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# Transformation of Rectangular Space Coordinates

ERWIN SCHMID

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Technical Bulletin No. 15 is a compilation of the basic geodetic concepts and the formulas necessary for the transformations of geodetic coordinates to three-dimensional rectangular coordinates.

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# Transformation of Rectangular Space Coordinates

ERWIN SCHMID, Mathematician

U. S. Coast and Geodetic Survey

## 1. INTRODUCTION

FIXING the position of points on the earth's surface by means of the two variables latitude and longitude is one of the most familiar examples of the use of a curvilinear coordinate system, leading to a (two-dimensional) non-Euclidean geometry, the geometry of curved spaces. This description of points is not only more elegant, in a mathematical sense, than would be a three-dimensional rectangular coordinate system but is a practical necessity as well. A rectangular (Cartesian) coordinate system implies the measurement of distances along straight lines, whereas the geodesist is, in effect, restricted to measuring along the curved surface of the earth.

On the other hand, in dealing with points not lying on the assumed surface of the earth, it is necessary to introduce a third dimension, and for many purposes it is convenient to introduce a three-dimensional rectangular coordinate system, familiarly known as an $x, y, z$ or $i, j, k$ system, with its concomitant Euclidean metric — the Pythagorean theorem.

The conversion of geodetic coordinates to $x, y, z$ coordinates requires only the most elementary notions of analytic geometry in addition to some basic geodetic concepts, but as the latter are not universally known to scientists whose fields have in recent years, with the development of missiles, been broadened to include geodesy, it was felt that a compilation of such formulas would serve a useful purpose.

## 2. LATITUDE AND LONGITUDE

Latitude $\phi$ and longitude $\lambda$ are known as spherical coordinates, and have been used by astronomers since antiquity to designate the position of points on the celestial sphere, such as stars, as well as of points on the earth, assumed spherical — which it was, within the limits of error of the observational methods in use until Newton's time. Newton's conjectures, and subsequent refinements in observational techniques, have shown that an oblate ellipsoid of revolution is a much closer approximation to the figure of the earth, but the original concept of latitude, as the elevation of the pole, has been retained because this is a physical, measurable quantity. The latitude thus *observed* is in fact the astronomic latitude and not the geodetic latitude which is a mathematical fiction arising from *computation* on an assumed ellipsoid; but as a first approximation the two are reasonably close. At any rate, after having computed a geodetic position on the ellipsoid by means of triangulation or traverse, we set the geodetic latitude there equal to the elevation of the pole relative to the ellipsoid normal or, equivalently, to the angle at which the normal meets the plane of the equator (fig. 1).

The longitude $\lambda$ on the ellipsoid is, as in the case of the sphere, the angle which the meridian plane of a point makes with an arbitrarily designated, fixed meridian plane.

## 3. THE MERIDIAN ELLIPSE

Figure 1 shows the meridian ellipse, a cross section of the reference ellipsoid through the axis of revolution, this latter being the $y$ axis. The $x$ axis lies in the plane of the equator; the origin $O$ is the center ot the ellipse and of the ellipsoid generated by revolving the

meridian ellipse about the y axis. With semimajor axis a and semiminor axis b as indicated in figure 1, the equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$. From this equation all the quantities needed for the geometry of the ellipsoid can be computed.

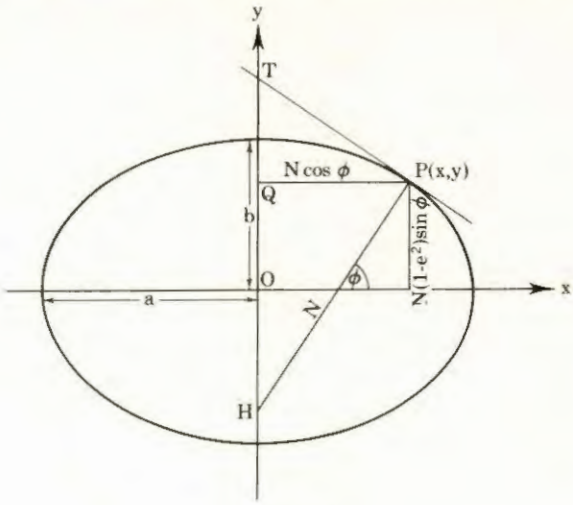


FIGURE 1

In particular, by differentiation, the slope of the normal PH is $-dx/dy = a^2y/b^2x$. From the figure, on the other hand, this slope is $\tan\phi$. Hence

$$\frac{a^2y}{b^2x} = \tan\phi \tag{3.1}$$

where x and y are the abscissa and ordinate, respectively, of P. Designating the length PH by N, it follows from figure 1, that

$$x = N\cos\phi\,. \tag{3.2}$$

Substituting (3.2) in (3.1) and setting $e^2 = 1 - b^2/a^2$,

$$y = N(1-e^2)\sin\phi\,. \tag{3.3}$$

With these values of x and y, the equation of the ellipse is

$$\frac{N^2\cos^2\phi}{a^2} + \frac{N^2(1-e^2)^2\sin^2\phi}{b^2} = 1$$

whence, solving for N,

$$N = \frac{a}{(1-e^2\sin^2\phi)^{1/2}} \tag{3.4}$$

## 4. DEFLECTION OF THE VERTICAL

With each geodetic position $(\phi_G, \lambda_G)$ on the ellipsoid can be associated the point $(\phi, \lambda)$ on the celestial sphere to which the ellipsoid normal at the geodetic position is directed—the

so-called geodetic zenith. The initial assumption in geodetic computation is that this normal coincides with the direction of the gravity vector. Actually, however, the gravity vector is not normal to any artificially created mathematical surface but to the level surfaces of its potential field. To the extent that a designated one of these equipotential surfaces, called the geoid, deviates at a point from the ellipsoid, will the direction of the gravity vector at that point fail to coincide with the direction of the ellipsoid normal. Thus the gravity vector at $(\phi_G, \lambda_G)$ points toward the point $(\phi_A, \lambda_A)$ on the celestial sphere, known as the astronomical zenith, which differs quite generally from the geodetic zenith. The angle between these two diverging normals is called the deflection, or deviation, of the vertical.

