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Analytic Absolute Orientation in Photogrammetry

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Analytic Absolute Orientation in Photogrammetry

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1. INTRODUCTION

THE subject of absolute orientation consists of a geometric transformation of coordinates of a set of points from one three-dimensional rectangular system into another similar system where (1) the two systems differ in scale (dilation), orientation (rotation), and position (translation); and (2) the transformation data consist of the coordinates in both systems for a few common "control" points. The problem is to first determine a set of transformation coefficients by means of which the coordinates of the control points in the first system can be converted into those of the second, so that any residual discrepancies satisfy the condition of least squares; and thereafter to use these coefficients to convert many other point coordinates given in terms of the first system. The type of transformation is specified as orthogonal, linear, or affine where the dilation is identical along all three rectangular axes.

This problem is in a sense a sequel to the paper "Analytic Aerotriangulation in the Coast and Geodetic Survey."[1] This problem applies to the special case of analytic aerotriangulation in which the number of photographs is only two (or where a very few photographs are reduced to a single coordinate system and then treated as though they were like the two-photograph model). The special case arises only occasionally in topographic mapping but frequently in construction and cadastral surveys. Insofar as the author knows, the specific solution is not treated elsewhere although similar treatments exist;[2] moreover the method of solution differs from that of other similar problems. Although the solution is carried through the programing stage for electronic computation, it has not been actually programed nor tested, nor is the author an experienced programer.

The general treatment here is quite analogous to the instrumental solution. An operator of a stereoscopic mapping instrument (consisting of two projectors containing glass transparencies of two adjacent overlapping aerial photographs) adjusts by a routine, iterative, mechanical procedure the second projector in position and orientation relative to the first one until all corresponding image rays in the overlap area are in spacial coincidence. The projected images in the overlap area are then said to form a "model." We shall be dealing with the rectangular coordinates of image intersections in this model, the coordinate system being any convenient instrumental frame of reference. In the analytic solution, the arbitrarily adopted reference system consists of the perspective center of the first camera (projector) as the origin, the plane of the first transparency as the horizontal, and the distance between the two perspective centers as unity.

The second operational step in the instrumental solution is (a) to "level" the model by rotating the two projectors as a unit until a few projected images intersect at known geodetic heights, (b) to "scale" the model to fit known positions of two horizontal geodetic control points by changing the unit base distance, and (c) to "shift" the map sheet so that the image rays of known points intersect at their proper positions. This is absolute orientation. Then the operator can measure or plot the positions and heights of other image intersections in the map replica of the geodetic control system. This paper treats of an analytical method which accomplishes the same general result as the instrumental absolute orientation, but based entirely on numerical data rather than the intersections of optically projected image rays.

Mr. Tewinkel is a member of the Office of Research and Development.

[1] Analytic Aerotriangulation in the Coast and Geodetic Survey by Wm. D. Harris, G. C. Tewinkel, and Charles A. Whitten, Photogrammetric Engineering, March 1962.

[2] Construction of orthogonal matrices and their application in analytic photogrammetry by G.H. Schut, *Photogrammetria*, Delft, Netherlands, Vol. 15, No. 4, 1958-59. An exact linear solution of the problem of absolute orientation by E.H. Thompson, *Ibid.* A stereocomparator technique for aerial triangulation by D.W.G. Arthur, Professional Papers, New Series, No. 20, Ordnance Survey, London, 1955. Analytical Orientation Methods by C.M.A. Van Den Hout, *Estralto dal Bollettino di Geodesia e Scienze affini, Revista dell' Instituto Geografico Militare*, Florence, Italy, Vol. XX, No. 3, 1961.

The choice of the iterative method may seem at first to be awkward or unnecessary, but experience gained with the relative orientation problem has instilled confidence in the speed and applicability of the system.

2. MATHEMATICAL BASIS

The following equations are arbitrarily selected from analytic geometry texts as constituting a simultaneous dilation s, a rotation **A** and translation **B** for transforming X-coordinates into x-coordinates:[3]

$$\begin{aligned} x_1 &= s(a_{11} X_1 + a_{12} X_2 + a_{13} X_3) + b_1 \\ x_2 &= s(a_{21} X_1 + a_{22} X_2 + a_{23} X_3) + b_2 \\ x_3 &= s(a_{31} X_1 + a_{32} X_2 + a_{33} X_3) + b_3; \end{aligned}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix};$$

or

$$\mathbf{x} = s\,\mathbf{AX} + \mathbf{B}. \tag{2.1}$$

(x_1, x_2, x_3 are identical, respectively, to the x, y, z notation of coordinate axes.)

The inverse transformation of 2.1 can be denoted as:

$$\mathbf{X} = (1/s)\ \mathbf{A}^{-1}\ (\mathbf{x} - \mathbf{B});$$

or

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = (1/s) \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 - b_1 \\ x_2 - b_2 \\ x_3 - b_3 \end{bmatrix};$$

or

$$\begin{aligned} X_1 &= (1/s)\left[a_{11}(x_1 - b_1) + a_{21}(x_2 - b_2) + a_{31}(x_3 - b_3)\right] \\ X_2 &= (1/s)\left[a_{12}(x_1 - b_1) + a_{22}(x_2 - b_2) + a_{32}(x_3 - b_3)\right] \\ X_3 &= (1/s)\left[a_{13}(x_1 - b_1) + a_{23}(x_2 - b_2) + a_{33}(x_3 - b_3)\right]. \end{aligned} \tag{2.2}$$

