite. Since the interferometer is insensitive to changes of a whole number of wave-lengths in the path difference, there are ambiguities in the measurement of the angle to the satellite which become more numerous as the antenna separation is increased. They can only be resolved from an approximate knowledge of the satellite's track obtained from another source, such as a second interferometer with a pair of antennas set closer together to give a "course" reading, or set at right angles to the first pair. In the latter case, it is possible to define then the direction of the line joining satellite and station during the satellite's transit. But to deduce the orbit, distance and relative velocity as supplied by Doppler data are needed, or the directions from other interferometers at different places. If neither is available, then some assumptions about the orbit must be made.

The Minitrack and Microlock Interferometer systems are standard electronic trackers for American satellites. (See figs. 14 and 15.) Although corrections for air drag and ionospheric refraction were included in orbit computations from the Minitrack system, average residuals

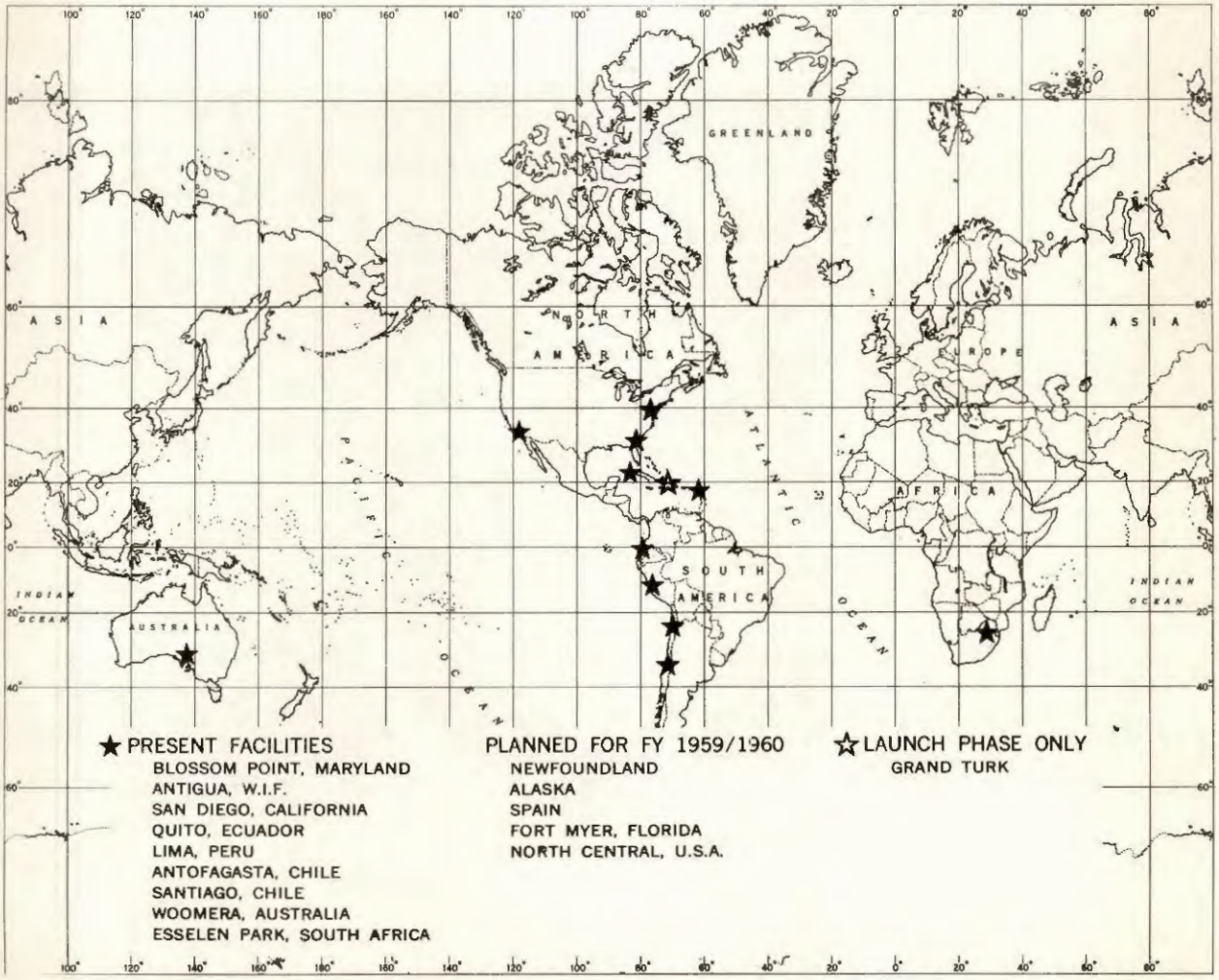

FIG. 15.—The Minitrack network.

were about 4 minutes of arc at each station, which is not considered good enough for geodetic accuracy.

Unfortunately, if the transmitter fails to operate in a satellite, so do the Doppler and interferometer stations on the earth. Hence, if possible, other types of tracking should always be used in addition to these.

Radar Tracking.—When the satellite is used as a reflector with a large radio-telescope, the advantages are freedom from use of the satellite-borne equipment and of ambient meteorological conditions, at the ground radar installation, which is not true of Doppler, interferometer, or optical techniques. Disadvantages are the small reflecting area of the satellite, with consequent narrow beam power concentration from a large antenna system with a small field of view. Unless the orbit is known fairly well, it is no easy matter to direct the radar beam to the right part of the sky at the right time. Large radio-telescopes are obviously not mobile.

Tracking radars to give azimuth and elevation angles to a satellite and distance are located in Bermuda, Hawaii, Texas, Woomera in Australia, and South Africa. These have moderately sized dishes (antennas) and are known as the FPS-16. Tracking radars are also located at the Pacific Missile Range on the west coast; White Sands Proving Ground, N. Mex.; and at Eglin Air Force Base and Cape Canaveral, Fla. The Naval Research Laboratory is operating tracking radars having 50-, 60-, and 84-foot diameter antennas. The National Science Foundation is sponsor for construction of 85- and 140-foot diameter instruments at Greenbank, W. Va.

An 85-foot radio-telescope is in use at the University of Michigan, under sponsorship of the Office of Naval Research, and a similar one has been installed at Goldstone, Calif. Present plans call for installation of 85-foot types at Woomera in Australia, South Africa, and in Spain. The largest operating radio telescope has a 250-foot antenna and is located at Jodrell Bank, England.

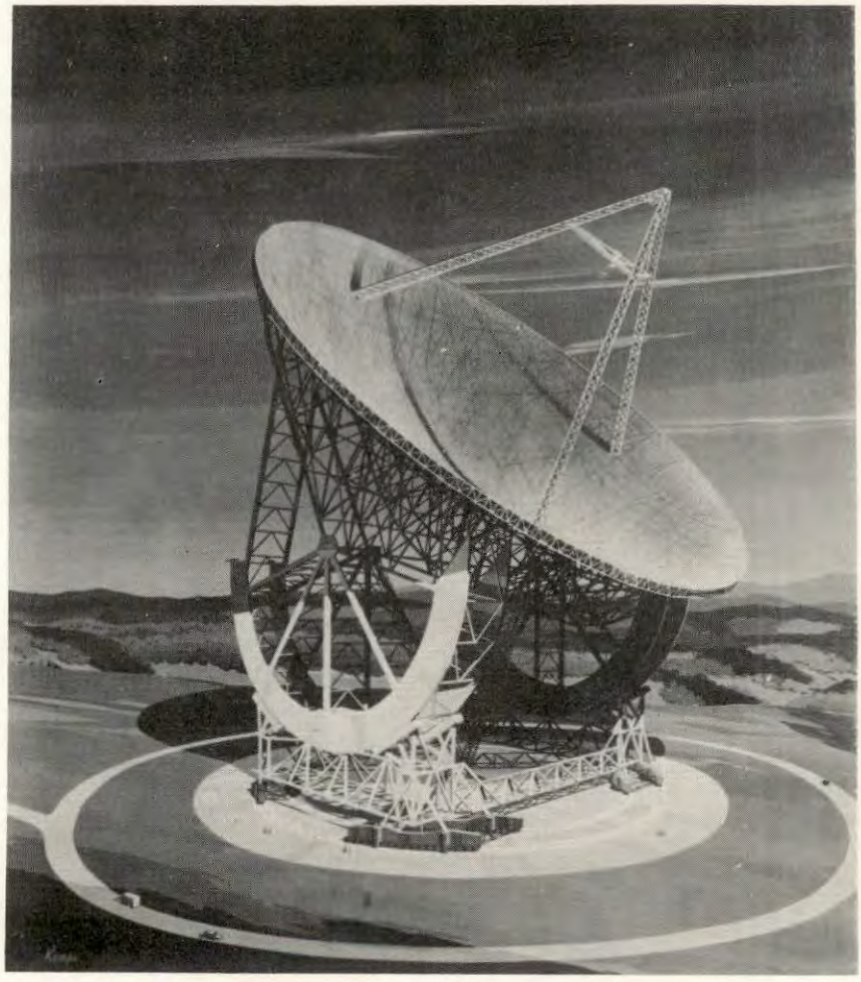

FIG. 16.—Steerable radio telescope under construction at Sugar Grove, W. Va.

The largest radio telescope in the world is under construction at Sugar Grove, W. Va., for the Office of Naval Research. The steerable parabolic antenna will be 600 feet in diameter and capable of altitude rotation from the horizon to the zenith. The entire structure will be able to rotate up to 450 degrees in a horizontal plane. Figure 16 shows this radio telescope as it will appear when completed.

Other types of satellite tracking are being investigated, for instance, infrared techniques where television techniques are being applied to improve sensitivity, selectivity, and rapid readout characteristics of the tracker (to provide selectivity to "chop off" the sky background and permit tracking in daylight as well as tracking of fainter objects at night). An investigation is being made of the shoran or hiran techniques for possible application to near-earth satellites.

SUMMARY CONCLUSION

The orbit of an artificial near-earth satellite has been discussed and the dependence of geodetic parameters, such as the flattening, on the nodal and perigee motions, which are deduced mathematically from time-correlated satellite position data; hence, the more exacting requirements for orbital data from a purely geodetic satellite. The requirements for a geodetic satellite have been discussed and tracking methods enumerated to obtain position data. While optical instrumentation still provides the most accurate and useful geodetic orbit information, improvements in electronic techniques may approach the optical results before the launching of the first geodetic satellite by NASA in 1961. The observational program should be on a coordinated international basis to make maximum use of the geodetic satellite. This will entail the provision of large numbers of mobile tracking stations.

APPENDIX

An artificial earth satellite is maintained in a circular (elliptic) orbit by precisely balancing the earth's gravitational attraction by centrifugal force produced by the satellite's tangential velocity. The kinetic energy behind this force is provided by the rocket propulsion system. The rocket vehicle must get through the denser air layer (about 100,000 feet) before accelerating to satellite speed to avoid excessive kinetic heating and the enormous drag forces.

The chief structural problem is the design of the integral propellant tanks, which are thin-walled cylindrical shells designed to withstand compressive end loads (from the rocket motor, missile inertia, and drag), shear forces and bending moments (from the guidance system), and internal pressure. The thin walls of these tanks may be stabilized by internal pressurization or by mechanical means or both. Obviously the materials used must have a high specific strength and stiffness and the ability to maintain these properties over a wide range of temperatures.

Some improvements in rocket engine performance over that of liquid oxygen-kerosene systems may be obtained by using various combinations of the oxidants (liquid oxygen, ozone, and fluorene) and the fuels liquid hydrogen, ammonia, hydrazine, and the boron hydrides, to increase the specific impulse. (The specific impulse is the thrust force delivered per unit weight of propellant consumed per second.) It is questionable whether the gain obtained in this way is worth the engineering effort required to make use of these "exotic" fuels almost all of which are either toxic, corrosive, or liable to detonation. Solid propellants, in light of recent developments, give specific impulses near those of liquid propellants, and simplify the engineering problem, but the associated structure weights are appreciably higher. Nuclear rocket propulsion, if feasible, will enable space travel beyond the capabilities of conventional rockets.

Table 1.—Specific impulses of several fuels (Sea level)

Fuel	Chamber Pressure	
	300 p.s.i.	1,000 p.s.i.
	seconds	*seconds*
Solid (best projected)	235	265
N_2O_4 - N_2H_4	250	280
H_2O_2 - N_2H_4	255	285
O_2 - Kerosene	260	300
F_2 - NH_2	312	355
O_2 - H_2	350	390
F_2 - H_2	365	410
O_3 - H_2	375	425

Note: The efflux velocity of the rocket nozzles is $V_E = g_0 I$, where g_0=32.2 ft./sec.^{2}, hence for V_E to be in ft./sec., the impulse I, which is actually a force, must be in seconds.

Guidance of a satellite vehicle consists of the determination of the position and velocity during its powered flight, and of computing the corrections to be made to the flight path to achieve the required satellite injection conditions. There are in general two kinds of guidance; radar measurement of position and velocity; measurement of acceleration by means of an inertial navigation system in the rocket.

The center of an inertial guidance system is the space-stabilized inertial platform (gyroscopic) carrying three orthogonally mounted accelerometers. A launch point reference is placed in the rocket by aligning this stable platform to the local vertical and local north. After inserting the data for satellite injection, the rocket can be directed to fly between launch site and point of injection. The accelerometers sense deviations from the precomputed flight path and through an associated system of integrators, commands are issued to the control system as steering commands, propulsion orders, firing signals, etc. Radio-inertial combination may be used. Command signals from the earth may be used to cause changes in the attitude of the rocket and consequent change in the trajectory.

With radar guidance systems there is the problem of noise in the system (due to variations in motor thrust and in the measuring system itself, and to ionization in the rocket flame).