## 5. THE GAUSS SPHERE

A simple way to deal with these concepts is to think of the celestial sphere as shrunk to unit radius and centered at the center of the ellipsoid, as Gauss did in his spherical representation of a surface. He assigned to each point of a surface with its given surface normal, that point on the unit sphere in which the radius parallel to the normal pierces the sphere. It is obvious from this construction that the relative positions of points on the ellipsoid or, for that matter, on the geoid, mapped on the Gauss sphere are the same on this latter sphere as the corresponding images on the celestial sphere. Hence, for example, the deflection of the vertical is numerically equal to the arc length of the great circle segment connecting the images of the astronomic and geodetic positions on this unit sphere, and the direction cosines of the ellipsoid normal at point P are the $x, y, z$ coordinates of the image $P'$ of P on the sphere. It also makes clear why, given the direction cosines of the ellipsoid normal in terms of the geodetic position of a point, we merely substitute the corresponding astronomic position to get the direction cosines of the gravity vector at the same point.

## 6. RECTANGULAR COORDINATES OF A GEODETIC POINT

We are now in a position to establish a rectangular coordinate system with prescribed orientation and an origin $P_0$ whose position in space is given relative to a geodetic system, using the formulas of analytic geometry or vector analysis.

In figure 2 let the meridian plane of $P_0$ be the XZ plane of a right-handed Cartesian coordinate system, with origin O at the center of the ellipsoid, the Z axis coinciding with the axis of rotation pointing north, and the equatorial plane the XY plane. The X axis points from O toward the side of the meridian on which $P_0$ lies. Given are the geodetic coordinates $(\phi_0, \lambda_0)$ of $P_0$ and its elevation $h_0$ above the reference ellipsoid. (Elevations with respect to the ellipsoid are not directly measurable, but can be approximated from the corresponding geoid elevations, if sufficient data are available to determine the distance between the two surfaces.)

Let P $(\phi, \lambda)$ be any point on the ellipsoid, and $\lambda' = \lambda_0 - \lambda$, i.e., $\lambda'$ is longitude with reference to the meridian of $P_0$, measured positive eastward. The rectangular coordinates of P are therefore, from figure 2 and figure 1:

$$X = OK = N \cos \phi \cos \lambda'$$

$$Y = OL = N \cos \phi \sin \lambda'$$

$$Z = MP = N (1 - e^2) \sin \phi$$

where N is the length of the normal PH at P, and e the eccentricity of the meridian ellipse. The quantity N is one of the principal radii of curvature of the ellipsoid and hence extensively tabulated in the geodetic literature together with the second principal radius of

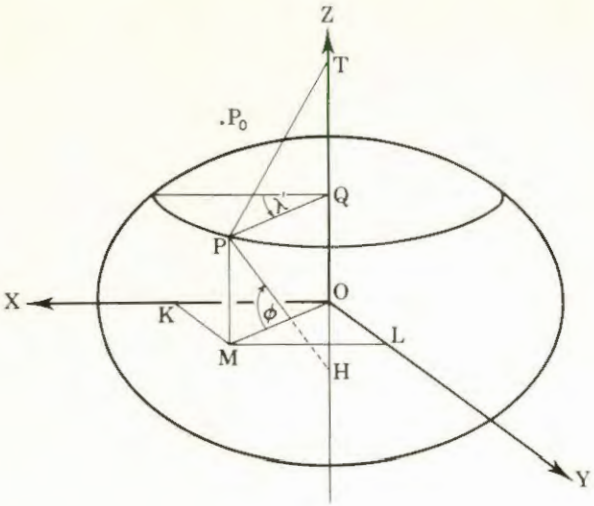


FIGURE 2

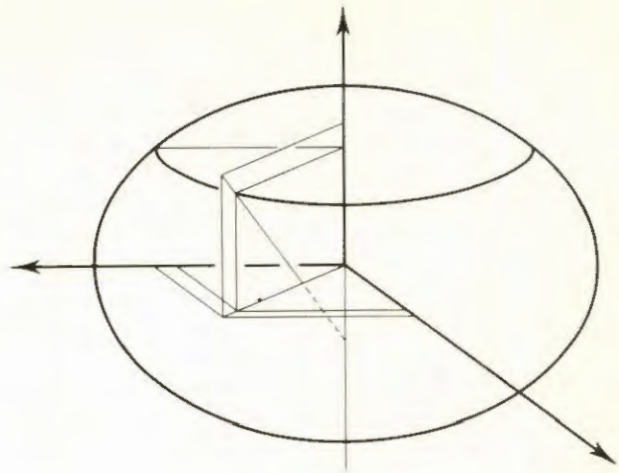

FIGURE 3

curvature R. R is also the radius of curvature of the meridian ellipse. Its length $a(1-e^2)/(1-e^2 \sin^2\phi)^{3/2}$ can easily be derived from the equation of the meridian ellipse, using the formula for curvature of a plane curve, but does not enter into the formulas used here.