The scale factor s is common to all three coordinate axes. The a's are direction cosines in the orthogonal rotation matrix which is defined[4] in terms of three sequential plane rotations ω, ϕ, κ (omega, phi, kappa) about the x_1, x_2, x_3 axes, respectively, as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\cos\kappa & \begin{array}{c}\cos\omega\sin\kappa \\ +\sin\omega\sin\phi\cos\kappa\end{array} & \begin{array}{c}\sin\omega\sin\kappa \\ -\cos\omega\sin\phi\cos\kappa\end{array} \\ -\cos\phi\sin\kappa & \begin{array}{c}\cos\omega\cos\kappa \\ -\sin\omega\sin\phi\sin\kappa\end{array} & \begin{array}{c}\sin\omega\cos\kappa \\ +\cos\omega\sin\phi\sin\kappa\end{array} \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{bmatrix}. \tag{2.3}$$

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[3] See footnote 1 above, Equation 11, 12, and 14.

[4] See footnote 1 above, Section 3.5.

Although the angles ω, ϕ, κ have special significance with regard to photogrammetric instruments, any such meaning is irrelevant in *absolute* orientation. The notation is retained nevertheless because it serves a useful purpose and because the details of its manipulation have already been developed.

Seven independent, unknown, elementary parameters are implied in Equations 2.1, which are linear: s, ω, ϕ, κ, b_1, b_2, b_3. Logically, at least seven pieces of pertinent known data are needed to establish at least seven equations from which the unknown can be evaluated. At least four of the known terms must necessarily consist of the coordinates of two horizontal control stations (X_{11}, X_{21}), (X_{12}, X_{22}); and at least three of them must consist of three appropriate, non-collinear, vertical control stations (X_{33}), (X_{34}), (X_{35}). The vertical and horizontal stations need not be distinct. Actually several more pieces of data will always be present for practical reasons, requiring a least-squares adjustment inasmuch as the coordinates x_1, x_2, x_3 in the first reference system are *observed* values.

Several methods[5] are available for treating the orthogonal rotation matrix, but in this paper an iterative solution is utilized because (1) close approximations are ordinarily available, (2) the system converges rapidly, and (3) a detailed pattern for such a procedure is already developed and in use for the previous problem of relative orientation.

Upon partial differentiation, Equations 2.1 yield a set of linear observation equations in terms of corrections to the seven approximate parameters:

$$\begin{aligned} v_1 &= p_{11} + p_{12}d\omega + p_{13}d\phi + p_{14}d\kappa + p_{15}ds + p_{16}db_1 \\ v_2 &= p_{21} + p_{22}d\omega + p_{23}d\phi + p_{24}d\kappa + p_{25}ds + p_{26}db_2 \\ v_3 &= p_{31} + p_{32}d\omega + p_{33}d\phi + p_{34}d\kappa + p_{35}ds + p_{36}db_3 \end{aligned} \tag{2.4}$$

It is noted that there are a total of 18 coefficients which need to be evaluated in terms of the approximate values of the parameters. A routine for forming these coefficients is now derived.

The differential of $\mathbf{x}$ in Equation 2.1 can be expressed as

$$d\mathbf{x} = d(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B}). \tag{2.5}$$

But a small change in $\mathbf{x}$ is also

$$d\mathbf{x} = \mathbf{x}_{\text{correct}} - \mathbf{x}_{\text{approximate}}. \tag{2.6}$$

where $\mathbf{x}_{\text{cor}}$ is construed to be the given value which may have been computed earlier, and $\mathbf{x}_{\text{app}}$ is the coresponding value obtained from Equation 2.1 through the use of the approximated parameters. By transposition

$$0 = \mathbf{x}_{\text{app}} - \mathbf{x}_{\text{cor}} + d\mathbf{x} \tag{2.7}$$

In practice Equation 2.7 is not zero for all points because of observational discrepancies and because the number of points selected is normally greater than the number of unknown parameters, whence a finite residual $\mathbf{v}$ exists:

$$\mathbf{v} = \mathbf{x}_{\text{app}} - \mathbf{x}_{\text{cor}} + d\mathbf{x}. \tag{2.8}$$

The criterion of the solution consists of determining values for the seven parameters so that $\mathbf{v}$'s are made small, or minimized.

Equations 2.1 gives a formula for finding $\mathbf{x}_{\text{app}}$:

$$\mathbf{x}_{\text{app}} = s\,\mathbf{A}\,\mathbf{X} + \mathbf{B}$$

where s, $\mathbf{A}$, $\mathbf{B}$ are approximations. $\mathbf{x}_{\text{cor}}$ is a known, measured, or previously computed quantity. Substituting into 2.8,

$$v = (s\,\mathbf{A}\,\mathbf{X} + \mathbf{B}) - \mathbf{x} + d(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B}), \tag{2.9}$$

which is a statement of the observation Equations 2.4 in matrix notation. The term $(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B} - \mathbf{x})$ constitutes the "constant" term p_{i1} (which diminishes upon iteration) of Equation 2.4 whereas the other coefficients are derived from $d(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B})$:

$$\begin{aligned}
pi_1 &= s\,\mathbf{A}\,\mathbf{X} + \mathbf{B} - \mathbf{x} \\
pi_2 &= \partial(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B})/\partial\omega = s(\partial A/\partial\omega)\mathbf{X} \\
pi_3 &= \partial(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B})/\partial\phi = s(\partial A/\partial\phi)\,\mathbf{X}\ i = 1, 2, 3 \\
pi_4 &= \partial(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B})/\partial\kappa = s\,\partial A/\partial\kappa)\,\mathbf{X} \\
pi_5 &= \partial(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B})/\partial s = \mathbf{A}\,\mathbf{X} \\
pi6 &= \partial(s\,\mathbf{A}\,\mathbf{X} + \mathbf{B})/\partial b_i = 1
\end{aligned} \tag{2.10}$$