The main problem with inertial guidance systems is in the design of gyroscopes and accelerometers of sufficient accuracy and stability.

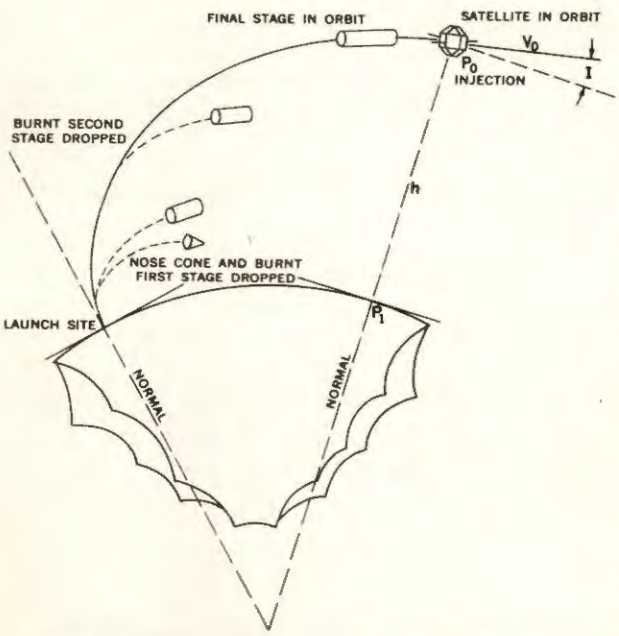

FIG. 17.—The satellite vehicle trajectory from launch to injection.

The wander of the gyroscopes, which usually have a spin rate of 12,000 to 24,000 r.p.m., can be minimized by using flotation gyroscopes.

The error characteristics of the radar and inertial guidance systems are to some extent complementary. The radar system has excellent long term stability but suffers from noise at higher frequencies, while the inertial system has little high frequency noise but suffers from long-term drift.

The main function of the control system in rocket fired satellites is to apply the corrections given by the guidance system to the flight path. The couples necessary to change the attitude of the rocket are usually produced by deflecting the efflux jets from the main rocket motors or by movable vanes in the jets. The control problem is complicated by any flexibility of the structure and by fuel sloshing. Small retrorockets or rockets placed on the satellite itself at proper angles (and fired from the earth by radio control) may be used to alter its orbit, raise the apogee, change its tangential velocity, etc.

SATELLITE INJECTION ACCURACY

The flight pattern of a multistage satellite vehicle, as depicted in figure 17, consists of a short stage of powered flight during which the missile climbs almost vertically out of the earth's atmosphere. At the end of this, the first stage and nose cone are dropped, the second stage fires and is dropped at its burnout, the process continuing to the final stage which may eject the satellite into orbit and then assume an orbit of its own.

The accuracy of placing a satellite in a desired orbit depends on its position and velocity at burnout of the final stage—in figure 17, the values of h, I, and V_o which are respectively the height above the earth at burnout, the angle which the trajectory makes with the horizontal at altitude h, and the speed at burnout. For instance, if h is 300 miles and a circular orbit is desired, V_o must be just over 17,000 m.p.h., and the angle I must be zero. An error in speed of ± 35 m.p.h. will cause a deviation in the orbit of 30 miles. An error of 1.5 degrees in I, when h is 300 miles, will cause an orbital deviation of 100 miles. With an h of 300 miles, a speed of 17,000 m.p.h. means success while 16,800 m.p.h. means failure! The allowable errors to achieve an orbit are 2 to 3 degrees in I, the injection angle, and 100 to 500 m.p.h. in the speed V_o. For specific elliptic orbits the error in I must be no more than 0.5 to 2 degrees and in Vo no more than 25 to 50 miles per hour. For precision orbits, as a 24-hour circular orbit, the error in I cannot be more than 0.1 to 0.25 degree and in V_o not more than 1 to 2 m.p.h.!

In summary, to launch a satellite vehicle so that the orbit will lie between altitudes of 200 and 1,500 miles, an injection speed, V_o, of about 17,400 m.p.h. is required and the injection angle, I, must be at least within 3 degrees of horizontal. If launching speed is 340 m.p.h. slower, the tolerance on the injection angle is approximately halved—it would have to be within 1.5 degrees of horizontal! ***At present any orbit within rather wide limits is a success!***

MINIMUM ENERGY REQUIRED TO LIBERATE A UNIT MASS FROM THE EARTH

The gross takeoff weight of Vanguard 1, launched March 17, 1958 (and still in orbit), was 22,600 pounds. The satellite itself is 6.4 inches in diameter and weighs 3.25 pounds. (Also placed in orbit was the 50-pound third stage rocket casing.) So it required 22,600 pounds of takeoff weight (most of this was fuel), to place 53.25 pounds in orbit with perigee 353 n.m., and apogee 2,140 n.m. Why does it require so much energy to place a satellite in orbit?

Newton's fundamental law of gravitation may be written

$$F = \frac{G m M}{r^2} \tag{1}$$

where

m, M are the masses of two bodies in space

r is the distance between them

G is a constant.

If M and R are respectively the mass and mean radius of the earth (the earth is considered a radially homogeneous sphere) then at sea level the force, F_o, on a body of mass m is from (1)

$$F_0 = G m M / R^2 \tag{2}$$

But from Newton's laws of motion, a force exerted by the earth on a mass m at earth's sea level is

$$F_o = m g_o \tag{3}$$

where g_o is the acceleration of gravitation at sea level. Hence from (2) and (3)

$$G = R^2 g_o / M \tag{4}$$

and (1) may be written with the value of G from (4) as

$$F = g_o m \left(\frac{R}{r}\right)^2 \tag{5}$$

If $m = 1$, F is the force of gravitation at some point in space also the acceleration of gravitation, g, on a unit mass located at that point

$$F = g = g_o \left(\frac{R}{r}\right)^2 \tag{6}$$

Work or energy, E, is equal to force times distance, $dE = F\,dr$, or from (6)

$$E = g_o \int_{r_1}^{r_2} \left(\frac{R}{r}\right)^2 dr \tag{7}$$

To escape earth's gravitation, assuming the start at sea level, $r_1 = R$, $r_2 = \infty$ and (7) becomes

$$E = g_o R^2 \int_R^\infty \frac{dr}{r^2} = g_o R \tag{8}$$

Equation (8) states that the minimum energy required to liberate a mass of 1 slug (32.2 lbs.) from the earth's attraction, is equal to that required to move a unit mass a distance equal to the mean radius of the earth under a constant force equal to the acceleration of gravitation at earth's sea level. If in (8) one places $g_o = 32.2$ ft./sec.^{2} (this value is actually that of gravity at sea level—gravity being defined as the resultant of the gravitational force and the centrifugal force of earth's rotation, the latter force is quite small and negligible as far as the stated numerical value of g_o is concerned), $R = 21 \times 10^6$ ft., then $E = 6.79 \times 10^8$ ***ft. lbs. per unit mass, the minimum energy required to liberate a unit mass from the earth.*** Actually the required energy is somewhat higher since air resistence has been ignored. But it is less for a satellite since its velocity will be less than escape velocity.

ESCAPE VELOCITY WITH MINIMUM EXPENDITURE OF ENERGY

If a body originally at rest at a point P in space falls to the earth, its striking velocity is V_{P_0} and the kinetic energy per unit mass when it strikes the earth will be $(1/2)V^2_{P_0}$ ignoring air resistance. This must be equal to the potential energy of the body at the point P, that is

$$\tfrac{1}{2} V_{P_0}^{\,2} = \int_R^{R+h} g\,dr \tag{9}$$

where h is now the vertical distance from the surface of the earth to the point P in space. With the value of g from (6) placed in (9) find

$$V_{P_0}^2 = 2 g_0 R^2 \int_R^{R+h} \frac{dr}{r^2} = 2 g_0 R h / (R+h) = g_0 H,$$

or

$$V_{P_0} = \sqrt{g_0 H}, \quad H = 2Rh/(R+h). \tag{10}$$

Hence V_{P_0} *(the geometric mean* between g_0 and H, where H is *the harmonic mean* of h and R) is the velocity for a body to leave the earth in order to coast to a height h. If one lets $h \to \infty$ then $H = 2R$, and from (10)

$$V_{e_0} = \sqrt{2 g_0 R}. \tag{11}$$

With $g_0 = 32.2$ ft./sec.^{2}, $R = 21 \times 10^6$ ft. find from (11) that $V_{e_0} = 36{,}700$ ft./sec. which is about 7 miles/sec. or 25,200 m.p.h. as the minimum velocity a body must attain at sea level if it is to escape the earth's attraction ***with the minimum expenditure of energy.***

The gravitational velocity of escape, V_e, at any distance h from the surface of the earth is then from (10) and (11)

$$V_e^2 = V_{e_0}^2 - V_{P_0}^2 = 2 g_0 R - 2 g_0 R h/(R+h) = 2 g_0 R^2/(R+h)$$

or

$$V_e = R\sqrt{2 g_0/(R+h)} = \sqrt{g_0 R H/h}, \tag{12}$$

where H is the ***harmonic mean*** of h and R.

VACUUM ORBIT OF A NEAR EARTH SATELLITE

Beginning with the elements at the injection point P_0, as shown in figure 17, one can establish the differential equation of the motion (consider the earth a sphere and neglect air resistance). Kepler's second law may be used which states that the rate of description of area by a moving radius vector (r in fig. 18) is constant for all parts of the orbit, and write (from fig. 18)

$$r^2 \frac{d\omega}{dt} = k$$

Now when P is at P_0,

$$r \frac{d\omega}{dt} = \frac{ds}{dt} \cos I = V_0 \cos I$$

and $r = R + h$, hence

$$r^2 \frac{d\omega}{dt} = (R+h) V_0 \cos I = k \tag{13}$$

If V is the speed of the satellite at P, note from figure 18 that

$$V^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\omega}{dt}\right)^2. \tag{14}$$

But $\frac{dr}{dt} = \frac{dr}{d\omega}\frac{d\omega}{dt}$ and from (13) $\frac{d\omega}{dt} = \frac{k}{r^2}$

so that $\frac{dr}{dt} = \frac{k}{r^2}\frac{dr}{d\omega}$ and (14) may be written

$$V^2 = \left(\frac{ds}{dt}\right)^2 = \frac{k^2}{r^4}\left(\frac{dr}{d\omega}\right)^2 + \frac{k^2}{r^2}. \tag{15}$$

From the integral used to obtain (10), with the sign changed, one has the integral of the velocity from P_0 to P

$$\left[V^2\right]_{V_0}^{V} = -2 g_0 R^2 \int_{R+h}^{r} \frac{dr}{r^2} \tag{16}$$

or

$$V^2 = V_0^2 + 2\mu/r - 2\mu/(R+h) = q + 2\mu/r, \tag{17}$$

where $q = V_0^2 - 2\mu/(R+h)$, $\mu = g_0 R^2$.

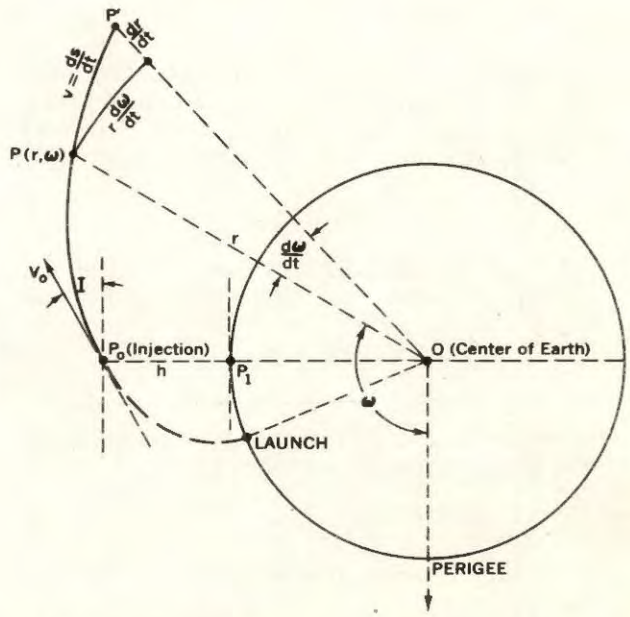

FIG. 18.—Polar representation of trajectory elements from injection.

From (15) and (17) one may now write

$$V^2 = \frac{k^2}{r^4}\left(\frac{dr}{d\omega}\right)^2 + \frac{k^2}{r^2} = q + \frac{2\mu}{r},$$

and solving for $d\omega$, one finds

$$d\omega = k\,dr/r^2\,(q + 2\mu/r - k^2/r^2)^{1/2}, \qquad (18)$$

which is the polar differential equation of the orbit.