With P a distance h, measured along the normal, above the ellipsoid surface it is apparent from figure 3 that its coordinates will be

$$
\begin{aligned}
X &= (N+h)\cos\phi\cos\lambda' \\
Y &= (N+h)\cos\phi\sin\lambda' \qquad (6.1)\\
Z &= [N(1-e^2)+h]\sin\phi .
\end{aligned}
$$

In particular the coordinates of $P_0$ are

$$
\begin{aligned}
X_0 &= (N_0+h_0)\cos\phi_0 \\
Y_0 &= 0 \qquad (6.2)\\
Z_0 &= [N_0(1-e^2)+h_0]\sin\phi_0 .
\end{aligned}
$$

Now translate the origin from O to $P_0$ and rotate the X and Z axes in the XZ plane, as indicated in figure 4, so as to make the direction of the Z axis coincide with the outward direction of the ellipsoid normal at $P_0$. This leaves the direction of the Y axis and the Y coordinates unchanged and rotates the X and Z axes through an angle of $-(90^\circ-\phi_0)$. Substituting this angle in the familiar linear transformation formulas and calling the coordinates with respect to the new axes, x, y, and z the transformed coordinates are

$$
\begin{aligned}
x &= (X-X_0)\sin\phi_0 - (Z-Z_0)\cos\phi_0 \\
y &= Y \qquad (6.3)\\
z &= (X-X_0)\cos\phi_0 + (Z-Z_0)\sin\phi_0 .
\end{aligned}
$$

This result is also readily apparent by inspection of figure 5 in which an arbitrary point P is projected on both sets of axes.

Substituting the quantities (6.1) and (6.2) into (6.3), the rectangular coordinates of a point in space in terms of its geodetic coordinates, and its perpendicular distance $h$ from the ellipsoid are

$$
\begin{aligned}
x &= (N+h)[\sin(\phi_0-\phi) - \sin\phi_0 \cos\phi\,(1-\cos\lambda')] \\
&\quad + e^2N \sin\phi \cos\phi_0 - e^2N_0 \sin\phi_0 \cos\phi_0 \\
y &= (N+h)\cos\phi \sin\lambda' \\
z &= (N+h)[\cos(\phi_0-\phi) - \cos\phi_0 \cos\phi\,(1-\cos\lambda')] \\
&\quad - e^2N \sin\phi_0 \sin\phi - (N_0+h_0) + e^2N_0 \sin^2\phi_0
\end{aligned}
\tag{6.4}
$$

The quantities with zero subscripts are the constant functions of the origin $P_0$ of the system, and $\lambda' = \lambda_0 - \lambda$ is longitude of $P$ with reference to the meridian of $P_0$.

## 7. DIRECTION OF THE GRAVITY VECTOR

Since equations (6.4) give rectangular coordinates of a point on the ellipsoid normal at $(\phi, \lambda')$ in terms of the parameter $h$, the coordinates of a second point on this normal can be found by, say, setting $h = 0$ in (6.4). The direction cosines of the line joining two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ being proportional to the three differences $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$, we have for the ellipsoid normal

$$
\begin{aligned}
x_2 - x_1 : y_2 - y_1 : z_2 - z_1 &= h[\sin(\phi_0-\phi) - \sin\phi_0 \cos\phi\,(1-\cos\lambda')] : \\
&\quad h\cos\phi \sin\lambda' : h[\cos(\phi_0-\phi) - \cos\phi_0 \cos\phi\,(1-\cos\lambda')].
\end{aligned}
$$

Dividing through by $h$, the length of the segment, the direction cosines of the normal are found to be

$$
\begin{aligned}
&\sin(\phi_0-\phi) - \sin\phi_0 \cos\phi\,(1-\cos\lambda') : \cos\phi \sin\lambda' : \\
&\cos(\phi_0-\phi) - \cos\phi_0 \cos\phi\,(1-\cos\lambda')
\end{aligned}
\tag{7.1}
$$

where $\phi$ and $\lambda'$ correspond to the geodetic position of the point in question.

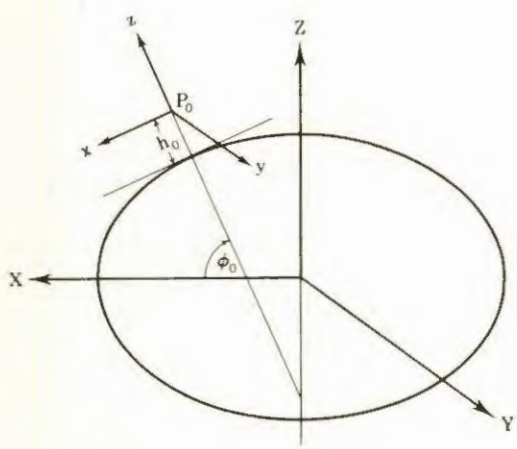


FIGURE 4

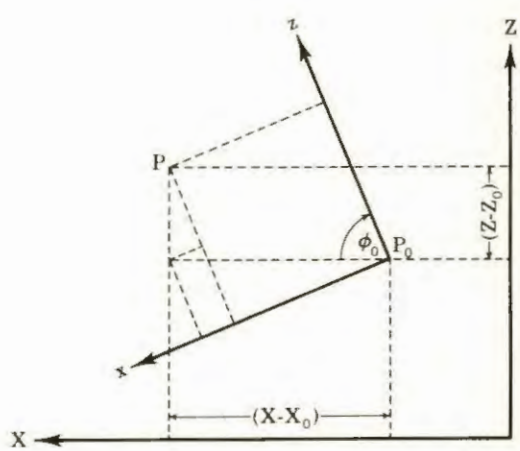


FIGURE 5

The direction of the gravity vector, as determined by astronomic observation, is parallel to the ellipsoid normal at the astronomic position and its direction cosines are, therefore,