The three partial derivatives of A were stated quite thoroughly in Analytic Aerotriangulation in the Coast and Geodetic Survey.[6] In order to make this paper more complete, however, the following formulas are repeated from that paper based on Equation 2.3 herein:

$$\partial A/\partial\omega = \begin{bmatrix} 0 & -a_{13} & a_{12} \\ 0 & -a_{23} & a_{22} \\ 0 & -a_{33} & a_{32} \end{bmatrix};$$

$$\partial A/\partial\phi = \begin{bmatrix} -\sin\phi\cos\kappa & \sin\omega\cos\phi\cos\kappa & -\cos\omega\cos\phi\cos\kappa \\ \sin\omega\sin\kappa & -\sin\omega\cos\phi\sin\kappa & \cos\omega\cos\phi\sin\kappa \\ \cos\phi & \sin\omega\sin\phi & -\cos\omega\sin\phi \end{bmatrix};$$

$$\partial A/\partial\kappa = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ -a_{11} & -a_{12} & -a_{13} \\ 0 & 0 & 0 \end{bmatrix}$$

Moreover, it was shown in that paper[7] how these values, as well as the products $(\partial A/\partial\omega)\mathbf{X}$, etc., might be computed in a very systematic arrangement and stored for future use. It will be noted, however, that in the present case no corrections need be applied to the initial p-values: formulas 2.10 need no further treatment.

To be more explicit, the formula for p_{i3} is expanded as an example:

$$pi_3 = s(\partial A/\partial\phi)\,\mathbf{X} = s\begin{bmatrix} \partial a_{11}/\partial\phi & \partial a_{12}/\partial\phi & \partial a_{13}/\partial\phi \\ \partial a_{21}/\partial\phi & \partial a_{22}/\partial\phi & \partial a_{23}/\partial\phi \\ \partial a_{31}/\partial\phi & \partial a_{32}/\partial\phi & \partial a_{33}/\partial\phi \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\begin{aligned}
p_{13} &= s\,[(\partial a_{11}/\partial\phi)X_1 + (\partial a_{12}/\partial\phi)X_2 + (\partial a_{13}/\partial\phi)X_3] \\
p_{23} &= s\,[(\partial a_{21}/\partial\phi)X_1 + (\partial a_{22}/\partial\phi)X_2 + (\partial a_{23}/\partial\phi)X_3] \\
p_{33} &= s\,[(\partial a_{31}/\partial\phi)X_1 + (\partial a_{32}/\partial\phi)X_2 + (\partial a_{33}/\partial\phi)X_3]
\end{aligned} \tag{2.11}$$

where, for example[8],

$$\begin{aligned}
a_{11} &= \cos\phi\cos\kappa \\
\partial a_{11}/\partial\phi &= -\sin\phi\cos\kappa, \text{ etc.}
\end{aligned}$$

Thus, matrix formulas for the coefficients of the observation equations have been derived. In Sections 5 and 6 a routine is indicated for an orderly evaluation of these coefficients.

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[5] See footnote 2 above.

[6] See footnote 1 above, Equations 22, 23, and 24.

[7] See footnote 1 above.

[8] See footnote 1 above, Equation 23.

3. DETERMINATION OF APPROXIMATE PARAMETERS

Equations 2.4 for the coefficients of the observation equations are expressed in terms of the seven unknowns themselves: ω, ϕ, κ, s, b_1, b_2, b_3. It is therefore necessary to devise some method for obtaining approximate values for them. The solutions to the equations then yield corrections to these approximations.

Referring to the fundamental Equation 2.1, $\mathbf{x} = s\,\mathbf{A}\,\mathbf{X} + \mathbf{B}$, $\mathbf{x}$ and $\mathbf{X}$ are considered as known control data, ω, ϕ, κ are involved in $\mathbf{A}$, s is the unknown scale factor, and the translations b_1 b_2, b_3 are involved in $\mathbf{B}$. If the equation is applied to two distinct horizontal control points,

$$\mathbf{x}' = s\,\mathbf{A}\,\mathbf{X}' + \mathbf{B}$$
$$\mathbf{x}'' = s\,\mathbf{A}\,\mathbf{X}'' + \mathbf{B}$$

Subtracting,

$$\mathbf{x}' - \mathbf{x}'' = s\,\mathbf{A}(\mathbf{X}' - \mathbf{X}'')$$
$$s = (\mathbf{x}' - \mathbf{x}'')/\,\mathbf{A}(\mathbf{X}' - \mathbf{X}''). \tag{3.1}$$

The terms $(\mathbf{x}' - \mathbf{x}'')$ and $(\mathbf{X}' - \mathbf{X}'')$ convey the meaning of the linear distances between the two control points expressed in terms of the two different coordinate systems. Moreover, Equation 3.1 implies that the rotation $\mathbf{A}$ needs to be determined before s can be evaluated. In practice, since both ω and ϕ are near zero, s can be determined closely by applying Equation 3.6 without considering the rotation ($\mathbf{A} = 1$). (A special case of this sequence exists if the elevations X_3 are known for both of the horizontal control points. Then the final correct value of s can be determined immediately from Equation 3.6, eliminating s as an unknown and reducing the number of unknowns from seven to six. However, this special case does not ordinarily occur in work of the Coast and Geodetic Survey; even in the instances where it did, the scheme as presented here is applicable.)