By completing the square on the terms in r under the radical of (18) one may write the radical as

$$\left[q + \mu^2/k^2 - (k/r - \mu/k)^2\right]^{1/2}$$

$$= (q + \mu^2/k^2)^{1/2}\left[1 - \left\{\frac{k/r - \mu/k}{(q + \mu^2/k^2)^{1/2}}\right\}^2\right]^{1/2}$$

whence (18) may be written

$$d\omega = -\,d\left(\frac{k/r - \mu/k}{q + \mu^2/k^2)^{1/2}}\right)\Bigg/\left[1 - \left\{\frac{k/r - \mu/k}{(q + \mu^2/k^2)^{1/2}}\right\}^2\right]^{1/2},$$

which integrates directly to give

$$\omega - \omega_0 = \arccos\left[(k/r - \mu/k)/(q + \mu^2/k^2)^{1/2}\right]$$

or

$$r = p/\left[1 + e\cos(\omega - \omega_0)\right], \qquad (19)$$

where

$$p = k^2/\mu,\ e = (1 + q\,k^2/\mu^2)^{1/2},$$

$$q = V_0^2 - 2\mu/(R + h),\ \mu = g_0 R^2$$

Equation (19) is clearly the polar equation of a conic section with one focus at the center, o, of the earth (see fig. 18). If $\omega_0 = o$, then the polar axis of the coordinate system is the line joining the perigee to the center of the earth and equation (19) may be written

$$r = p/(1 + e\cos\omega) \qquad (20)$$

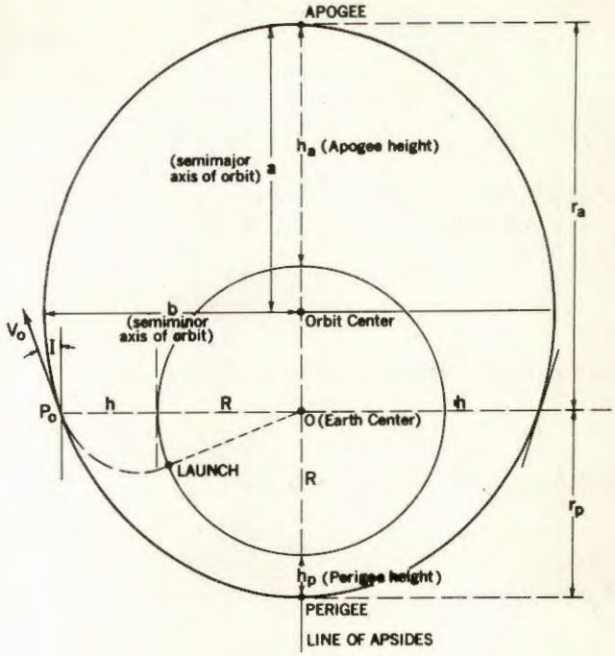

FIG. 19.—Elliptic orbit elements.

Equation (20) represents a circle if $e = o$; an ellipse, parabola, or hyperbola according as $e \lesseqgtr 1$. From figure 19 and equation (20) it is seen that when $\omega = 0, \pi$ one has

$$r_p = p/(1 + e) = R + h_p,$$
$$r_a = p/(1 - e) = R + h_a, \qquad (21)$$

where r_p and r_a are the values of r at perigee and apogee; h_p and h_a are perigee and apogee heights. Also $r_p + r_a = 2a$ where a is the semimajor axis of the orbit. Hence from (21) $p/(1 + e) + p/(1 - e) = 2a$, whence

$$p = a(1 - e^2), \qquad (22)$$

and (20) may be written

$$r = a(1 - e^2)/(1 + e\cos\omega). \qquad (23)$$

From (23) when $\omega = \pi/2, 3\pi/2$, then $r = a(1 - e^2)$ and from figure 19 and equation (23) it is seen that

$$r_{\pi/2} = r_{3\pi/2} = a(1 - e^2) = R + h. \qquad (24)$$

From (19) $p = k^2/\mu$ where $k = (R + h)\,V_0\cos I$ from (13) and $\mu = g_0 R^2$. From these values and equations (22) and (24) one finds

$$
\begin{aligned}
p &= a(1-e^2) = R + h = k^2/\mu \\
&= (R+h)^2 V_0^2 \cos^2 I / g_0 R^2 \\
&= 2(R+h)\cos^2 I\, V_0^2 / V_e^2 \\
&= 2(R+h)\cos^2 I / k_0^2,
\end{aligned} \tag{25}
$$

where $V_e^2 = 2 g_0 R^2/(R+h)$ as given by (12), and $k_0 = V_e/V_0$. From the third and last member of (25) find

$$\cos I = k_0/2^{1/2}, \quad k_0 = V_e/V_0 \tag{26}$$

Again from (21) and the second term of (25) find

$$h_p = a(1-e) - R, \quad h_a = a(1+e) - R \tag{27}$$

and from (27) solving for a and e

$$a = R + \tfrac{1}{2}(h_a + h_p), \quad e = \tfrac{1}{2}(h_a - h_p)/a. \tag{28}$$

Now from (19)

$$
\begin{aligned}
e &= (1 + q\, k^2/\mu^2)^{1/2} \\
&= \left\{ 1 + (R+h)^2 V_0^2 \cos^2 I \left[V_0^2 - 2\mu/(R+h) \right] / \mu^2 \right\}^{1/2} \\
&= (1 + 4 V_0^2 \cos^2 I (V_0^2 - V_e^2)/V_e^4)^{1/2} \\
&= (2 - k_0^2)^{1/2}/k_0
\end{aligned} \tag{29}
$$

where

$$V_e = R\,[2 g_0/(R+h)]^{1/2}, \quad k_0 = V_e/V_0, \quad \cos I = k_0/2^{1/2}.$$

From (29), $e = 0$ if $k_0 = 2^{1/2}$, whence $\cos I = 1$, and $I = 0$.

$$V_0 = V_e/2^{1/2}, \quad V_e = R\,[2 g_0/(R+h)]^{1/2}. \tag{30}$$

Equations (30) are the conditions for a circular orbit at altitude h. For instance suppose $h = 300$ miles $= 1.584 \times 10^6$ ft. and place $g_0 = 32.2$ ft./sec.^{2}, $R = 21 \times 10^6$ ft. Then $V_e = \sqrt{12.58} \times 10^4 = 3.55 \times 10^4$ ft./sec. or 6.72 miles/sec. or 24,190 m.p.h. Then $V_0 = 0.7071 \times 24{,}190 = 17{,}100$ m.p.h. This value was cited earlier for a circular orbit at 300 miles altitude under Satellite Injection Accuracy.

Again from (29) $e = 1$ when $k_0 = 1$, then $V_0 = V_e$ and ***the satellite escapes*** from the earth ***along a parabolic orbit.*** When $0 < k_0 < 1$, $e > 1$, $V_0 > V_e$ and ***the satellite escapes*** the earth ***along a hyperbolic orbit.*** When $1 < k_0 \leq 2^{1/2}$ then $0 \leq e < 1$ and the satellite describes an ***elliptic orbit*** including the circular limiting orbit ($e = 0$). Not all values of $1 < k_0 \leq 2^{1/2}$ will give useful elliptic orbits (from a geodetic view point) since the orbit may intercept the earth.

An absolute lower limit for k_0 may be obtained from (24) and (27), $h + R = a(1 - e^2)$ and $h_p + R = a(1 - e)$ which by division gives $1 + e = (h + R)/(h_p + R)$ and if $h_p = 0$ (the elliptic orbit tangent to the earth), then $e = h/R$. From (29) $e = (2 - k_0^2)^{1/2}/k_0$ or $k_0 = [2/(1 + e^2)]^{1/2}$ and the value of $e = h/R$ gives $\left[2/\left(1 + \frac{h^2}{R^2}\right)\right]^{1/2}$ as the absolute lower limit (tangent trajectory).

THE PERIOD OF THE MOTION AND THE ORBITAL VELOCITY

Since the area of an ellipse is $\pi a b$ one may write from equation (13), Kepler's second law,

$$
\begin{aligned}
r^2 \frac{d\omega}{dt} = k &= \frac{2\pi a b}{T} \\
&= (R+h) V_0 \cos I \\
&= a(1-e^2) V_0 \cos I,
\end{aligned} \tag{31}
$$

and with the aid of equations (24) to (30) one may write from (31)

$$
\begin{aligned}
T &= \frac{2\pi a b}{(R+h) V_0 \cos I} \\
&= 2\pi \frac{[R + \tfrac{1}{2}(h_a + h_p)]^{3/2}}{(R+h)^{1/2} V_0 \cos I} \\
&= \frac{\pi(R+h)}{V_e}\left(1 - \frac{V_0^2}{V_e^2}\right)^{-3/2} \\
&= \frac{2\pi g_0 R^2}{V_e^3}\left(1 - \frac{V_0^2}{V_e^2}\right)^{-3/2} \\
&= \frac{2\pi g_0 R^2}{k_0^3 V_0^3}\left(1 - \frac{1}{k_0^2}\right)^{-3/2}
\end{aligned} \tag{32}
$$

From (17), (23), and useful relations in equations (24) to (30) one can write the orbital velocity as

$$
\begin{aligned}
V &= \left(V_0^2 + \frac{2 g_0 R^2}{r} - \frac{2 g_0 R^2}{R+h}\right)^{1/2} \\
&= [1 + 2e\cos\omega/(1+e^2)]^{1/2} V_0,
\end{aligned} \tag{33}
$$

where

$$
\begin{aligned}
e &= (2 - k_0^2)^{1/2}/k_0 = \tfrac{1}{2}(h_a - h_p)/a, \\
V_0 &= V_e/k_0 \\
a &= R + \tfrac{1}{2}(h_a + h_p).
\end{aligned}
$$

Now the maximum and minimum values of V occur at perigee and apogee respectively, that is when $\omega = 0, \pi$ respectively. (This can be determined simply from (33) by placing $\frac{dV}{d\omega} = 0$). From

(33) with $\omega = 0$, π and $\cos I = k_0/2^{1/2} = 1/(1+e^2)^{1/2}$ find

$$V_p = (1+e)\, V_0/(1+e^2)^{1/2} = (1+e)\, V_0 \cos I,$$
$$V_a = (1-e)\, V_0/(1+e^2)^{1/2} = (1-e)\, V_0 \cos I. \qquad (34)$$

For the circular case, equations (30), $a = b = R + h$, $e = 0$, $\cos I = 1$ and from (32) and (33),

$$V = V_0 = 2\pi(R+h)/T. \qquad (35)$$

SUMMARY OF VACUUM ELLIPTIC ORBIT FORMULAS

From equations (21) to (35) one may write:

$$\cos I = k_0/2^{1/2} = 1/(1+e^2)^{1/2}$$
$$= (2R + h_a + h_p)^{1/2}/2^{1/2}(R + h_a + h_p - h)^{1/2},$$
$$V_0 = V_e/k_0 = (V_e \sec I)/2^{1/2},$$
$$k_0 = 2^{1/2}/(1+e^2)^{1/2},$$
$$V_e = R(2g_0)^{1/2}/(R+h)^{1/2}$$
$$= R(2g_0)^{1/2}/a^{1/2}(1-e^2)^{1/2},$$
$$h = a(1-e^2) - R,$$
$$h_p = a(1-e) - R,$$
$$h_a = a(1+e) - R,$$
$$r = a(1-e^2)/(1+e\cos\omega),$$
$$e = (2-k_0^2)^{1/2}/k_0 = \tfrac{1}{2}(h_a - h_p)/a,$$
$$a = (R+h)\,k_0^2/2(k_0^2 - 1)$$
$$= (R+h)/(1-e^2)$$
$$= R + \tfrac{1}{2}(h_a + h_p),$$
$$b = a(1-e^2)^{1/2} = a^{1/2}(R+h)^{1/2}$$
$$= (R+h)^{1/2}[R + \tfrac{1}{2}(h_a + h_p)]^{1/2}$$
$$T = 2\pi ab/(R+h)V_0 \cos I$$
$$= 2\pi[R + \tfrac{1}{2}(h_a + h_p)]^{3/2}/R\,g_0^{1/2} \qquad (36)$$
$$= \frac{\pi(R+h)}{V_e}\left(1 - \frac{V_0^2}{V_e^2}\right)^{-3/2}$$
$$= \frac{2\pi g_0 R^2}{V_e^3}\left(1 - \frac{V_0^2}{V_e^2}\right)^{-3/2}$$
$$= \left(\frac{2}{1-e^2}\right)^{1/2}\frac{2\pi a}{V_e}$$
$$= \frac{2\pi g_0 R^2}{k_0^3 V_0^3}\left(1 - \frac{1}{k_0^2}\right)^{-3/2}$$
$$= \frac{2\pi}{\sqrt{V_a V_p}}[R + \tfrac{1}{2}(h_a + h_p)],$$
$$V = [1 + 2e\cos\omega/(1+e^2)]^{1/2} V_0,$$
$$V = V_0 = 2\pi(R+h)/T \quad \text{(circular orbit)}$$
$$V_p = (1+e)\, V_0 \cos I,$$
$$V_a = (1-e)\, V_0 \cos I,$$
$$V_a V_p = g_0 R^2/[R + \tfrac{1}{2}(h_a + h_p)],$$
$$V_p/V_a = (1+e)/(1-e) = (h_a + R)/(h_p + R),$$

and where for useful elliptic orbits

$$[2/(1 + \frac{h^2}{R^2})]^{1/2} \le k_0 \le 2^{1/2}.$$

A GEODETIC SATELLITE ORBIT (VACUUM)

Suppose that it is desired to have a geodetic satellite with a perigee of 1,000 miles and apogee of 1,500 miles. Assume the radius of the earth to be 3,959 statute miles. (This is the radius of the sphere equivalent in area to the International Ellipsoid of reference.) Then from equations (36)

$$a = R + \tfrac{1}{2}(h_a + h_p) = 3{,}959 + 1{,}250 = 5{,}209 \text{ miles}$$
$$e = \tfrac{1}{2}(h_a - h_p)/a = (250)/(5209) = 0.048,$$
$$e^2 = 0.0023$$
$$k_0 = [2/(1+e^2)]^{1/2} = 2/(1.0023)^{1/2} = 1.4126$$
$$h = a(1-e^2) - R = (5209)(0.9977) - 3{,}959$$
$$= 5{,}197 - 3{,}959 = 1{,}238 \text{ miles}$$
$$V_e = R[2g_0/a(1-e^2)]^{1/2}$$
$$= (3{,}959)[(64.4)/(5{,}280)(5{,}197)]^{1/2}$$
$$= 6.0665 \text{ miles/sec} = 21{,}840 \text{ m.p.h.}$$
$$V_0 = V_e/k_0 = (21{,}840)/(1.4126) = 15{,}460 \text{ m.p.h.}$$
$$\cos I = 0.7071\, k_0 = (0.7071)(1.4126) = 0.99885,$$
$$I = 2°45'.$$
$$T = [2/(1-e^2)]^{1/2}(2\pi a/V_e)$$
$$= [2/(0.9977)]^{1/2}[(6.2832)(5{,}209)/(6.0665)]$$
$$= (1.4158)(5{,}395) = 7{,}638.2 \text{ sec.}$$
$$= 2 \text{ hours } 7.3 \text{ min.}$$
$$V_p = (1+e)\, V_0 \cos I = (1.048)(15{,}460)(0.99885)$$
$$= 16{,}185 \text{ m.p.h.}$$
$$V_a = (1-e)\, V_0 \cos I = (0.952)(15{,}460)(0.99885)$$
$$= 14{,}700 \text{ m.p.h.}$$