$$\sin(\phi_0 - \phi_A) - \sin\phi_0 \cos\phi_A (1 - \cos\lambda'_A):$$
$$\cos\phi_A \sin\lambda'_A : \cos(\phi_0 - \phi_A) - \cos\phi_0 \cos\phi_A (1 - \cos\lambda'_A) \tag{7.2}$$

the subscript A indicating that astronomic coordinates are used. Specifically, $\phi_A$ is the astronomic latitude of point P, and $\lambda'_A = \lambda_A - \lambda_0$, $\lambda_0$ still being the geodetic longitude of the origin $P_0$.

## 8. DIRECTIONS OF THE AXES OF A SECOND COORDINATE SYSTEM

The rectangular coordinate system established at $P_0$ with the ellipsoid normal (or, if preferred, the gravity vector) for its $z$ axis is designated the principal, or master, system and coordinates with respect to it as, for example, the geodetic control points (6.4) are designated $x_0, y_0, z_0$.

We consider next a secondary coordinate system with origin at the point $P_1$ $(\phi_1, \lambda_1)$. The $z_1$ axis is to be the ellipsoid normal at $P_1$ pointing outward, and its direction cosines with respect to the principal system have already been computed in (7.1). If, on the other hand, the gravity vector is to be the $z_1$ axis, the direction cosines follow from (7.2), where $\phi_A$ and $\lambda_A$ are astronomic latitude and longitude of $P_1$ respectively.

In either case the $x_1y_1$ plane shall pass through $P_1$ and be perpendicular to the $z_1$ axis, with the $x_1$ axis pointing south, and the $y_1$ axis completing a right-handed system in the sequence $x_1y_1z_1$.

The next step is to compute all the cosines of the angles which each axis in a set forms with the three axes of the second set, nine in number. For this purpose the origin $P_1$ may be assumed to be on the ellipsoid since the axes are similarly oriented for all values of h. $P_1$ is the general point P in figure 2. The ellipsoid normal at $P_1$ cuts the polar axis at H and $HP_1$ is therefore the direction of the $z_1$ axis. A tangent to the ellipsoid at $P_1$ cuts the polar axis in T, hence $TP_1$ is the direction of the $x_1$ axis. In the right triangle $P_1QT$ the angle $QTP_1 = \phi_1$ (see also fig. 1), and $QT = P_1Q \cot\phi_1 = N_1 \cos\phi_1 \cot\phi_1$. The geodetic coordinates of $P_1$ and $P_0$ are $(\phi_1, \lambda_1)$ and $(\phi_0, \lambda_0)$. The longitude difference $\lambda_0 - \lambda_1$ is again designated $\lambda'$. In the XYZ system of figure 2 the coordinates of $P_1$, H, and T are therefore

$$P: \begin{cases} X_P = N_1 \cos\phi_1 \cos\lambda' \\ Y_P = N_1 \cos\phi_1 \sin\lambda' \\ Z_P = N_1 (1 - e^2) \sin\phi_1 \end{cases}$$

$$H: \begin{cases} X_H = 0 \\ Y_H = 0 \\ Z_H = -N_1 e^2 \sin\phi_1 \end{cases}$$

$$T: \begin{cases} X_T = 0 \\ Y_T = 0 \\ Z_T = 0Q + QT = Z_P + N_1 \cos\phi_1 \cot\phi_1. \end{cases}$$

Hence the direction cosines of the $x_1$ axis, i.e., of the tangent TP are

$$\frac{X_P - X_T}{TP} : \frac{Y_P - Y_T}{TP} : \frac{Z_P - Z_T}{TP} = -\frac{N_1 \cos\phi_1 \cos\lambda'}{N_1 \cot\phi_1} : \frac{N_1 \cos\phi_1 \sin\lambda'}{N_1 \cot\phi_1} : -\frac{N_1 \cos\phi_1 \cot\phi_1}{N_1 \cot\phi_1}$$

$$= \sin\phi_1 \cos\lambda' \quad : \quad \sin\phi_1 \sin\lambda' \quad : -\cos\phi_1$$

Similarly, using the coordinates of P and H, the direction cosines of the $z_1$ axis are

$$\frac{X_P - X_H}{HP} : \frac{Y_P - Y_H}{HP} : \frac{Z_P - Z_H}{HP} = \cos\phi_1 \cos\lambda' : \cos\phi_1 \sin\lambda' \ : \sin\phi_1$$

The direction cosines of the $y_1$ axis are the three two-rowed determinants from the matrix

$$\begin{pmatrix} \cos\phi_1 \cos\lambda' & \cos\phi_1 \sin\lambda' & \sin\phi_1 \\ \sin\phi_1 \cos\lambda' & \sin\phi_1 \sin\lambda' & -\cos\phi_1 \end{pmatrix}$$

consisting of the cosines of the $z_1$ and $x_1$ axes, or

$$\begin{vmatrix} \cos\phi_1 \sin\lambda' & \sin\phi_1 \\ \sin\phi_1 \sin\lambda' & -\cos\phi_1 \end{vmatrix} : - \begin{vmatrix} \cos\phi_1 \cos\lambda' & \sin\phi_1 \\ \sin\phi_1 \cos\lambda' & -\cos\phi_1 \end{vmatrix} : \begin{vmatrix} \cos\phi_1 \cos\lambda' & \cos\phi_1 \sin\lambda' \\ \sin\phi_1 \cos\lambda' & \sin\phi_1 \sin\lambda' \end{vmatrix} = -\sin\lambda' \ : \cos\lambda' \ : 0 .$$

In vector analysis the direction cosines of a line are interpreted as the components of the unit vector parallel to that line and with suitable orientation. In this type of analysis the unit vector along the y axis is obtained as the cross, or exterior, product of the z and x unit vectors in that order for a right-hand system, with computations similar to the above.