Ordinarily zeros are close initial approximations for the angles ω and ϕ, but κ may have any angular value. The sine of the angle is needed. The angle κ represents the angle in the xy-plane from the $+x$ to the $+X$ axis (figure 1) where the xy- and XY-planes are approximately coplanar if ω and ϕ are each approximately zero. It is convenient to express the coordinate differences:

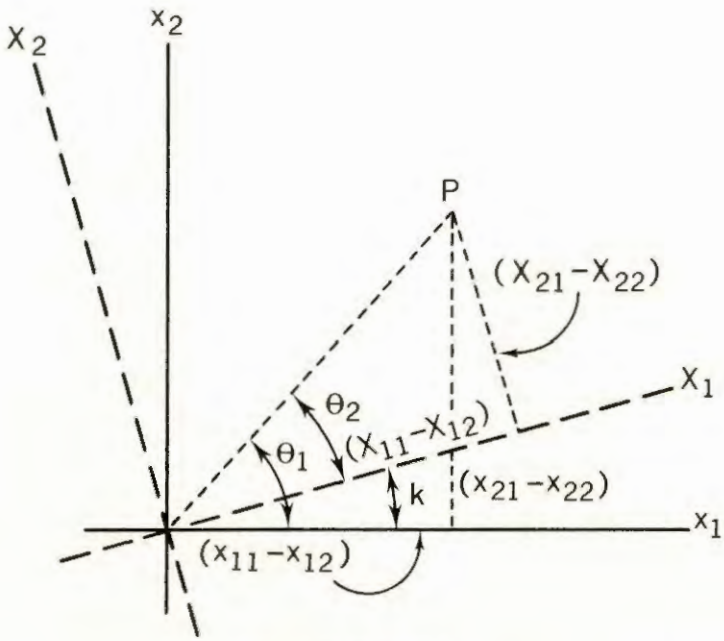

FIG. 1.—The angle κ.

$$\begin{array}{lll} \Delta x_1 = x_{11} - x_{12} & \Delta x_2 = x_{21} - x_{22} & \Delta x_3 = x_{31} - x_{32} \\ \Delta x_1 = X_{11} - X_{12} & \Delta X_2 = X_{21} - X_{22} & \Delta X_3 = X_{31} - X_{32} \end{array} \tag{3.2}$$

Referring to the figure,

$$\tan \theta_1 = \Delta x_2 / \Delta x_1 \qquad \tan \theta_2 = \Delta X_2 / \Delta X_1 \tag{3.3}$$

$$\kappa = \theta_1 - \theta_2$$

$$\tan \kappa = (\tan \theta_1 - \tan \theta_2) / (1 + \tan \theta_1 \tan \theta_2) \tag{3.4}$$

$$\sin \kappa = \tan \kappa / (1 + \tan^2 \kappa)^{1/2}. \tag{3.5}$$

Also,

$$s^2 = (\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2) / (\Delta x_1^2 + \Delta X_2^2 + \Delta X_3^2) \tag{3.6}$$

Thus approximate values for κ and s can be obtained from the data for two horizontal control points. It is hence expedient, insofar as concerns the minimization of the number of subsequent iterations, that (a) data for the two points be reliable and (b) that the two points be a relatively long distance apart. Usually the elevations X_{31} and X_{32} are not known, whereupon they can be considered both zero unless it is known that the two points differ greatly in elevation relative to their distance apart. If this difference $(X_{31} - X_{32})$ is known to be significantly large, photogrammetrists ordinarily have various means at their disposal for obtaining a rough estimate.

If Equation 2.1 is solved for B,

$$\mathbf{B} = \mathbf{x} - s\,\mathbf{A}\,\mathbf{X}$$

which can be evaluated inasmuch as s and A are known approximately. These formulas are spelled out more explicitly for x_1, x_2, x_3 by Equation 2.1. Thus, each horizontal control point yields values for b_1 and b_2 and each vertical control point yields a value for b_3. Inasmuch as s and **A** are approximate, each control point will yield values different from the others. It is therefore suggested, in order to minimize the number of subsequent iterations, that one adopt for each of the three terms the value which is the average obtained from using all the appropriate control points:

$$\begin{aligned} b_1 &= (b_{11} + b_{21} + \ldots + b_{m1}) / m \\ b_2 &= (b_{12} + b_{22} + \ldots + b_{m2}) / m \\ b_3 &= (b_{13} + b_{23} + \ldots + b_{n3}) / n \end{aligned} \tag{3.7}$$

where m and n are the numbers of horizontal and vertical points, respectively.

Thus approximate values for the seven unknown parameters can be determined for use (1) in evaluating the coefficients of the observation Equations 2.10 and (2) as values to which to apply the iterative corrections implied by the observation equations.

4. SOLUTION OF THE SIMULTANEOUS EQUATIONS

Presumably one will use a subroutine computer program for solving the set of simultaneous linear Equations 2.4 in seven unknowns for the corrections and will not need to concern himself with the details of the solution. Nevertheless, a computational routine is given here in the event one may wish to use it to make the absolute orientation program more of a "package" operation. Moreover, the routine, including the formation of the normal equations, only requires programing four different formulas which are used in loop routines. The scheme also allows one to pass immediately from an observation equation into the normal equation matrix without storing any of the observation equations. However, the scheme outlined here does not provide any "check" columns nor any provision for determining the residuals and the standard deviation.