Hence for a satellite orbit of perigee 1,000 miles and apogee 1,500 miles, one must have at injection h = 1,238 miles, V_0 = 15,460 miles,

Table 2

k_o	V_o	V_p	V_a	$\cos I$	I		h_p	h_a	a	e	T	
	m.p.h.	m.p.h.	m.p.h.		°		miles	miles	miles		hrs.	min.
1.3498	16,180	20,270	10,615	0.95444	17	21.5	0	3,600	5,760	0.3127	2	28.2
1.3552	16,115	20,050	10,835	0.95826	16	36.5	50	3,460	5,715	0.2984	2	26.4
1.3606	16,050	19,815	11,065	0.96208	15	50.0	100	3,310	5,665	0.2834	2	24.6
1.3681	15,965	19,490	11,400	0.96738	14	40.5	200	3,150	5,635	0.2618	2	23.1
1.3810	15,815	18,850	12,040	0.97651	12	26.5	300	2,710	5,465	0.2205	2	16.8
1.4000	15,600	17,650	13,235	0.98994	8	08.0	588	2,105	5,305	0.1429	2	10.2
1.4082	15,510	16,875	14,010	0.99574	5	17.5	800	1,770	5,245	0.0927	2	08.6
1.4126	15,460	16,185	14,700	0.99885	2	45.0	1,000	1,500	5,209	0.0480	2	07.3
1.4140	15,445	15,710	15,175	0.99984	1	01.0	1,150	1,330	5,198	0.0173	2	06.9
1.4142	15,443	15,443	15,443	1.0000	0	0	1,238	1,238	5,197	0	2	06.9

Based on an injection altitude of 1,238 miles and velocity of escape at 1,238 miles of 21,840 m.p.h.

$I = 2°45'$ and correspondingly a = 5,209 miles, e = 0.048, T = 2 hours 7.3 min., V_p = 16,185 m.p.h., V_a = 14,700 m.p.h.

If the injection altitude, h = 1,238 miles, is attained but the required values of I and V_o are not, a useful orbit may or may not be achieved. If a useful one is to be obtained then with h = 1,238, R = 3,959 one finds from (36) that $1.3498 \leq k_o \leq 1.4142$. Table 2 lists the elements of several possible orbits from the tangent orbit to the limiting circular orbit.

While the figures of table 2 are only approximate their orders of magnitude are correct and indicate again the precision with which the injection quantities I and V_o for a particular h must be controlled in order to have any usable orbit and particularly to achieve a specified orbit. Figure 20 shows the orbits drawn to scale from table 2 for the values of h_p = o, 1,238 miles. All geodetically useful orbits must lie between these. (Obviously the tangent orbit, for which h_p = o, would be of no practical value. It is doubtful if much useful information could be obtained for an h_p under 100 miles.)

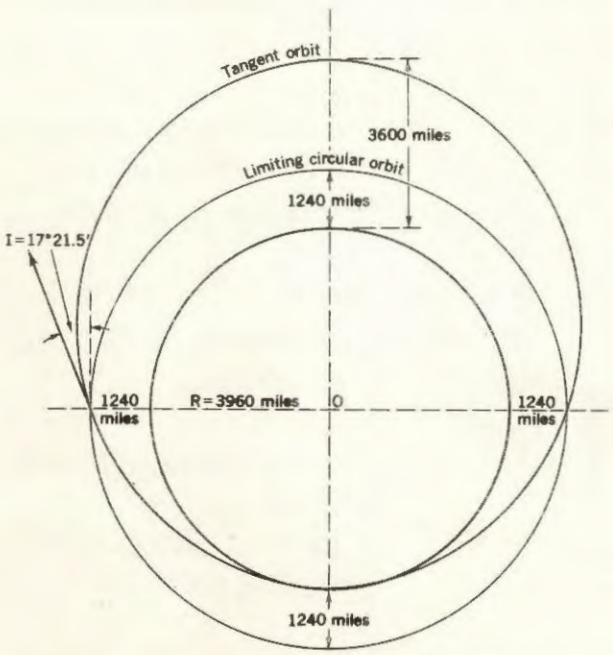

FIG. 20.—Range of achievable orbits based on injection elements of a desired orbit for geodetic purposes. (See Table 2.)

AN ESTIMATE OF ROCKET TAKEOFF WEIGHT

Let us suppose that the geodetic satellite, as shown in figure 5, weighs 200 lbs., and that the orbit desired is that already discussed—perigee 1,000 miles, apogee 1,500 miles. Suppose also that the fuel to be used is oxygen-kerosene which has a specific impulse of 260 (assuming a chamber pressure of 300 p.s.i.) as shown in table 1.

Now if V_E is the efflux velocity or discharge velocity through the rocket motor nozzles, then $V_E = g_o I$. (This is not quite true since V_E actually increases with altitude because of decrease in atmospheric pressure.) I, the specific impulse, is a force measured in seconds since g_o = 32.2 ft./sec.^{2} and V_E is in ft./sec. If the rocket has mass m and velocity V then

$$m\frac{dv}{dt} = -V_E\frac{dm}{dt} = g_o\left(-I\frac{dm}{dt}\right) \qquad (37)$$

Equation (37) states that the change in the momentum of the rocket is equal to the momentum of the discharged gas which is the product of the efflux velocity V_E of the gas relative to the rocket and mass rate of discharge of the gas $\left(-\frac{dm}{dt}\right)$

Also from (37) since $V_E = g_o I$, the quantity

$$F = \left(-I\frac{dm}{dt}\right)$$

is a force called the thrust.

Integrating equation (37) considering the jet velocity V_E as constant

$$V_0 = \int_0^{V_0} dv = -V_E \int_{m_0}^{m_f} \frac{dm}{m}$$

$$= V_E \ln \frac{m_0}{m_f} = g_0 I \ln \frac{m_0}{m_f}, \qquad (38)$$

Where m_0 is the initial mass of the rocket and m_f the final or residual mass after the propellant has burned. For simplicity consider a multistage rocket of n stages equal in size, with equal structure to total stage weight in each of K. If W_T is the gross rocket vehicle weight, W_P the useful payload weight, one may write in the light of (38)

$$V_0 = n g_0 . I \ln m_n$$

$$W_T = \left[\frac{m_n (1-K)}{1 - m_n K}\right]^n W_p, \qquad (39)$$

where $m_n = m_0/m_f$ is now the mass ratio per stage. Now the injection velocity required is V_0 = 15,460 m.p.h. or 4.295 mi./sec. Assume m_n = 2.5 and K = 0.317. From the first of (39), placing $g_0 = [(32.2)/(5{,}280)]$ mi./sec.^{2} = 6.1 x 10^{-3} mi./sec.^{2}

$$n = V_0/g_0 I \ln m_n$$

$$= (4.295)/(6.1 \times 10^{-3})(260)(0.91629)$$

n = 2.96 or 3 stages are required.

From the second of (39)

$$W_T = 200\left(\frac{0.683 \times 2.5}{1-(2.5)(0.317)}\right)^3$$

$$= 200(8.2)^3 = 200 \times 551$$

$$= 110{,}200 \text{ lbs.}$$

These are of course hypothetical figures. The actual Thor-Delta configuration with which NASA plans to place the geodetic satellite in orbit is a three stage rocket system. The first stage is a production Thor missile with the nose cone and guidance removed. The second stage uses a modified version of the Vanguard second stage rocket engine. The Thor guidance system is installed in the second stage to provide attitude control. The coast-phase guidance thus affords much higher orbits than with previous vehicles since a prescribed vehicle attitude can be maintained up to 2,000 seconds after second-stage burnout. An ABL X-248 solid propellant rocket motor is used as the Delta third stage. Prior to ignition, this stage is spun up to 150 r.p.m. to obtain spin stability after separation, since neither guidance nor autopilot is carried in the third stage. The take-off weight is estimated at 112,000 lbs.

ESTIMATES OF THE MEAN RADIUS AND MASS OF THE EARTH FROM RADAR AND DOPPLER MEASUREMENTS

Suppose that the maximum and minimum heights on the orbit have been determined say by radar measurements, and the corresponding minimum and maximum orbital velocities have been obtained from Doppler data. That is h_a, h_p, V_p, V_a have been determined. From (36) the relation $V_p/V_a = (h_a + R)/(h_p + R)$ gives

$$R = (V_a h_a - V_p h_p)/(V_p - V_a) \qquad (40)$$

Now the mass M of the earth can be estimated from the energy conservation relation

$$\frac{G M m}{h_a + R} + \tfrac{1}{2} V_p^2 m = \frac{G M m}{h_p + R} + \tfrac{1}{2} V_a^2 m,$$

where G is the gravitational constant. Solving for M find

$$M = \frac{1}{2G} \frac{(V_p^2 - V_a^2)(h_p + R)(h_a + R)}{(h_a - h_p)} \qquad (41)$$

M may be written in terms of the period T and R, h_a, h_p. From (36)

$$(h_p + R)(h_a + R) = a^2(1 - e^2),$$

$$V_p^2 - V_a^2 = \frac{8\pi^2 a (h_a - h_p)}{T^2 (1 - e^2)},$$

and these quantities placed in (41) give

$$M = \frac{1}{2G} \frac{8\pi^2}{T^2} a^3 = \frac{1}{2G}\left(\frac{\pi}{T}\right)^2 (2R + h_a + h_p)^3, \qquad (42)$$

where G is the gravitational constant.

THE FLATTENING OR ELLIPTICITY

The French mathematician, Jean Picard, began in 1669 the determination of the length of a meridian arc from a point near Corbeil to one near Amiens employing an elaborate system of triangulation. He used for the first time in geodetic surveying, a quadrant graduated in minutes and seconds, mounting telescopes whose oculars had cross hairs for sighting instead of the usual pinhole alidades. The Cassinis (also French mathe-

maticians but of Italian descent) improved Picard's methods and extended his arc southward to the Pyrenees and northward to Dunkerque. Because of certain inaccuracies in their work, it appeared that the length of the degree of the meridional arc decreased from the equator to the pole, which of course implied that the earth was elongated at the poles (like a football). But these conclusions were contrary to the developments of mechanics by Newton, Huyghens, and their contemporaries which showed that a surface of revolution as the earth would necessarily have to be flattened at the poles (like a door knob). The result was a controversy between two groups of geodesists—one favoring elongation at the poles, the other flattening at the poles. To settle the dispute, the Royal Academy of Paris organized two geodetic expeditions, one to Lapland, the other to Peru (the site of the present country Ecuador). The arc measurements of the two expeditions were finished in 1737 and 1743, respectively; the results proved inclusively that the earth was flattened at the poles and incited Voltaire's quip that Maupertuis (the leader of the Lapland expedition) had "flattened the poles and the Cassinis!" Since that time a principal objective of geodesy has been the determination of the flattening from triangulation arc lengths, gravity or combinations of these. Presently it is being determined from near earth satellite orbits.

If a_e, b are the semimajor, semiminor axes of the earth, then the flattening is defined as $f = (a_e - b)/a_e$, and $F = 1/f$. Table 3 lists the principal published determinations from 1799 to 1960. Principal values of F as obtained from satellite data are:

Author	*F*
O'Keefe	298.38±0.07
Eckels	298.32±0.05
Cook	298.24
King-Hele	298.4
Jacchia	298.29±0.11

These values suggest the adoption of $F = 298.3 \pm 0.1$.

Now table 3 will be seen to contain six determinations near this value and two, that of Helmert and Krassovsky, are exactly this value. Helmert's value, obtained in 1907 from pendulum gravity measurements alone, was considered by him to be more accurate than any value which can be obtained from geodetic arc measurements even though the arcs of more recent decades be used.[1]

In 1956, in the determination of the Hough ellipsoid from arcs[2], F was assumed to be 297 ± 1

Table 3

Year	Author	F = 1/f	a_e
			m
1799	Gen. Comm. for weights and measures.	334.29	6,375,739
1810	DeLambre.....	311.5	6,376,428
1810	DeLambre.....	308.65	6,376,523
1819	Walbeck.....	302.78	6,376,895
1828	Schmidt.....	297.65	6,376,959
1838	Everest.....	300.80	6,377,253
1841	Bessel.....	299.15	6,377,397
1847	Everest.....	311.04	6,376,634
1849	Airy.....	299.32	6,377,491
1856	James and Clarke	297.72	6,377,936
1858	Clarke.....	294.26	6,378,294
1863	Clarke.....	294.36	6,378,288
1863	Pratt.....	295.36	6,378,245
1866	Clarke.....	294.98	6,378,207
1868	Fischer.....	288.50	6,378,338
1880	Clarke.....	293.47	6,378,249
1891	Harkness.....	300.20	6,377,972
1907	Helmert.....	298.3 ± 0.7	6,378,140
1907	Hayford.....	297.8	6,378,283
1909	Hayford (1) (International).	297.0	6,378,388
1909	Hayford (2).....	298.2	6,378,062
1942	Krassovsky.....	298.3	6,378,245
1948	Jeffreys.....	297.1	6,378,099
1956	Hough.....	297.0 ± 1	6,378,270
1958-1959	Satellite Determinations.	297.9 to 298.32	

and the value of a_e as computed on this basis was taken to be 6,378,270 ± 100 meters, the uncertainty in large measure due to the uncertainty in F. In the published report of this work[3], a graph of a_e versus F is given and if one enters this graph with $F = 298.3$, and interpolates (linearly) between the isostatic and free air curves (lines), a_e is found to be 6,378,145 meters. Now table 3 shows that Helmert had 6,378,140 meters in 1907. This illustrates how greatly the determination of a_e from arcs depends upon F. And satellites give values of the flattening quite independent of any surface measurements! It is interesting to note that the value of $a_e = 6{,}378{,}175$ meters was obtained from radar measurements of the distance to the moon.[4] In the next few sections the methods by which the flattening is determined from orbital data will be described.