Combining the above results the three direction cosines of the $x_1, y_1$, and $z_1$ axes through $P_1$, relative to the XYZ system of figure 2, are respectively

$$\begin{aligned} x_1 \text{ axis:} &\quad \sin\phi_1 \cos\lambda' , \ \sin\phi_1 \sin\lambda' , \ -\cos\phi_1 \\ y_1 \text{ axis:} &\quad -\sin\lambda' \quad , \quad \cos\lambda' \quad , \quad 0 \\ z_1 \text{ axis:} &\quad \cos\phi_1 \cos\lambda' , \ \cos\phi_1 \sin\lambda' , \ \sin\phi_1 \end{aligned} \tag{8.1}$$

Similarly, with $\phi = \phi_0$, $\lambda' = 0$, the three axes at $P_0$ have the direction cosines

$$\begin{aligned} x_0 \text{ axis:} &\quad \sin\phi_0, \ 0 \ , -\cos\phi_0 \\ y_0 \text{ axis:} &\quad 0 \quad , \ 1 \ , \ 0 \\ z_0 \text{ axis:} &\quad \cos\phi_0, \ 0 \ , \ \sin\phi_0 \end{aligned} \tag{8.2}$$

## 9. ANGLES BETWEEN THE TWO SETS OF AXES

The cosine of the angle which a line A whose direction cosines are $(\alpha_1, \alpha_2, \alpha_3)$ makes with a second line B $(\beta_1, \beta_2, \beta_3)$ is

$$\cos(A,B) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 .$$

This quantity is also the length of the orthogonal projection of a unit length of either line upon the other. Hence, for example, a unit length on the $x_0$ axis projects on the $x_1$ axis, from (8.1) and (8.2), as

$$\cos(x_0, x_1) = \sin\phi_0 \sin\phi_1 \cos\lambda' + \cos\phi_0 \cos\phi_1$$

$$= \cos(\phi_0 - \phi_1) - \sin\phi_0 \sin\phi_1 (1 - \cos\lambda').$$

The form in the second line is preferable for most computing purposes since many significant figures are needed in the trigonometric functions and these are more likely to be available for small angles.

Computing similarly the other projections and arranging them in the following tabular form

$$\begin{matrix} \cos(x_0x_1) & \cos(y_0x_1) & \cos(z_0x_1) \\ \cos(x_0y_1) & \cos(y_0y_1) & \cos(z_0y_1) \\ \cos(x_0z_1) & \cos(y_0z_1) & \cos(z_0z_1) \end{matrix} \tag{9.1}$$

it can be seen that these nine quantities are the direction cosines of the three axes of each of the two systems of coordinates relative to the other. Thus, for example, the first row in (9.1) represents the three direction cosines of the $x_1$ axis relative to the $x_0y_0z_0$ system, and the second column contains the three direction cosines of the $y_0$ axis relative to the $x_1y_1z_1$ system of coordinates.

The table (9.2) below gives the numerical values, arranged in the order indicated in (9.1)

$$\begin{matrix} \cos(\phi_0-\phi_1) - \sin\phi_0 \sin\phi_1 (1-\cos\lambda'), & \sin\phi_1 \sin\lambda', & -\sin(\phi_0-\phi_1) - \sin\phi_1 \cos\phi_0 (1-\cos\lambda') \\ -\sin\phi_0 \sin\lambda', & \cos\lambda', & -\cos\phi_0 \sin\lambda' \\ \sin(\phi_0-\phi_1) - \sin\phi_0 \cos\phi_1 (1-\cos\lambda'), & \cos\phi_1 \sin\lambda', & \cos(\phi_0-\phi_1) - \cos\phi_0 \cos\phi_1 (1-\cos\lambda') \end{matrix} \tag{9.2}$$