The method is that shown by Hildebrand[9] which may be called the Gauss-Crout algorithm, and which is also noted as being attributed to Cholesky.

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9 Introduction to numerical analysis by F.B. Hildebrand, McGraw-Hill, New York, 1956, p. 431.

The observation Equations 2.4 are rewritten somewhat differently to conform to usual practice:

$$p_1\, d\omega + p_2\, d\phi + p_3\, d\kappa + p_4\, ds + p_5\, db_1 + p_6\, db_2 + p_7\, db_3 = p_8 \tag{4.1}$$

This is a perfectly general type of linear equation in seven unknowns, applying to any kind of work. For each of the three types of equation for this particular problem, one of the db-coefficients is unity and the other two are zeros. (It may be interesting that the eight coefficients can ordinarily be punched conveniently onto a single IBM card. But in this special application, the coefficients may never need to appear on a card at all, occurring only in the computer memory.)

The total number of these p-equations is $(2m + n)$ where m is the number of horizontal control points and n is the number of vertical points.

The coefficients of each are formed independent from the others and at different times. As soon as one of the equations is formed, it will be shown that its contribution to the terms of the normal equation matrix is computed and stored, and then the particular observation or p-equation is "forgotten."

Let the form of the normal equations be

$$c_{i1}\, d\omega + c_{i2}\, d\phi + c_{i3}\, d\kappa + c_{i4}\, ds + c_{i5}\, db_1 + c_{i5}\, db_2 + c_{i7}\, db_3 = c_{i8}$$
$$i = 1 \ldots 7 \tag{4.2}$$

where there are always exactly seven equations, making a 7 by 8 coefficient matrix of 56 terms. We shall denote this normal equation system as the C-matrix. (Figure 2.)

Another auxiliary 7 x 8 matrix is utilized in the computation: it is denoted as the D-matrix. The "answers" consist of a column E-matrix of the values of the seven unknowns.

Now formulas are presented for forming the elements of the C-, D-, and E-matrices, beginning with the p-coefficients of the observation equations.

The formation of each element of the normal equation matrix may be expressed as[10]

$$c_{ij} = \sum_{\kappa = 1}^{2m+n} p_i\, p_j \qquad i = 1 \ldots 7,\ j = 1 \ldots 8 \tag{4.3}$$

In Fortran programing, this operation can be performed by the sequence of instructions

```
  DO    1 I = 1,7
  DO    1 J = 1,8
1 C(I,J) = 0.
.............................
  DO    2 I = 1,7
  DO    2 J = 1,8
2 C(I,J) = C(I,J) + P(I)*P(J).
```

This program states that, after initializing each element to zero, the value of an element of the normal equation system is equal to the value it had previously plus the product of two specific coefficients from an observation equation. In this way, when a p-equation is formed, it can immediately be applied to the accumulating system of normal equations. Then the program can continue to compute some other items until eventually another p-equation is formed when again its coefficient products can be added to the normal system.

After all the $(2m + n)$ p-equations have been treated to form the normal equation system, the elements of the D-matrix can be computed. The sequence of operations is shown in figure 2. The general formula is

$$D_{ij} = C_{ij} - \sum_{\kappa = 1}^{i-1} D_{i\kappa} D_{\kappa j} \qquad i<j,\ i = 1 \ldots 7,\ j = 1 \ldots 8 \tag{4.4}$$

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[10] Manual of geodetic astronomy by A.J. Hoskinson and J.A. Duerksen, Special Pub. No. 237, U.S. Coast and Geodetic Survey, Washington, 1952, p. 138.

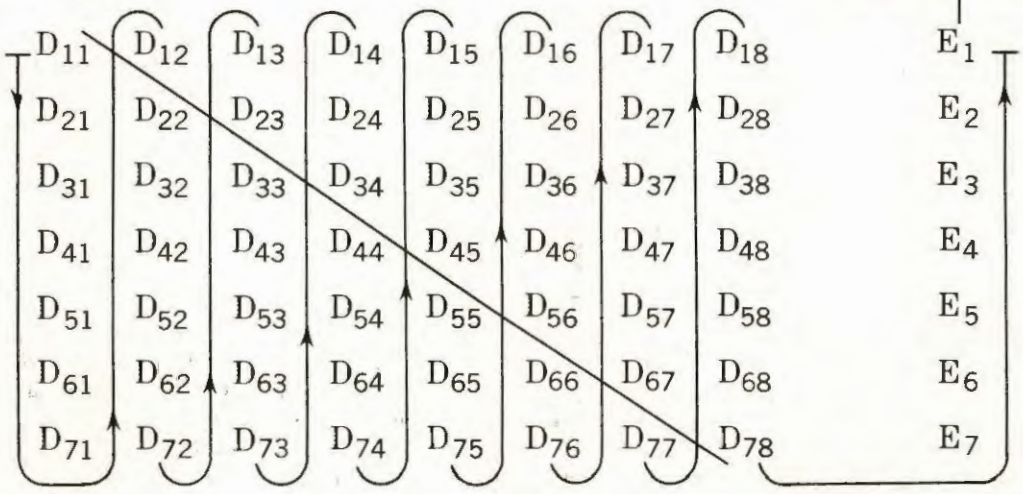
Observation Equations

$$
\begin{array}{cccccccc}
P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} \\
P_{21} & P_{22} & P_{23} & \cdots & \cdots & \cdots & \cdots & P_{28} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
P_{i1} & P_{i2} & P_{i3} & \cdots & \cdots & \cdots & \cdots & P_{i8}
\end{array}
\qquad i = 2m + n
$$