[1]The figure of the earth and isostasy from measurements in the United States by J. F. Hayford, *U. S. Gov. Print. Off.*, 1909.

[2]A new determination of the earth from arcs by B. Chovitz, I. Fischer; *Transactions, American Geophysical Union*, October 1956.

[3]*Ibid.*, page 542.

[4]Echos from the moon at a wavelength of 10 cm. by B. S. Yaplee, R. H. Bruton, K. J. Craig, and N. G. Roman, *Proceedings of the Institute of Radio Engineers*, January 1958.

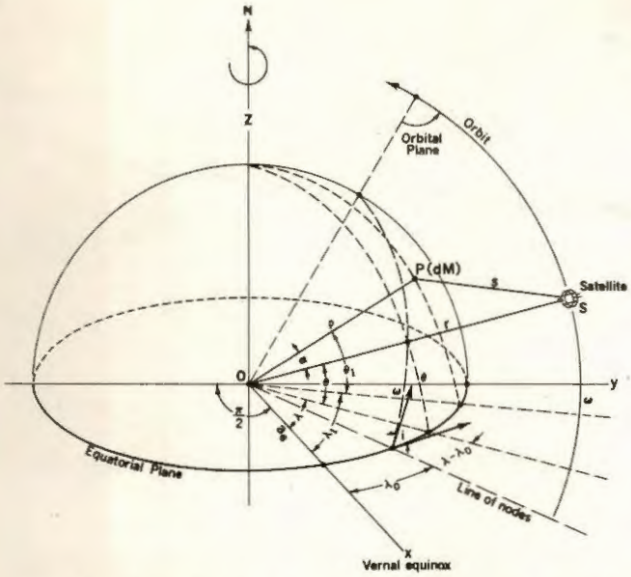

FIG. 21.—The satellite referred to a geocentric inertial system of coordinates.

THE GRAVITATIONAL POTENTIAL

Actually the flattening, as determined from gravity data (although the measurements are made on the topographical surface which is not an equipotential surface), is that of the geoid which is the equipotential surface at sea level and is very closely an exact spheroid. The geoid is somewhat irregular, deviating from a mean ellipsoid at any one of its points by probably not more than $\pm$ 50 meters. The mean ellipsoid is very near the equilibrium figure of a rotating body. Now the potential, of which the geoid is an equipotential surface, is the geopotential W, the sum of the purely gravitational potential U and a centrifugal term, that is

$$W = U + (x^2 + y^2)\,\omega^2/2$$

The geopotential W is appropriate to a coordinate system rotating with the earth at the angular velocity ω and hence for geodesy. The gravitational potential U is appropriate for inertial coordinates and thus for satellite orbits.

To find an expression for the gravitational potential in terms of observable elements or elements which can be inferred from observations, the position of the satellite and an attracting mass dM in the earth are referred to an inertial geocentric system of coordinates which for simplicity have been taken as follows (fig. 21).

The x-axis is in the plane of the equator and directed toward the vernal equinox. The y-axis is 90° from the x-axis in the plane of the equator, and the axis of rotation, the z-axis, is orthogonal to the x and y axes at their intersection. The assumptions are that these axes are the principal axes of inertia of the earth; that the axis of rotation passes through the center of mass of the geoid and that the earth displays rotational symmetry about the polar axis. In figure 21, θ is the geocentric latitude, λ is right ascension, and r is the geocentric distance. The satellite orbit relative to the coordinate system is such that ω_0 is the argument of perigee and i is the inclination of the plane of the orbit to the equator (the dihedral angle between the plane of the equator and the plane of the orbit). From figure 21, one has the coordinates of a point P of mass dM and those of the satellite at S:

$$\begin{aligned} P(dM):\ & (x_m,\ y_m,\ z_m) \\ S:\ x_s &= r\cos\theta\cos\lambda \\ y_s &= r\cos\theta\sin\lambda \\ z_s &= r\sin\theta \end{aligned} \tag{43}$$

From (43) one has

$$\begin{aligned} x_s^2 + y_s^2 &= r^2 - z_s^2 = r^2(1-\sin^2\theta), \\ \rho^2 &= x_m^2 + y_m^2 + z_m^2 \end{aligned} \tag{44}$$

If s is the distance PS as shown in figure 21,

$$\begin{aligned} s^2 &= (x_s - x_m)^2 + (y_s - y_m)^2 + (z_s - z_m)^2 \\ &= \rho^2 + r^2 - 2\rho r\cos\alpha, \end{aligned} \tag{45}$$

$$\cos\alpha = (x_s x_m + y_s y_m + z_s z_m)/\rho r \tag{46}$$

From (45)

$$\frac{1}{s} = \frac{1}{r}\left[1 + \left(\frac{\rho}{r}\right)^2 - 2\frac{\rho}{r}\cos\alpha\right]^{-1/2} \tag{47}$$

One has but to place $t = \left(\frac{\rho}{r}\right)^2 - 2\frac{\rho}{r}\cos\alpha$ in the binominal expansion

$$(1+t)^{-1/2} = 1 - \frac{1}{2}t + \frac{3}{8}t^2 - \frac{5}{16}t^3 + \frac{35}{128}t^4 \cdots$$

to obtain the following series expansion for (47)

$$\frac{1}{s} = \frac{1}{r}\left[P_0(\cos\alpha) + \frac{\rho}{r}P_1(\cos\alpha) + \left(\frac{\rho}{r}\right)^2 P_2(\cos\alpha) + \left(\frac{\rho}{r}\right)^3 P_3(\cos\alpha) + \left(\frac{\rho}{r}\right)^4 P_4(\cos\alpha) + \cdots\right] \tag{48}$$

where the coefficients P_0, P_1, P_2, P_3, P_4 are the so called Legendre polynominals: $P_0(\cos\alpha) = 1$, $P_1(\cos\alpha) = \cos\alpha$, $P_2(\cos\alpha) = (1/2)(3\cos^2\alpha - 1)$, $P_3(\cos\alpha) = (1/2)(5\cos^3\alpha - 3\cos\alpha)$, $P_4(\cos\alpha) = (1/8)(35\cos^4\alpha - 30\cos^2\alpha + 3)$, ...

Let the potential U be given as usual by

$$U = G \int dM/s. \tag{49}$$

Then from (48) and (49) one has (with the equatorial radius of the earth a_e)

$$U = U_0 + a_e U_1, + a_e^2 U_2 + a_e^3 U_3 + a_e^4 U_4 + \dots \tag{50}$$

where

$$U_0 = \frac{G}{r} \int P_0 (\cos \alpha) dM = \frac{G}{r} M$$

$$U_1 = \frac{G}{r^2} \int \rho P_1 (\cos \alpha) dM = \frac{G}{r^2} \int \rho \cos \alpha \, dM$$

$$U_2 = \frac{G}{r^3} \int \rho^2 P_2 (\cos \alpha) dM = \frac{G}{2r^3} \int \rho^2 (3\cos^2 \alpha - 1) dM$$

$$U_3 = \frac{G}{r^4} \int \rho^3 P_3 (\cos \alpha) dM$$

$$= \frac{G}{2r^4} \int \rho^3 (5 \cos^3 \alpha - 3 \cos \alpha) dM, \text{etc.}$$

U_0 is the potential of the given mass to spherical order (approximates a sphere that is homogeneous in layers). U_1, U_2, ... U_4, ... represent the departures and will produce accelerations which may be regarded as perturbations. The pertubations will be small if spherical asymmetries are small, or if the distance r to the satellite is great.

Because of the assumptions made as to the rotating mass, the first harmonic term U_1 is necessarily zero. Since the coordinate axes are also the principal axes of inertia, the products of inertia are zero, namely

$$\int x_m y_m \, dM = \int x_m z_m \, dM = \int y_m z_m \, dM = 0. \tag{51}$$

Define arbitrarily

$$\begin{aligned} A &= \int x_m^2 \, dM = \int y_m^2 \, dM, \\ B &= \int z_m^2 \, dM, \\ C &= \int z_m^3 \, dM, \\ D &= \int z_m x_m^2 \, dM = \int z_m y_m^2 \, dM. \end{aligned} \tag{52}$$

The moments of inertia about the x, y, z axes are then $I_x = A + B = I_y$, $I_z = 2A$.

Since $U_1 = 0$, one begins with U_2, whence from (44), (46) and (50)

$$\begin{aligned} 2\frac{r^3}{G} U_2 &= \int \rho^2 (3 \cos^2 \alpha - 1) dM \\ &= \int \left[\frac{3(x_m x_s + y_m y_s + z_m z_s)^2}{r^2} - \left(x_m^2 + y_m^2 + z_m^2\right) \right] dM \\ &= \left(\frac{3}{r^2} x_s^2 - 1\right) \int x_m^2 \, dM \\ &+ \left(\frac{3}{r^2} y_s^2 - 1\right) \int y_m^2 \, dM + \left(\frac{3}{r^2} z_s^2 - 1\right) \int z_m^2 \, dM \end{aligned} \tag{53}$$

+[3 product of inertia terms which are zero by (51)]

From (52), the three remaining integrals of (53) may be written

$$2\frac{r^3}{G} U_2 = \left[\frac{3}{r^2}\left(x_s^2 + y_s^2\right) - 2\right] A + \left(\frac{3}{r^2} z_s^2 - 1\right) B,$$

and from (43) and (44)

$$x_s^2 + y_s^2 = r^2 (1 - \sin^2 \theta),$$

$$z_s^2 = r^2 \sin^2 \theta$$

so that

$$\begin{aligned} 2\frac{r^3}{G} U_2 &= \left[\frac{3}{r^2} \cdot r^2 (1 - \sin^2 \theta) - 2\right] A \\ &\quad + \left(\frac{3}{r^2} \cdot r^2 \, \mathrm{sm}^2 \theta - 1\right) B \\ &= A - 3A \sin^2 \theta + 3B \sin^2 \theta - B \\ &= (A - B) - (A - B) 3 \sin^2 \theta \\ &= (A - B)(1 - 3 \sin^2 \theta) \end{aligned}$$

or finally

$$U_2 = \frac{G(A - B)}{2r^3} (1 - 3 \sin^2 \theta) = \frac{GM}{r} \cdot \frac{J}{3r^2} (1 - 3 \sin^2 \theta)$$

where

$$J = \frac{3}{2M} (A - B).$$

Continuing in this way find

$$\begin{aligned} U_3 &= \frac{1}{2} \frac{G}{r^4} \sin \theta [(3D - C)(3 - 5 \sin^2 \theta)] \\ &= \frac{GM}{r} \cdot \frac{H}{5r^3} (3 - 5 \sin^2 \theta) \sin \theta \end{aligned}$$

where

$H = \frac{5}{2M}(3D - C)$, etc. so that one may write (50) finally as

$$U = \frac{\tau}{r}\left[1 + \frac{J a_e^2}{3 r^2}(1 - 3\sin^2\theta) + \frac{H a_e^3}{5 r^3}(3 - 5\sin^2\theta)\sin\theta + \frac{K a_e^4}{30 r^4}(3 - 30\sin^2\theta + 35\sin^4\theta) + \cdots\right] \quad (54)$$

where

$$J = \frac{3}{2M}(A - B),$$

$$H = \frac{5}{2M}(3D - C),$$

$$\tau = GM,$$

r is the radial distance from the center of the earth to the satellite, and θ is the geocentric latitude of the satellite.

THE COEFFICIENTS OF THE GRAVITY POTENTIAL AS FUNCTIONS OF THE FLATTENING

The polar equation of an oblate ellipsoid of revolution in terms of the minor semiaxis b and geocentric latitude θ is

$$r = b(1 - e^2\cos^2\theta)^{-1/2}, \quad (55)$$

and from the elliptic parameters (f is the flattening), $b = a_e(1-f) = a_e(1-e^2)^{1/2}$, $e^2 = 2f - f^2$, so that (55) may be written

$$r = a_e(1-f)[1-(2f-f^2)\cos^2\theta]^{-1/2} \quad (56)$$

Using the binominal expansion

$$(1-k^2)^{-1/2} = 1 + (1/2)k^2 + (3/8)k^4 \cdots$$

where $k^2 = (2f - f^2)\cos^2\theta$, equation (56) may be written (ignoring powers of f above the second) as

$$r = a_e\left(1 - f\sin^2\theta - \frac{3}{2}f^2\sin^2\theta + \frac{3}{2}f^2\sin^4\theta\right), \quad (57)$$

and by division

$$a_e/r = 1 + \left(f + \frac{3}{2}f^2\right)\sin^2\theta - \frac{1}{2}f^2\sin^4\theta, \quad (58)$$

also to second order terms in the flattening one finds from (57)

$$r^2 = a_e^2(1 - 2f\sin^2\theta - 3f^2\sin^2\theta + 4f^2\sin^4\theta). \quad (59)$$

Assuming the spheroid is a bounding equipotential then (54) and (57) must satisfy the geopotential

$$W = U + \frac{1}{2}\omega^2 r^2(1 - \sin^2\theta) = \text{constant} \quad (60)$$

Hence if the value of U from (54) and the value of r from (57) are substituted in (60), retaining only powers of f as high as f^2 and ignoring small negligible terms; then if the sums of the coefficients of the second and fourth powers of $\sin\theta$ are equated to zero, one can solve the two equations for J and K in terms of the flattening.