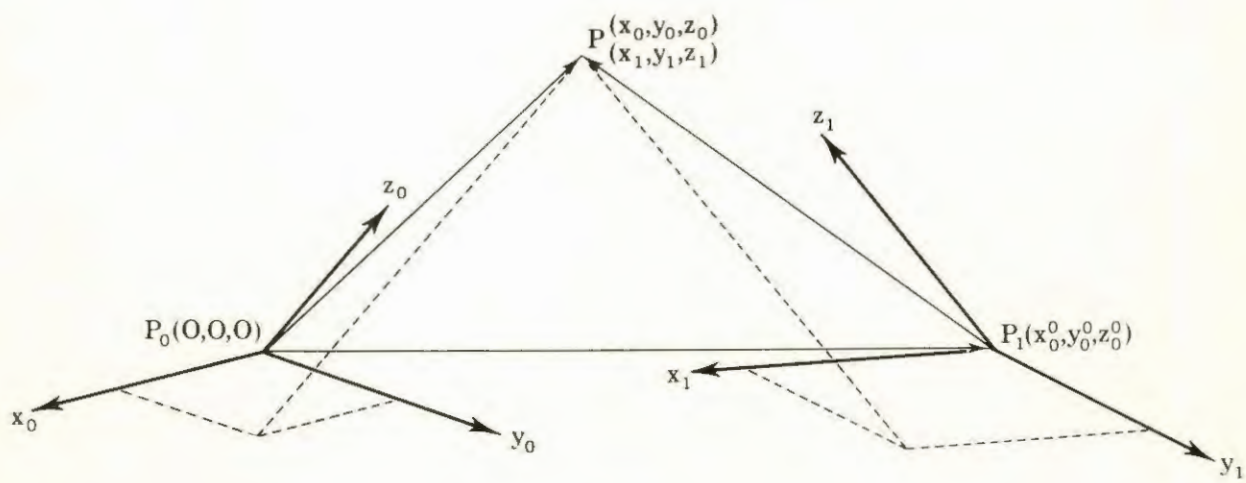


FIGURE 6

## 10. TRANSFORMATION OF COORDINATES FROM ONE SYSTEM TO THE OTHER

Given: the rectangular coordinates $(x_0, y_0, z_0)$ of a point P with respect to the principal coordinate system at $P_0$. Required: to find the coordinates $(x_1, y_1, z_1)$ of the same point with respect to the coordinate system of $P_1$.

$P_0$, being the origin, has coordinates $(x_0, y_0, z_0) = (0,0,0)$. The corresponding coordinates for $P_1$, designated $(x_0^0, y_0^0, z_0^0)$ in figure 6, are computed from equations (6.4).

The position vector $\overline{P_1P}$ in the new coordinate system is, from figure 6, equal to the difference $\overline{P_0P} - \overline{P_0P_1}$ of the indicated position vectors of P and $P_1$ respectively in the old coordinate system. The components of position vector $\overline{P_1P}$, i.e., the rectangular coordinates of its end point P, are therefore obtained by projecting the corresponding components of the vector $\overline{P_0P} - \overline{P_0P_1}$ onto the appropriate axes of the coordinate system in which $\overline{P_1P}$ is the position vector.

The resulting transformation equations are

$$x_1 = (x_0 - x_0^0)\cos(x_0x_1) + (y_0 - y_0^0)\cos(y_0x_1) + (z_0 - z_0^0)\cos(z_0x_1)$$

$$y_1 = (x_0 - x_0^0)\cos(x_0y_1) + (y_0 - y_0^0)\cos(y_0y_1) + (z_0 - z_0^0)\cos(z_0y_1)$$

$$z_1 = (x_0 - x_0^0)\cos(x_0z_1) + (y_0 - y_0^0)\cos(y_0z_1) + (z_0 - z_0^0)\cos(z_0z_1)$$

which may be written, for computational purposes,

$$\begin{aligned} x_1 &= x_0\cos(x_0x_1) + y_0\cos(y_0x_1) + z_0\cos(z_0x_1) - x^* \\ y_1 &= x_0\cos(x_0y_1) + y_0\cos(y_0y_1) + z_0\cos(z_0y_1) - y^* \\ z_1 &= x_0\cos(x_0z_1) + y_0\cos(y_0z_1) + z_0\cos(z_0z_1) - z^* \end{aligned} \tag{10.1}$$

where

$$x^* = x_0^0\cos(x_0x_1) + y_0^0\cos(y_0x_1) + z_0^0\cos(z_0x_1)$$

$$y^* = x_0^0\cos(x_0y_1) + y_0^0\cos(y_0y_1) + z_0^0\cos(z_0y_1)$$

$$z^* = x_0^0\cos(x_0z_1) + y_0^0\cos(y_0z_1) + z_0^0\cos(z_0z_1)$$

are constant for all points, and the cosines are the quantities (9.2).

## 11. GRAVITY SYSTEMS

Either, or both, of the coordinate systems involved in the transformation (10.1) may be gravity systems, that is to say the $z$ axis does not point in the direction of the ellipsoid normal corresponding to the geodetic position but in the actual and observable direction of gravity at the origin. It is convenient in questions involving *direction* (but not distance) to think of the astronomic coordinates $(\phi_A, \lambda_A)$ as specifying a point on the ellipsoid whose ellipsoid coordinates are $(\phi_A, \lambda_A)$ and therefore different, in general, from the point $(\phi_G, \lambda_G)$. A gravity system at a point $(\phi_G, \lambda_G)$ has axes which are parallel to the axes of a normal-to-the-ellipsoid system at the assumed point $(\phi_A, \lambda_A)$ of the ellipsoid. Hence for a gravity system at $P_0$ or $P_1$ it is necessary to substitute the appropriate astronomic coordinates in equations (8.1) and/or (8.2), and also in (9.2) and (10.1). However, the coordinates $x_0^0, y_0^0, z_0^0$ which appear in (10.1) are functions of the geodetic *distance* between $P_0$ and $P_1$ and must in every case be computed from (6.4) with *geodetic* coordinates.

As an illustration, consider the transformation of coordinates from a gravity system to an ellipsoidal system with the same origin. The geodetic coordinates of the common origin $P_0$, as determined from triangulation, are $\phi_G, \lambda_G$; its astronomic position, resulting from direct astronomic observation, is $\phi_A, \lambda_A$. In the transformation equations (10.1) $x_0, y_0, z_0$ are coordinates of a point in space referred to the gravity system, and are assumed to be given. The latitude $\phi_0$ in the formulas is the astronomic latitude $\phi_A$ since the $z_0$ axis has the direction of the gravity vector. Likewise $\phi_1 = \phi_G$ and $\lambda' = \lambda_0 - \lambda_1 = \lambda_A - \lambda_G$. With these values of $\phi_0$, $\phi_1$, and $\lambda'$ we can compute the elements of the matrix (9.2) which are the cosines (9.1) required as coefficients in (10.1).