Normal Equations

$$
\begin{array}{cccccccc}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} & C_{18} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} & C_{28} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} & C_{57} & C_{58} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} & C_{67} & C_{68} \\
C_{71} & C_{72} & C_{73} & C_{74} & C_{75} & C_{76} & C_{77} & C_{78}
\end{array}
$$

Auxilliary Matrix — Answer Matrix



Sequence of computation and location of the diagonal

FIG. 2.—Tabular systems for solution of the set of simultaneous linear equations in seven unknowns.

which applies to all the terms on and below the diagonal; but if the term is above the diagonal, the term must be divided by the value of the diagonal term D_{ii}. This can be accomplished by automatic computer by (1) initially setting all the elements equal to zero, (2)

applying Equation 4.4, and (3) then if the term is above the diagonal (i.e., if i is less than j), divide by the diagonal term D_{ij}. In Fortran notation, the program might be:

```
  DO    3 I = 1,7
  DO    3 J = 1,8
3 D(I,J) = 0.
.................................
  DO    6 J = 1,8
  DO    6 I = 1,7
  DO    4 K = 1,6
4 DUMMY = C(I,J) - D(I,K)*D(K,J)
  D(I,J) = DUMMY
  IF (I-J) 5,6,6
5 D(I,J) = D(I,J)/D(I,I)
6 CONTINUE
```

The formation of the E column is called the "back solution" in which each element can be obtained from

$$E_i = D_{i,8} - \sum_{j=1}^{7} D_{ij}E_j \quad i = 7,6,5\ldots1 \tag{4.5}$$

which uses descending values of i. In Fortran notation.

```
  DO 7 I = 1,7
7 E(I) = 0.
.............................
  DO 8 L = 1,7
  I = 8 - L
  DO 8 J = 1,7
  DUMMY = D(I,8) - D(I,J)*E(J)
8 E(I) = DUMMY
```

This completes the solution of the equations but no attempt is made here to indicate the use of check columns nor conditioned equations and the subject of residuals and the standard deviation is discussed in the next section.

5. RESIDUALS

A residual is the amount by which an observation Equation 2.4 fails to be satisfied by the values of the seven parameters after the final iteration. These values are the terms p_{11}, p_{12}, p_{13} in Equation 2.4 and the p_{i1} term in Equation 2.10:

$$\text{Residual} = \mathbf{v} = s\,\mathbf{A}\,\mathbf{X} + \mathbf{B} - \mathbf{x}. \tag{5.1}$$

All the other terms of the equations are insignificantly small due to the very small differential corrections to the parameters obtained on the final iteration. Thus each given control coordinate value has a residual, and the values of the residuals are distributed in magnitude according to the principle of least squares.

The root-mean-square (rms) value of the residuals is given by the formula

$$\text{rms} = ((\Sigma v^2)/n)^{1/2} \tag{5.2}$$

where n is the total number of given control coordinates.

It is expected that all of the residuals and the rms-value will be printed for visual inspection for the purpose of detecting any gross mistakes in the input data and also of displaying an indication of the relative accuracy of the analysis. One can eliminate or correct mistaken data and then repeat the solution. In any case, gross mistakes and blunders should not be allowed to influence the accuracy of the final results.

It may be argued that the decision for the acceptance or rejection of data can be made by the computer. For example, the computer can be instructed to eliminate all data from a control point which has a residual greater than three times the rms-value. However, this is considered here to be an undesirable method of obtaining the most accurate overall results because the computer might eliminate a geometrically important control point due to a simple error in data handling whereas a person might be able to correct the fault so as not to exclude the point.