From (54), neglecting the term in H,

$$U = \frac{\tau}{r}\left[1 + \frac{J}{3}\frac{a_e^2}{r^2}(1 - 3\sin^2\theta) + \frac{K}{30}\frac{a_e^4}{r^4}(3 - \sin^2\theta + 35\sin^4\theta)\right]. \quad (61)$$

Since J and K are small quantities and $K < J$, it is not necessary to include terms of orders Jf^2, Kf, or Kf^2. Hence for the term in J one can let $\frac{a_e^2}{r^2} = 1 + 2f\sin^2\theta$ from either (57) or (59), and in the K term $\frac{a_e^4}{r^4} = 1$. Equation (61) can be written then with the value of $1/r$ from (58) as

$$U = \frac{\tau}{a_e}\left[1 + \left(f + \frac{3}{2}f^2\right)\sin^2\theta - \frac{1}{2}f^2\sin^4\theta\right]\left[1 + \frac{J}{3}(1 - 3\sin^2\theta)(1 + 2f\sin^2\theta) + \frac{K}{30}(3 - 30\sin^2\theta + 35\sin^4\theta)\right]$$

and in multiplying by the external factor, multiply only the first two terms retaining all in the first, and only those to f in the second term, that is

$$U = \frac{\tau}{a_e}\left[1 + \left(f + \frac{3}{2}f^2\right)\sin^2\theta - \frac{1}{2}f^2\sin^4\theta + \frac{J}{3}(1 + 2f\sin^2\theta)(1 + f\sin^2\theta)(1 - 3\sin^2\theta) + \frac{K}{30}(3 - 30\sin^2\theta + 35\sin^4\theta)\right]$$

or finally

$$U = \frac{\tau}{a_e}\left\{1 + \left(f + \frac{3}{2}f^2\right)\sin^2\theta - \frac{1}{2}f^2\sin^4\theta + J\left[\frac{1}{3} + (f-1)\sin^2\theta - 3f\sin^4\theta\right] + K\left(\frac{1}{10} - \sin^2\theta + \frac{7}{6}\sin^4\theta\right)\right\} \quad (62)$$

For the term $\frac{\omega^2 r^2}{2}(1 - \sin^2\theta)$ of W, use the value of r^2 from (59) and write

$$\frac{\omega^2 r^2}{2}(1 - \sin^2\theta) = \frac{\omega^2}{2}a_e^2(1 - 2f\sin^2\theta - 3f^2\sin^2\theta + 4f^2\sin^4\theta)(1 - \sin^2\theta)$$

$$= \frac{\omega^2}{2}a_e^2[1 - (1 + 2f + 3f^2)\sin^2\theta + (2f + 7f^2)\sin^4\theta]. \quad (63)$$

Now

$$\omega^2 a_e^2 = \frac{\tau}{a_e}\left(\frac{\omega^2 a_e^3}{\tau}\right) = \frac{\tau}{a_e} m,$$

hence (63) can be written

$$\frac{\omega^2}{2} r^2 (1-\sin^2\theta)$$
$$= \frac{\tau}{a_e}\frac{m}{2}\left[1-(1+2f+3f^2)\sin^2\theta + (2f+7f^2)\sin^4\theta\right],$$

and neglecting in this last, terms in mf^2, one has

$$\frac{\omega^2 r^2}{2}(1-\sin^2\theta)$$
$$= \frac{\tau}{a_e}\frac{m}{2}\left[1-(1+2f)\sin^2\theta + 2f\sin^4\theta\right]. \qquad (64)$$

From (62) and (64) one has finally

$$W = \frac{\tau}{a_e}\left\{1+\left(f+\frac{3}{2}f^2\right)\sin^2\theta - \frac{1}{2}f^2\sin^4\theta \right.$$
$$+ J\left[\frac{1}{3}+(f-1)\sin^2\theta - 3f\sin^4\theta\right]$$
$$+ K\left(\frac{1}{10} - \sin^2\theta + \frac{7}{6}\sin^4\theta\right)$$
$$\left. + \frac{m}{2}\left[1-(1+2f)\sin^2\theta + 2f\sin^4\theta\right]\right\} \qquad (65)$$

From (65) add the coefficients of the $\sin^2\theta$ terms and place the sum equal to zero; similarly for the $\sin^4\theta$ terms to obtain the two equations:

$$f+\frac{3}{2}f^2 + J(f-1) - K - \frac{m}{2}(1+2f) = 0,$$
$$-\frac{1}{2}f^2 - 3fJ + \frac{7}{6}K + mf = 0. \qquad (66)$$

From the first of equations (66), with $K = fm = f^2 = 0$, one has to a first approximation $J = f - \frac{m}{2}$ and this value of J placed in the second of (66) gives $K = 3f^2 - \frac{15}{7}fm$. Now this value of K placed in the first of (66) gives

$$J = \frac{1}{1-f}\left(f - \frac{3}{2}f^2 + \frac{8}{7}fm - \frac{m}{2}\right)$$
$$= (1+f+f^2)\left(f - \frac{3}{2}f^2 + \frac{8}{7}fm - \frac{m}{2}\right)$$

and retaining terms up to f^2, neglecting terms in mf^2, find

$$J = f - \frac{3}{2}f^2 + \frac{8}{7}mf - \frac{m}{2} + f^2 - \frac{fm}{2} = f - \frac{1}{2}f^2 + \frac{9}{14}mf - \frac{m}{2}$$

or finally

$$J = f - \frac{1}{2}f^2 + \frac{9}{14}mf - \frac{m}{2}, \quad K = 3f^2 - \frac{15}{7}mf \qquad (67)$$

where $m = \omega^2 a_e^3/\tau$

CORRESPONDING GRAVITY FORMULAS

Now gravity on the surface is given by

$$g^2 = \left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2, \qquad (68)$$

and from (60)

$$\frac{\partial W}{\partial r} = \frac{\partial U}{\partial r} + \omega^2 r\cos^2\theta,$$
$$\frac{\partial W}{\partial \theta} = \frac{\partial U}{\partial \theta} - \omega^2 r^2 \sin\theta\cos\theta. \qquad (69)$$

From (68) and (69)

$$g^2 = \left(\frac{\partial U}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial U}{\partial \theta}\right)^2$$
$$+ 2\omega^2\cos\theta\left(r\cos\theta\frac{\partial U}{\partial r} - \sin\theta\frac{\partial U}{\partial \theta}\right)$$
$$+ r^2\omega^4\cos^2\theta, \qquad (70)$$

and from (54), neglecting the H term,

$$-\frac{\partial U}{\partial r} = \frac{\tau}{r^2}\left[1 + J\frac{a_e^2}{r^2}(1-3\sin^2\theta)\right.$$
$$\left. + \frac{K a_e^4}{6r^4}(3-30\sin^2\theta+35\sin^4\theta)\right], \qquad (71)$$

$$-\frac{\partial U}{\partial \theta} = \tau\frac{2a_e^2}{r^3}\sin\theta\cos\theta\left[J\right.$$
$$\left. + K\frac{a_e^2}{r^2}\left(1-\frac{7}{3}\sin^2\theta\right)\right]. \qquad (72)$$

From (72), it is seen that when $\theta = 0$, $\frac{\partial U}{\partial \theta} = 0$, and gravity on the equator is then from (70)

$$g_e^2 = \left(\frac{\partial U}{\partial r}\right)^2 + 2\omega^2 a_e\frac{\partial U}{\partial r}$$
$$+ a_e^2\omega^4 = \left(\frac{\partial U}{\partial r} + \omega^2 a_e\right)^2,$$

or

$$g_e = -\left(\frac{\partial U}{\partial r}\right)_{\left(\substack{r=a_e\\ \theta=0}\right)} - \omega^2 a_e. \qquad (73)$$

From (71)

$$-\left(\frac{\partial U}{\partial r}\right)_{\left(\substack{r=a_e\\ \theta=0}\right)} = \frac{\tau}{a_e^2}\left[1 + J + K/2\right]. \qquad (74)$$

From (73) and (74)

$$g_e = \frac{\tau}{a_e^2}\left[1 + J + K/2\right] - \omega^2 a_e$$

$$= \frac{\tau}{a_e^2}\left[1 + J + K/2 - \frac{\omega^2 a_e^3}{\tau}\right]$$

$$g_e = \frac{\tau}{a_e^2}\left[1 + J + K/2 - m\right],\ m = \omega^2 a_e^3/\tau. \quad (75)$$

Substituting the values of J and K from (67) in (75) gives

$$g_e = \frac{\tau}{a_e^2}(1 + f + f^2) - \frac{\tau}{a_e^2}\ m\left(\frac{3}{7}f + \frac{3}{2}\right)$$
$$= \frac{\tau}{a_e^2}(1 + f + f^2) - \omega^2 a_e\left(\frac{3}{7}f + \frac{3}{2}\right),$$

and solving this last equation for τ

$$\tau = \frac{a_e^2}{1 + f + f^2}\left[g_e + \omega^2 a_e\left(\frac{3}{7}f + \frac{3}{2}\right)\right]$$

$$= (1 - f)\, a_e^2\left[g_e + \omega^2 a_e\left(\frac{3}{7}f + \frac{3}{2}\right)\right]. \quad (76)$$

To derive the general formula for gravity from (68), the formulas (69) through (72) are used with the values of r/a_e and a_e/r from (57) and (58), negligible terms being rejected in the process to find

$$-\frac{\partial W}{\partial r} = \frac{\tau}{a_e^2}\left[1 + f + f^2 - \frac{3}{7}mf - \frac{3}{2}m\right.$$
$$+\left(\frac{5}{2}m - f - f^2 + \frac{2}{7}mf\right)\sin^2\theta$$
$$\left.-\frac{1}{8}(11f^2 - 15mf)\sin^2 2\theta\right],$$

and to order f, (77)

$$\frac{1}{r}\frac{\partial W}{\partial \theta} = \frac{\tau}{a_e^2} f \sin 2\theta$$

From (68) by taking the square root and expanding to two terms by the binominal formula one finds

$$g = -\frac{\partial W}{\partial r}\left[1 + \frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)^2 \Big/ \left(\frac{\partial W}{\partial r}\right)^2\right].$$

From the first of (77)

$$\left(\frac{\partial W}{\partial r}\right)^2 = \tau^2/a_e^4$$

ignoring small terms and this value with the value of

$$\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)^2 = \tau^2 f^2 \sin^2 2\theta / a_e^4$$

from the second of (77), one may write

$$g = -\frac{\partial W}{\partial r}\left(1 + \frac{1}{2}f^2\sin^2 2\theta\right), \quad (78)$$

and then from the first of (77) substituted in (78) one finds

$$g = \frac{\tau}{a_e^2}\left[1 + f + f^2 - \frac{3}{7}mf - \frac{3}{2}m\right.$$
$$+\left(\frac{5}{2}m - f - f^2 + \frac{2}{7}mf\right)\sin^2\theta$$
$$\left.+\frac{1}{8}(15mf - 7f^2)\sin^2 2\theta\right]. \quad (79)$$

When $\theta = 0$, (79) gives

$$g = \frac{\tau}{a_e^2}\left(1 + f + f^2 - \frac{3}{7}mf - \frac{3}{2}m\right) = g_e$$

as seen from (67) and (75), and (79) may be written

$$g = g_e\left[1 + C_1\sin^2\theta + C_2\sin^2 2\theta\right],$$

where

$$C_1 = \frac{1}{g_e}\left(\frac{5}{2}m - f - f^2 + \frac{2}{7}mf\right) = \frac{5}{2}m + \frac{15}{4}m^2 - f - \frac{26}{7}mf$$
$$C_2 = \frac{1}{g_e}\left(\frac{15}{8}mf - \frac{7}{8}f^2\right) = \frac{15}{8}mf - \frac{7}{8}f^2, \quad (80)$$

Now formulas (80) are in terms of geocentric latitude, θ, and to change to geodetic latitude ϕ, in an expression like (80), one has

$$g/g_e = 1 + C_1\sin^2\theta + C_2\sin^2 2\theta$$
$$g/g_e = 1 + C_1\sin^2\phi + (C_2 - fC_1)\sin^2 2\phi$$

where C_1, is of order f and C_2 of order f^2, hence

$$C_2 - fC_1 = \frac{15}{8}mf - \frac{7}{8}f^2 - \frac{5}{2}fm + f^2 = \frac{1}{8}f^2 - \frac{5}{8}fm$$

and

$$g = g_e [1 + A \sin^2 \phi + B \sin^2 2\phi], \tag{81}$$

where

$$A = C_1 = \frac{5}{2} m + \frac{15}{4} m^2 - f - \frac{26}{7} m f$$

$$B = C_2 - f C_1 = \frac{1}{8} f^2 - \frac{5}{8} f m$$

Now the geodetic formulas (54), (57), (67), (75), (76), and (81) are essentially the ones used in the determination of the flattening from the orbit of 1958 β_2 (Vanguard 1, instrumented sphere)[5].