The quantities $x^*, y^*, z^*$ in this case are all zero because $x_0^0, y_0^0, z_0^0$ are all zero. This follows from equation (6.4) which, when dealing with gravity systems must be interpreted carefully. We have, up to this point in the example, been dealing only with directions of the various axes relative to each other, and have found it convenient for this purpose to interpret the astronomic position of $P_0$ as a point $P_0'$ on the ellipsoid whose geodetic coordinates $\phi_0, \lambda_0$ are numerically equal to $\phi_A, \lambda_A$. This concept is not admissible in using equations (6.4) which are, in effect, formulas for *distance* between two points on the earth in terms of their geodetic coordinates, $\phi, \lambda$ and $\phi_0, \lambda_0$, and their ellipsoid heights $h$ and $h_0$. In all cases, whether either or both coordinate systems are gravity systems, the computation of $x_0^0, y_0^0, z_0^0$ from (6.4) requires the geodetic coordinates $\phi, \lambda$ and $\phi_0, \lambda_0$ of the two origins. In the present case the two origins are one and the same point; hence $\phi = \phi_0$, $\lambda = \lambda_0$, $h = h_0$ in (6.4), making the right hand side of those equations identically zero.

## 12. ASTRONOMIC AND GEODETIC AZIMUTHS: THE LAPLACE CORRECTION

Figure 7 shows the $x_0, y_0, z_0$ axes of the gravity system at $P_0$ discussed in 11. The directions of these axes are fixed, at the time of astronomic observation, relative to the celestial sphere. In particular, the $z_0$ axis points to the astronomic zenith, and the $x_0$ axis points to astronomic south, the zero point of astronomic azimuths which are measured in the plane of the horizon — the $x_0y_0$ plane. Also shown, in grossly exaggerated position, is the $x_1$ axis of the ellipsoidal system pointing in the direction of geodetic south. This latter direction is defined implicitly at a triangulation station by other visible points in the triangulation. Astronomic azimuth at $P_0$ are measured with reference to the $x_0z_0$ plane, while geodetic angles are observed with the plane of the $z_0$ and $x_1$ axes as initial plane so that geodetic azimuths are, in fact, also measured from the $x_1z_0$ plane. The difference between the two kinds of azimuth is therefore the angle between the two initial planes, i.e., the angle $\theta$ in the $x_0y_0$ plane of figure 7.

If the segment of the $x_1$ axis shown is assumed to be of unit length, then its indicated projections $\alpha_1$, $\alpha_2$, $\alpha_3$ on the $x_0$, $y_0$, and $z_0$ axes are the direction cosines of the $x_1$ axis or, from the matrix (9.2):

$$\alpha_1 = \cos(\phi_0 - \phi_1) - \sin\phi_0 \sin\phi_1 (1 - \cos\lambda') = \sin\phi_0 \sin\phi_1 \cos\lambda' + \cos\phi_0 \cos\phi_1$$

$$\alpha_2 = \sin\phi_1 \sin\lambda' \qquad = \sin\phi_1 \sin\lambda'$$

$$\alpha_3 = -\sin(\phi_0 - \phi_1) - \sin\phi_1 \cos\phi_0 (1 - \cos\lambda') = \sin\phi_1 \cos\phi_0 \cos\lambda' - \cos\phi_1 \sin\phi_0$$

where now, for the present purpose, the second form is preferable.

It is now a simple matter to get a close approximation to the angle $\theta$, the so-called Laplace correction. From the figure

$$\sin\theta = \frac{\alpha_2}{\sqrt{1 - \alpha_3^2}}. \tag{12.1}$$

$\alpha_3$ cannot exceed $(\phi_1 - \phi_0)$ numerically due to the obvious inequality

$$|\alpha_3| = |\sin \phi_1 \cos \phi_0 \cos \lambda' - \cos \phi_1 \sin \phi_0| \leq |\sin \phi_1 \cos \phi_0 - \cos \phi_1 \sin \phi_0|$$
$$= |\sin (\phi_1 - \phi_0)| \leq |\phi_1 - \phi_0|$$

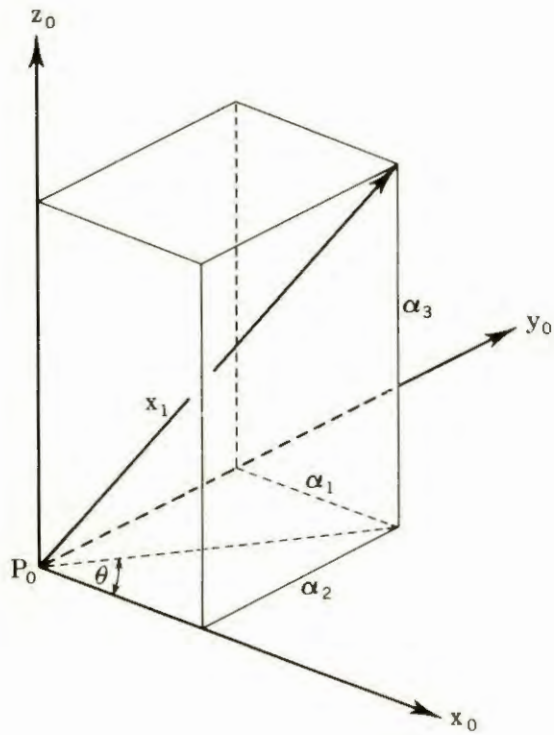


FIGURE 7

Since deflections of the vertical are at most, say, 1' or 0.0003, $\phi_1 - \phi_0$, which is a component of the deflection, and, a fortiori $\alpha_3$, cannot exceed this amount. Hence the denominator in (12.1) agrees with the number 1 to at least 7 significant figures, and