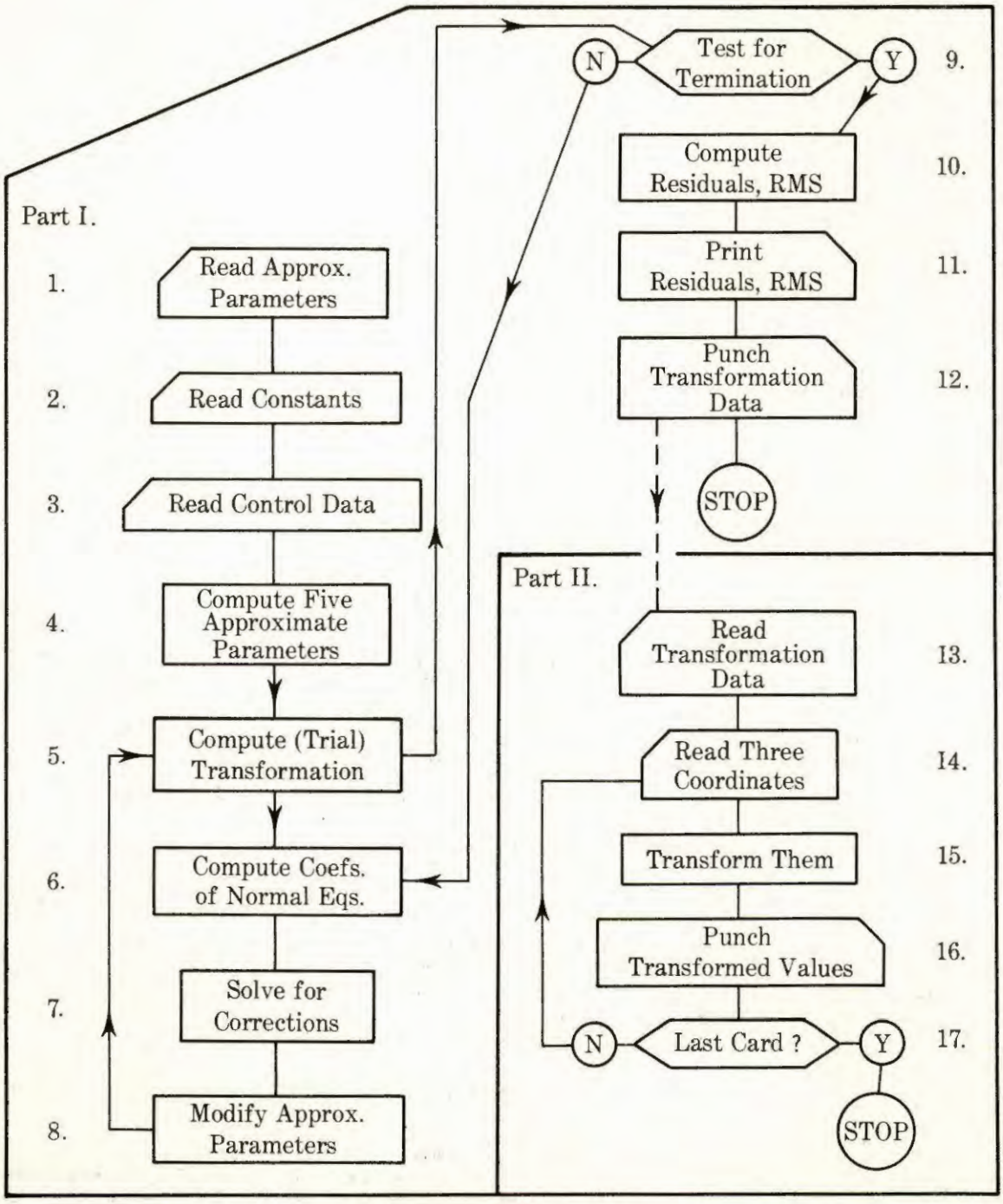


FIG. 3.—Computational scheme.

6. A COMPUTATIONAL SCHEME

A scheme is outlined for solving the complete problem of absolute orientation beginning with the coordinates of a finite number of control points (a minimum of two horizontal and three vertical, and a maximum of 30 in all) and terminating with the transformed coordinates of an unlimited number of "model" points. The accompanying block or flow diagram (figure 3) shows a suggested sequence of operations. The numbered items are discussed briefly here and then are considered in more detail in the next section. The scheme in general is to consider first the coordinates of the control points in the two systems (the ground control system and the model system) for determining by an iterative method the values of the seven unknown parameters which in turn define the coefficients of the transformation Equations 2.2, and second to apply, using a subsequent program, the transformation of the model coordinates for as many points as are desired.

1. The initial approximate parameters are all zeros and can be read in from one card containing eight words of zeros. The card format may be identical to the card output in Step 12 below so that if data are rejected and the problem resolved, the new computation need not start with zeros but with the final values used in the previous determination, which values should suffer only minor changes. The program in the next section is designed to sense the presence of a zero in the term b_1 $(X_{31,19})$ and change its procedure accordingly.

2. Operational constants which seem to be useful here are: (1) the total number M of distinct control stations being used; (2) the number N of elevations being used whether or not horizontal coordinates are also present; and (3) the number MN of pairs of horizontal coordinates being used whether or not the elevations are known.

3. The control data consist essentially of six elements associated with each control station and three in the model system. One or two of the control coordinates may not be known for a station whereupon the unknown coordinates *must* be indicated by zeros in the input for sensing purposes during the computation. The scheme allows for 30 distinct control stations which can be changed easily in the source program. The data for each station is considered to appear on one input card. The first two control stations must be especially reliable and appropriately separated horizontal stations (see Section 3 for the determination of initial approximations for κ and s, whereas the remaining horizontal and vertical control cards can be in any order whatsoever. The control coordinates should be scaled so that control and model coordinates have the same relative magnitude.

4. The five parameters are κ, s, b_1, b_2, b_3 where ω and ϕ are both considered to be zero initially.

5. The "trial" transformation is based on the approximate parameters and upon iteration eventually is the final transform according to Equation 2.1. The program is designed to proceed from Step 5 to Step 6 on the initial iteration and from Step 5 to Step 9 thereafter.

6. The coefficients of an observation Equation 2.10 are formed and immediately applied to the normal equation system as indicated by Equation 4.3. This is done for each element of given control data and is not done if a zero indicates that the data are unknown.

7. This is the standard solution of the set of simultaneous normal equations as indicated by Equations 4.4 and 4.5.

8. The results of the solution are added to the approximate parameters. Then the routine returns to Step 5 to compute a new trial transformation. The program is designed to proceed from Step 5 to Step 6 on the initial iteration but from Step 5 to Step 9 on succeeding iterations.

9. The test for termination of the iteration procedure is whether all of the ω, ϕ, κ corrections are smaller than 0.00001 in absolute value. If any one is greater, the program returns to repeat Steps 6 through 9. If they are all equal to or smaller than the criterion, then the iterations are complete and the program proceeds to Step 10.

10. Residual discrepancies are computed as shown in the first formula in Equation 2.10, and as Equation 5.1, and the root-mean-square value by Equation 5.2.