GEOCENTRIC COORDINATES OF A POINT ON THE SATELLITE ORBIT (KEPLERIAN)

Note in figure 21 the spherical right triangle whose sides are ω, θ, $\lambda - \lambda_0$. One can write

$$\begin{aligned} \sin\theta &= \sin\omega \sin i \\ \tan(\lambda - \lambda_0) &= \tan\omega \cos i \\ \cos\theta &= \cos\omega \sec(\lambda - \lambda_0) \\ &= \cos\omega (1 + \tan^2\omega \cos^2 i)^{1/2}, \end{aligned} \tag{82}$$

and with r from (19), considering ω_0 the argument of perigee, one can write from (43) and (82) the geocentric coordinates of the satellite, S, the orbit being considered a Keplerian ellipse:

$$\begin{aligned} x_s &= \frac{a(1-e^2)(\cos\lambda_0 \cos\omega - \sin\lambda_0 \sin\omega \cos i)}{1 + e\cos(\omega - \omega_0)} \\ y_s &= \frac{a(1-e^2)(\sin\lambda_0 \cos\omega + \cos\lambda_0 \sin\omega \cos i)}{1 + e\cos(\omega - \omega_0)} \\ z_s &= \frac{a(1-e^2)\sin\omega \sin i}{1 + e\cos(\omega - \omega_0)} \end{aligned} \tag{83}$$

Equations (83) are in terms of only one parameter, ω, since λ_0, ω_0, a, e, and i are considered constant for the undisturbed orbit. The Keplerian orbit lies in the plane

$$x \sin\lambda_0 - y \cos\lambda_0 + z \cot i = 0, \tag{84}$$

which is fixed, since λ_0 and i are considered fixed for an undisturbed orbit, and contains the line of nodes and the radius vector r. (See fig. 21.) Since the asymmetrical gravity field causes the line of nodes to regress the plane (84) continually rotates about the earth's axis, and its rate of rotation is simply related to the angular rate of travel of the satellite in the plane. From the second of equations (82), $\lambda_0 = \lambda - \arctan(\tan\omega \cos i)$, whence

$$\frac{d\lambda_0}{dt} = \frac{d\lambda}{dt} - \frac{\cos i}{1 - \sin^2 i \sin^2\omega} \frac{d\omega}{dt}. \tag{85}$$

Actually an instantaneous ellipse or osculating ellipse is used as a basis for defining the coordinates x_S, y_S, z_S of the satellite, although the actual trajectory of the satellite will not be in the plane of the orbit but inclined at a slight angle to it.

DIFFERENTIAL EQUATIONS OF MOTION

The satellite problem may be considered as essentially a ballistics problem and comparison orbits have been computed from this view point from the differential equations:

$$\begin{aligned} \ddot{x} &= U_0 \frac{x}{r} - B \rho V \dot{x} \\ \ddot{y} &= U_0 \frac{y}{r} - B \rho V \dot{y} \\ \ddot{z} &= U_1 \frac{z}{r} - B \rho V \dot{z} \end{aligned} \tag{86}$$

where U_0, U_1, are derived from the potential for a point external to the spheroid, B is the drag function, ρ is mass density of air at satellite altitude, r is the radial distance from center of the earth to satellite, V is the orbital velocity of the satellite; x, y, z are geocentric inertial coordinates.[6]

The commonly used method is to neglect the air resistance problem or account for it separately, and consider the motion of the satellite as a disturbed elliptic orbit, the oblateness effects disturbing or altering the Keplerian parameters a, e, λ_0, ω_0, i, t_0 where t_0 is the time when the satellite passes through perigee. That is, the continuous nonplanar space trajectory curve is approximated by an ellipse of best fit or osculating ellipse for each transit period, multiple or submultiple of the period. There are several methods of doing this and the literature on them is becoming quite voluminous.[7] The basic principles are found in any treatise on celestial mechanics.[8]

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[5] *Journal of Geophysical Research*, February 1959, page 211.

[6] The effect of the earth's oblateness and atmosphere on a satellite orbit by J. DeNike in Vistas in Astronautics, *Pergamon Press*, London, 1958.

[7] Some recent papers are: Application of Hansen's theory to the motion of an artificial satellite in the gravitational field of the earth by P. Musen, *Journal of Geophysical Research*, December 1959; Separate papers by B. Garfinkle, Y. Kozai, D. Brouwer, J. P. Vinti, *Astronomical Journal*, November 1959. *Journal of Research*, NBS, B. Math. and Math. Physics, Vol. 63B., Oct.-Dec. 1959; Motion of a satellite in the earth's gravitational field by G. V. Groves, *Proceedings of the Royal Society*, Series A, 19 January 1960.

[8] For instance: Celestial mechanics by W. M. Smart; *Longmans, Green and Co.*, New York, 1953.

If from (54) one writes

$$R = U - \frac{\tau}{r} = \tau a_e^2 \frac{J}{r^3}\left(\frac{1}{3} - \sin^2\theta\right), \qquad (87)$$

including only the term in J, and eliminating thus the purely spherical term $\frac{\tau}{r}$, where $\tau = GM$ (G the gravitational constant and M the mass of the earth), then R may be considered the gravitational "disturbing" function. Now in the first of equations (82), let $\omega = v + \omega_0$, or equivalently in (19) let $v = \omega - \omega_0$, (v is called the true anomaly), and write $\sin\theta = \sin(v + \omega_0)\sin i$, whence

$$\sin^2\theta = \sin^2 i \sin^2(v + \omega_0)$$
$$= \frac{1}{2}\sin^2 i\,[1 - \cos 2(v + \omega_0)]. \qquad (88)$$

with the value of $\sin^2\theta$ from (88) placed in (87)

$$R = \tau \frac{a_e^2}{a^3}\left(\frac{a}{r}\right)^3\left[\frac{1}{3} - \frac{1}{2}\sin^2 i + \frac{1}{2}\sin^2 i \cos 2(v + \omega_0)\right]. \qquad (89)$$

Now the mean angular velocity of the satellite is given by $n = 2\pi/T$ where T is the orbital period. If t_0 is the time when the satellite passes through perigee, and t is the time when the satellite is at some other point of its orbit, then the angle

$$M_0 = n(t - t_0), \qquad (90)$$

is called the mean anomaly. By means of the relation $\frac{dv}{dM_0} = \left(\frac{a}{r}\right)^2(1 - e^2)^{1/2}$, the disturbing function R of (89) may be transformed to a function of M_0 and ω_0. If this transformation is made then terms depending neither on M_0 or ω_0 are called ***secular*** (constant or cumulative), those depending on ω_0 but not on M_0 are ***long-periodic***, and terms depending on M_0 are ***short-periodic***. Hence from (89) the ***secular*** terms of R will be given by

$$R_s = \tau \frac{a_e^2}{a^3} J\left(\frac{a}{r}\right)^3\left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right), \qquad (91)$$

and

$$R_0 = \frac{1}{2\pi}\int_0^{2\pi} R_s\, dM_0$$
$$= \frac{J}{2\pi}\tau \frac{a_e^2}{a^3}\left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right)\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM_0. \qquad (92)$$

Now if $\left(\frac{a}{r}\right)^3$ is expanded in terms of M_0,[9] one has

$$\left(\frac{a}{r}\right)^3 = \left(1 + \frac{3}{2}e^2 + \frac{15}{8}e^4 + \frac{35}{16}e^6 + \ldots\right) + \sum_{j=1}^{\infty} f_j(e)\cos j M_0$$
$$= (1 - e^2)^{-3/2} + \sum_{j=1}^{\infty} f_j(e)\cos j M_0,$$

whence

$$\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM_0 = \int_0^{2\pi}(1 - e^2)^{-3/2} dM_0 + \sum_{j=1}^{\infty} f_j(e)\int_0^{2\pi}\cos j M_0\, dM_0$$
$$= 2\pi(1 - e^2)^{-3/2} + 0,$$

so finally

$$R_0 = J\tau \frac{a_e^2}{a^3}\left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right)(1 - e^2)^{-3/2}. \qquad (93)$$

The differential equation for $\frac{d\lambda_0}{dt}$ is of course not (85) but must be expressed in terms of R_0,

$$\frac{d\lambda_0}{dt} = \frac{1}{na^2(1 - e^2)^{1/2}\sin i}\frac{\partial R_0}{\partial i} \qquad (94)$$

and the argument of the perigee rate is [10]

$$\frac{d\omega_0}{dt} = -\frac{\cot i}{na^2(1 - e^2)^{1/2}}\frac{\partial R_0}{\partial i} + \frac{(1 - e^2)^{1/2}}{na^2 e}\frac{\partial R_0}{\partial e} \qquad (95)$$

From (93), (94), and (95) are then obtained the secular perturbations of first order. From (93)

$$\frac{\partial R_0}{\partial i} = -J\tau \frac{a_e^2}{a^3}(1 - e^2)^{-3/2}\sin i \cos i,$$
$$\frac{\partial R_0}{\partial e} = 3J\tau \frac{a_e^2}{a^3} e\left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right)(1 - e^2)^{-5/2} \qquad (96)$$

From the first of (96) and equation (94)

$$\frac{d\lambda_0}{dt} = -\frac{nJ\tau a_e^2 \cos i}{n^2 a^3 a^2(1 - e^2)^2} \qquad (97)$$

Now $n^2a^3 = GM = \tau$, and letting $p^2 = \frac{a^2}{a_e^2}(1 - e^2)^2$ then $\frac{d\lambda_0}{dt} = -\frac{nJ\cos i}{p^2}$, and since $n = 2\pi/T$, one can write (97) finally as

$$\frac{\delta\dot{\lambda}}{2\pi} = -\frac{J\cos i}{p^2}, \qquad (98)$$

[9] *Ibid.*, pages 41, 378.

[10] *Ibid.*, page 69.

where $\delta\dot{\lambda}$ is in radians per period.

From (95) and (96) one finds

$$\frac{d\omega_0}{dt} = \frac{n J a_e^2}{a^2(1-e^2)}\left(\cos^2 i + 1 - \frac{3}{2}\sin^2 i\right)$$
$$= \frac{nJ}{p^2}\left(2 - \frac{5}{2}\sin^2 i\right),$$

$n = 2\pi/T$, or finally

$$\frac{\delta\dot{\omega}_0}{2\pi} = \frac{J}{p^2}\left(2 - \frac{5}{2}\sin^2 i\right), \qquad (99)$$

where $s\dot{\omega}_0$ is in radians per period.

Note the factor $(2 - \frac{5}{2}\sin^2 i)$ in (99). This term is zero when i = arc sin (0.89444) = 63°26!2 = 63°4. That is when $i < 63°4$, the motion of perigee is in the direction of the motion of the satellite. When $i > 63°4$ the motion of perigee is opposite or retrograde with respect to that of the satellite.

Formulas (98), (99) together with additional ones to include secular terms of second order, were used to determine J and K and hence f from equations (67).[11] Analogous differential equations for the other elliptic elements are used with (94) and (95) and when the "osculating" or approximating ellipse is found for a particular time, the geocentric coordinates are found from equations analogous to (83).

Since the number of terms in considering higher order perturbations is large and their evaluation cumbersome, the more practical approach may be that from the ballistic problem as symbolized in equations (86). Air density at altitude is becoming better known and if satellites are spherical in shape (the drag function for spheres is well known), the equations should be more amenable to iterative numerical integration techniques. In attempting to avoid the more cumbersome perturbative techniques the method recently proposed by J. P. Vinti of the National Bureau of Standards should be investigated since it reduces the problem of satellite motion to quadratures, with use of a potential field that is much closer to the empirically accepted one for the earth than any used thus far as the starting point of calculations and may thus make possible the development of the gravitational theory of a satellite orbit very accurately without use of perturbation theory.[12] Air drag effects will have to be accounted for separately.

Equation (54) expresses U as a function of geocentric latitude, θ, and the radial distance, r, from the center of the earth. This form is adequate for studying the coefficients J, H, K relative to the flattening. But for detailed studies of the gravity field of the earth, U must be also a function of the right ascension (longitude) and higher order terms will have to be included. Their separation and evaluation will then depend on data from several satellites at different angles of inclination. Their evaluation will also depend on comparison of the observed and predicted positions of each satellite. While the motion of the line of nodes (the rotation of the plane (84) about the rotation axis of the earth) can be deduced from the observations, the motion of perigee in the instantaneous plane of the orbit cannot readily be determined from the observations and no detailed comparison of observation with theory has been published.

The inclination, i, is to a first approximation at least, theoretically unaffected by either atmospheric resistance or gravitational asymmetry. But the actual inclinations are observed to decrease gradually. This is attributed to a component of the air drag, normal to the direction of motion, caused by the rotation of the atmosphere with respect to the orbital plane. Such effects should be investigated and allowed for in the secular variations of the node and perigee ***before coefficients derived from the observations are adopted.***

SOME VALUES OBTAINED FROM SATELLITE OBSERVATIONS

Some published values of $F = 1/f$, where f is the flattening, which have been determined from satellite data were given in the discussion of table 3. The results indicated adoption of $F = 298.3 \pm 0.1$. The corresponding value of $J = 1623.4 \pm 4 \times 10^{-6}$ has been deduced.

The coefficients H and K in equation (54) have been determined by O'Keefe, Echels, and Squires[13] from the orbit of $1958\beta_2$ (Vanguard) as $H = 6 \times 10^{-6}$, $K = 6.4 \times 10^{-6}$. The undulation of the geoid corresponding to H is about 15 meters which is of course negligible when compared to a mean earth radius of about 6,371,000 meters. But the work is important from the standpoint of the methods used to eliminate the effects of air drag from the observations; and the possible indications of the crustal loadings of the earth.

By combining the data from Sputnik 2 and Vanguard 1, King-Hele and Merson[14] find $J = (1624.6 \pm 0.3) \times 10^{-6}$, $K = (6 \pm 1) \times 10^{-6}$ and with the aid of

[11] *Journal of Geophysical Research*, February 1959, pages 209-216.

[12] New method of solution for unretarded satellite orbits by J. P. Vinti, *Journal of the National Bureau of Standards*, B. Mathematics and Mathematical Physics, October-December 1959.