(12.2) $$\sin \theta = \alpha_2 = \sin \phi_1 \sin \lambda'$$

to the same number of significant figures. $\lambda'$ being small, and hence also $\theta$, $\sin \lambda'$ and $\sin \theta$ can be replaced by $\lambda'$ and $\theta$ in (12.2). The resulting relation is in the form given by Laplace

(12.3) $$\theta = \lambda' \sin \phi_1 = (\lambda_A - \lambda_G) \sin \phi_G$$

From figure 7 it is apparent that an observed astronomic azimuth increased by the quantity $\theta$ will give the geodetic azimuth to the same station. The geodetic azimuth so obtained, called a Laplace azimuth, is, in general, more accurate than the same azimuth computed through a chain of triangulation and is therefore, when available, held fixed in the adjustment of the triangulation.

## 13. COORDINATES OF THE ELLIPSOID CENTER

Many other arbitrarily oriented coordinate systems are of course possible. Consider, for example, a system in which the origin is a point with geodetic latitude $\phi_G = 28° \; 29' \; 28.''9451$ and an ellipsoid elevation $h = 6.255$ m. Let the $z$ axis have the direction of the inward pointing normal to the ellipsoid at the origin and let the $y$ axis have a geodetic azimuth of 262° 49' 07" (from south), with the $x$ axis completing a left-handed system. Required are the $x, y, z$ coordinates of the center of the Clarke 1866 ellipsoid ( $a$ = 6,378, 206.4 m, $e^2$ = 0.00676 86580).

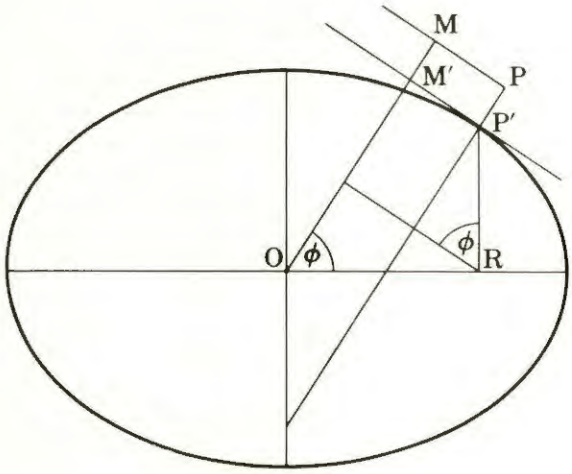


FIGURE 8

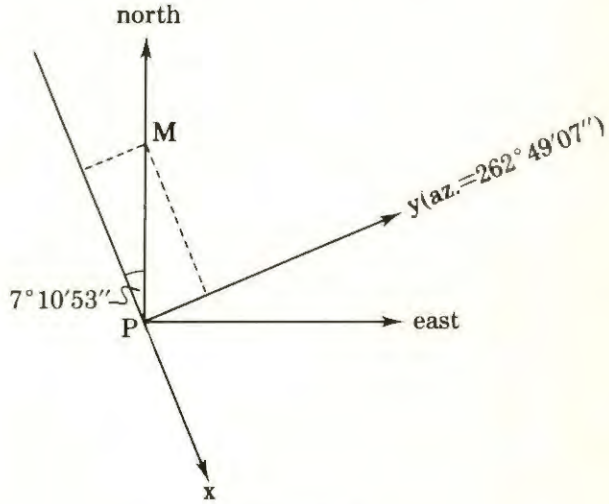


FIGURE 9

Figure 8 is the meridian ellipse of the origin P of the xyz system. P' and P are on the same ellipsoid normal, the $z$ axis of the coordinate system. A line through the ellipsoid center O, parallel to the normal PP', cuts the tangent at P' in M' and the trace of the xy plane in M. From the discussion in connection with figure 1 the abscissa and ordinate of P' are

$$OR = \frac{a \cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}$$

and

$$RP' = \frac{a(1 - e^2)\sin\phi}{\sqrt{1 - e^2 \sin^2\phi}}$$

Projecting these segments on OM' and M'P' respectively

$$OM' = OR \cos\phi + RP' \sin\phi = \frac{a \cos^2\phi + a(1 - e^2) \sin^2\phi}{\sqrt{1 - e^2 \sin^2\phi}} = a\sqrt{1 - e^2 \sin^2\phi}$$

$$= 6,373,292.533 \text{ m.}$$

$$M'P' = OR \sin\phi - RP' \cos\phi = \frac{a \, e^2 \sin\phi \cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}$$

$$= 18,113.914 \text{ m.}$$

Figure 9 shows the $xy$ plane and the trace $PM$ of the meridian plane of the origin $P$. Hence the direction $PM$ is due north and the $x$ and $y$ axes must have directions as indicated to meet the specifications of the given data. The projections of $MP$ ($= M'P'$) on the $x$ and $y$ axes give the $x$ and $y$ coordinates, respectively, of the ellipsoid center

$$x = -MP \cos 7°10'53'' = -17,971.817$$

$$y = MP \sin 7°10'53'' = 2,264.438$$

$$z = -OM = -(OM' + h) = -6,373,298.788$$



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