[13] The gravitational field of the earth by O'Keefe, Eckels, and Squires, *The Astronomical Journal*, September 1959.

[14] A new value for the earth's flattening derived from measurements of satellite orbits by D. G. King-Hele and R. H. Merson, *Nature*, March 28, 1959, page 882.

orbital data from Explorer 4 (1958 ϵ), they find as a preliminary estimate that the coefficient of the 6th harmonic in the potential is $(0.1 \pm 1.5) \times 10^{-6}$. Their revised estimate of F is 298.2 ± 0.03.

GENERAL REMARKS ON OBSERVATIONS

From figure 21, if an observer is at P, then in order to obtain the geocentric coordinates of the observing site, it will be necessary to know the geocentric latitude, the longitude and height above or below the geoid of the site. The observations from such a site are thus necessarily "topocentric" with reference to the local horizon or local equator system. For optical data obtained from such a site, where the position of the satellite is photographed against a star background, the geocentric coordinates of the site may have to be corrected for precession and nutation.[15]

ORBIT INFORMATION FROM DOPPLER, INTERFEROMETER, AND RADAR DATA

In the Doppler system if the received and transmitted frequencies are F and F_0, they are related by

$$F = F_0 \, (1 + \dot{d}/c) \tag{100}$$

where d is the distance between the satellite and the observer and c is the velocity of light (or radio waves). If the frequency is recorded as the satellite approaches and recedes, the radial velocity $\dot{d}$ can be calculated from

$$\dot{d} = c \, (F - F_0) / F_0 \tag{101}$$

At the instant of closest approach the radial velocity will be zero, for then the relative velicity, V, is orthogonal to the track line (see fig. 22). After t seconds the satellite will have moved a distance Vt from the point of closest approach whence

$$d^2 = d_0^{\,2} + V^2 t^2 \tag{102}$$

where d_0 is the minimum distance as shown in figure 22. If equation (102) is differentiated with respect to time one has

$$d\,\dot{d} = V^2 t,$$

and squaring this result

$$d^2 \dot{d}^2 = V^4 t^2,$$

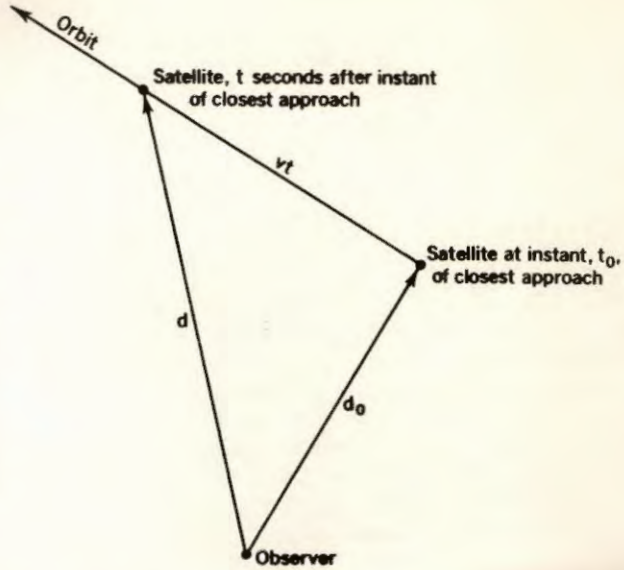

FIG 22.—Relation between distance and relative velocity in Doppler reductions.

or

$$\frac{t^2}{\dot{d}^2} = \frac{d^2}{V^4} \tag{103}$$

Substituting the value of d^2 from (102) in (103) obtain

$$\frac{t^2}{\dot{d}^2} = \frac{t^2}{V^2} + \frac{d_0^{\,2}}{V^4} \tag{104}$$

Now if in (104) one lets $y = \left(\frac{t}{s}\right)^2$ where from (101) $s = F_0 \, \dot{d}/c = F - F_0$, and $x = t^2$, then (104) may be written

$$y = m\,x + h, \tag{105}$$

where

$$m = \left(\frac{c}{F_0}\right)^2 \frac{1}{V^2}, \quad h = \left(\frac{C}{F_0}\right)^2 \frac{d_0^{\,2}}{V^4},$$

or (106)

$$V^2 = \left(\frac{C}{F_0}\right)^2 \frac{1}{m}, \quad d_0^{\,2} = \left(\frac{c}{F_0}\right)^2 \frac{h}{m^2}.$$

It is seen that by fitting the line (105) to the sets of values $x_i = t_i^2$, $y_i = \left(\frac{t_i}{s_i}\right)^2$, $i = 1, \ldots, n$ by the method of least squares the slope m and intercept h can be determined whence the relative velocity, V, and minimum distance, d_0, can be computed from (106). Initially the time, t_0, of closest approach and the transmitted frequency, F_0, are estimated from the symmetry of the frequency-time curve. The observed frequency, F, can then be reduced to a table of $\dot{d}$ versus t, where $\dot{d}$ is given by (101), and t is measured from t_0, the

[15]See Triangulation—A precise method for satellite tracking by P. G. Kirmser and I. Wakabayashi, *Journal of the Franklin Institute*, November 1959; and Geocentric coordinates from lunar and satellite observations by W. Markowitz, *Bulletin Geodesique*, No. 49, 1958.

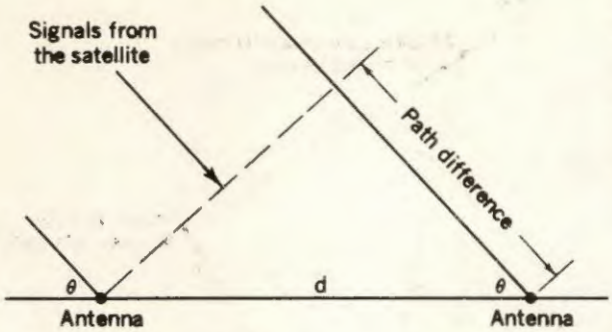

FIG. 23.—Principle of the interferometer technique.

estimated time of closest approach. In the least square fitting of the line (105), the time t_0 is varied until the mean square error is a minimum. Obviously the values of t_0, V, and d_0 are all the information that can be extracted by the Doppler technique from a single transit of the satellite at one station.

The technique suffers from the effects of ionospheric refraction particularly at lower frequencies. The orbital period can be found by observing successive transits at the same station but a correction must be applied to allow for the change in the observer's position relative to the satellite orbit due to the rotation of the earth in the intervening period.

If the orbit is to be obtained from Doppler data alone, the accuracy will be affected by the separation distances between the observing stations, and between the observing stations and the satellite; the curvatures of the earth and of the orbit must be taken into account in the analysis.

The interferometer technique makes use of the interference pattern between the signals received at a pair of antennas, the pattern determined by the difference in the distance between the satellite and each of the receiving antennas. The separation distance, d, is small compared with the satellite distance and hence the waves from the transmitter may be considered to traverse parallel paths as shown in figure 23.

When the satellite is at an angle, θ, such that the path difference is exactly equal to half a wave length,

$$\text{Path difference} = \frac{\lambda}{2} = d \cos \theta \qquad (107)$$

then the combined output from the two antennas is zero, since the separate waves interfere completely, and this will occur whenever the path difference is equal to an odd number of half wave lengths. As the satellite transits its orbit, the change in direction of incoming radiation (the angle θ) causes a succession of such nulls or zeros, the pattern depending on the direction of the track with respect to the line joining the two antennas. During the interval between successive nulls, the increase in $\cos \theta$ is λ/d, so that a record of the times at which the nulls occur gives immediately the rate of change of the direction cosine. If the phase difference of the signals at the two antennas is measured, instead of simply adding the signals at the two antennas, a continuout indication of $\cos\theta$ is available, and the measurement is practically independent of amplitude fluctuations.

The interferometer is insensitive to changes of a whole number of wave lengths in the path difference and ambiguities occur in the measurement of the angle θ. These amgibuities may be resolved by means of a second interferometer with a pair of antennas placed much closer together giving in effect coarse and fine indication.

A second direction cosine can be obtained from a similar set of antennas placed at right angles to the first set, whence it is possible to define the direction of the line joining the satellite to the observing station during the satellite transit. If the distance and relative velocity are obtained simultaneously from Doppler data, the orbit may then be deduced.

The principal American tracking systems employing the interferometer principle are the Minitrack and Microlock.[16] Although the accuracy of Minitrack data is only about 3 to 4 minutes of arc, the continuous orbit data can be averaged through several orbit transits and the results used in the determination of the flattening. Such data has been the major basis for the recently published determinations of the flattening.

The use of large radio telescopes for tracking purposes is very difficult because of the problem of locating the satellite even when its general position is predictable, and because of the confusion introduced by echos from other sources as meteors, aurorae, etc.

The geometrical limitations relative to intercepting the satellite are shown in figure 24 where E is the elevation of the radio telescope, 2θ its beam width, h the altitude of the satellite, R the ground range. Typical relative values are as follows:

Frequency *(mc./s.)*	θ *(deg.)*	E *(deg.)*	R *(km.)*	Δh *(km.)*	ΔR *(km.)*
120	1.25	10	1,500	30	75
36	4	30	500	30	70
36	4	90	500	∞	30

Hence for certain contact with a satellite its position must be predictable to approximately 30

[16]Tracking satellites by radio by J. T. Mengel and Paul Herget, *Scientific American*, January 1958. Microlock by H. L. Richter, Jr., et al in Vistas in Astronautics, *Pergamon Press*, London, 1958.

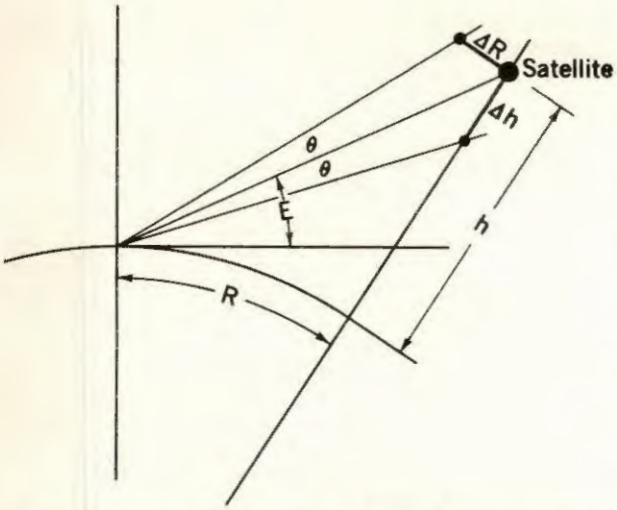

FIG. 24.--Geometry involved in locating a satellite using a radio telescope with beam width 2θ.

km. in height and to about 70 km. in ground range. (This data is relative to operation of the 250-foot radio telescope of the Jodrell Bank Experimental Station).[17] However, the radio telescope has the two unique advantages of being independent of satellite-borne equipment and of ambient meteorological conditions.

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Satellite	Code name	Launch date	Status	Period	Inclination	Apogee	Perigee	Transmitter frequencies
				(min.)	*(deg.)*	*(n.m.)*	*(n.m.)*	*(mc.)*
1958 δ_2	Sputnik 3	15 May 58	Down 6 Apr. 60					
1959 λ	Discoverer 8	20 Nov. 59	Down 8 Mar. 60					
1958 α	Explorer 1	31 Jan. 58	In orbit	108.1	33.22	1054	186	
1958 β_1	Rocket body	17 Mar. 58	do	138.2	34.26	2327	352	
1958 β_2	Vanguard 1	17 Mar. 58	do	134.0	34.27	2134	352	108.022
1959 α_1	Vanguard 2	17 Feb. 59	do	125.5	32.87	1784	302	
1959 α_2	Rocket body	17 Feb. 59	do	129.7	32.89	1986	300	
1959 ϵ_2	Capsule	13 Aug. 59	do	101.5	78.94	787	105	
1959 η	Vanguard 3	18 Sept. 59	do	129.9	33.33	2020	275	19.9904
1959 ι_1	Explorer 7	13 Oct. 59	do	101.2	50.31	585	299	
1959 ι_2	Rocket body	13 Oct. 59	do	101.1	50.30	580	298	
1960 β_1	Rocket body	1 Apr. 60	do	99.1	48.41	406	373	
1960 β_2	Tiros 1	1 Apr. 60	do	99.2	48.41	407	373	108.03; 107.997
1960 γ_1	Rocket body	13 Apr. 60	do	95.0	51.29	389	173	
1960 γ_2	Transit 1 B	13 Apr. 60	do	95.7	51.28	399	202	54; 162; 216; 324
1960 γ_3	Metal object	13 Apr. 60	do	94.0	51.29	361	151	
1960 δ	Discoverer 11	15 Apr. 60	Down 26 Apr. 60					
1960 ϵ_1	Sputnik 4	15 May 60	In orbit	94.3	64.89	357	164	19.995
ϵ_2	Rocket body	do	do	94.1	do	193	164	
ϵ_3	None	do	do	94.3	do	358	164	
ϵ_4	do	do	do	94.3	do	371	151	
ϵ_5	do	do	do	94.4	do	368	157	
ϵ_6	do	do	do	94.4	do	370	157	
ϵ_7	do	do	do	94.5	do	375	156	
ϵ_8	do	do	do	94.5	do	365	169	
ϵ_9	do	do	do	94.5	do	377	157	
1960 ζ	Midas 2	24 May 60	do	94.1	32.62	264	260	

NOTE: As of 1 June 1960 there were 24 artificial earth satellites in orbit, 9 of which (1960, ϵ_1-ϵ_9) resulted from the one launching of Sputnik 4.

FIG. 25.—Abstracted satellite status as of June 1, 1960. (From U. S. Air Force Cambridge Space Track report.) Compare with Figure 1.

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