11. The list of residuals and the rms-value need to be printed (but not necessarily punched) for a visual inspection for blunders as explained in Section 5.

12. The transformation data are punched out in two forms: one for the second program; and the other for Step 1 for the case where a control point is rejected or corrected because of a blunder. This completes the first part of the program.

13. This is the beginning of the second part of the program. A pair of input cards contain the output from Step 12 above consisting of all of the coefficients for the transformation of model coordinates into the control coordinate system as shown by Equation 2.2.

14. One card from a stack of input cards is read, computed, and the results punched before the next card is read. Each card contains the three model coordinates of a point. Any number of cards are admissible.

15. The transformation is computed according to Equation 2.2.

16. The output is punched as the three coordinates of the point in the control coordinate system. Whether they are punched or printed depends on their subsequent use: presumably the points may be used for subsequent processing, such as for the computation of earthwork, or the computation of bearings and distances for cadastral registry.

17. The program is allowed to continue as long as input cards are present.

7. A SUGGESTED PROGRAM

A computer program in Fortran notation is now devised to solve the problem developed in the foregoing pages. Inasmuch as the writer is a "rank amateur" at programing and the program has never been tested in any sense, the system should not be used without a close study for adequacy and accuracy. Nevertheless the program may be useful as it suggests a detailed routine for solving the problem, a routine which may have some merit from the standpoint of brevity and orderliness.

The program is based on the IBM-1620 computer, and the usual changes would necessarily have to be made for any other type of machine.

To give some idea of the size of the program, the source program for Part I specifies 880 words of dimensioned storage, 24 distinct symbols, uses 22 "formulas," and includes 184 statements. Part II has only 21 words of dimensioned storage, 8 symbols, 2 formulas, and 16 statements.

Little effort has been made to minimize the required storage. For example, no attempt is made to utilize the symmetry of the normal equations, resulting in a waste of 21 memory locations and some computer time. However, to avoid this waste might require more than the 21 words!

A few comments about the dimensioned tables may be worthwhile. The X- and A-Tables are shown at this point while the C- and D-Tables are shown in figure 2. The tables are designed and arranged to allow the evaluation of terms through the use of matrix operations insofar as possible. In general, formulas have not been included in the tables nor in the program list, but they are designated specifically in Section 6. The P-Table (fig. 2) is used first to store the coefficients of the observation equations and later also to store the results of the back solution, or the "answers."

Card formats are not specified as the reader or user can best do this to suit his particular computer and application.

The X-Table

I	J							
	1 2 3	4 5 6	7 8 9	10 11 12	13 14 15	16 17 18	19 20 21	22
1 2 3 4 5 6 7 . . . 28 29 30	Control Coordinates	Model Coordinates	Rotation AX	$(\partial A/\partial\omega)\ X\ s$	$(\partial A/\partial\phi)\ X\ s$	$(\partial A/\partial\kappa)\ X\ s$	Translation terms $B=x - A\ X\ s$	Point identification number.
31		Work storage space						

The A-Table

I	J		
	1	2	3
1	$\sin\omega$	$\sin\phi$	$\sin\kappa$
2	$\cos\omega$	$\cos\phi$	$\cos\kappa$
3	1	0	0
4	0	$\cos\omega$	$\sin\omega$
5	0	$-\sin\omega$	$\cos\omega$
6	$\cos\phi$	0	$-\sin\phi$
7	0	1	0
8	$\sin\phi$	0	$\cos\phi$
9	$\cos\kappa$	$\sin\kappa$	0
10	$-\sin\kappa$	$\cos\kappa$	0
11	0	0	1
12	Matrix product of the ϕ		
13	and κ		
14	constituents		
15	Matrix product of ω-constituent and the ϕ-ω		
16	product.		
17	— The Rotation Matrix A.		
18	0 Partial derivatives		
19	0 $(\partial A/\partial\omega)$		
20	0		
21			
22	$(\partial A/\partial\phi)$		
23			
24			
25	$(\partial A/\partial\kappa)$		
26	0	0	0

To assist in debugging the program, the addition of eight pairs of IF(SENSE SWITCH i) and WRITE statements may well be distributed in appropriate locations. Suggested places and data are:

After Statement 2, write tan κ

Before 10, sin κ and S

After 8, at least eight terms of the rotation matrix

After 27, the eight P-values

After 29, the seven diagonal terms and last term of the normal equation system

After 33, the bottom row of the auxiliary table

After 35, the seven results of the back solution

After 36, the seven corrected parameters

With the Sense Switch ON, all of these values will then be printed out for each iteration.

If memory storage space is critical, the A-Table can be reduced by 27 words by re-using rows 3 to 11 to store rows 18 to 26 through a change in indexing in the affected formulas.

SUGGESTED PROGRAM

```
C     ABSOLUTE ORIENTATION, ORTHOGONAL 3D TRANSFORM, ITERATIVE,
                                                          TEWINKEL

      DIMENSION X(31, 22), A(26, 3), P(1, 8), C(7, 8), D(7,8)
      READ, A(1,1), A(1,2), A(1,3), S, X (31,19), X(31,20), X(31,21), IDENT
      READ, M, N, MN, IDENT

      DO 1, I = 1,M
    1 READ, X(I,1), X(I,2), X(I,3), X(I,4), X(I,5), X(I,6), X(I,22)

      IF (X(31,19)) 101,10,101                    (Test for presence/absence)
  101 DO 2 I= 1,6
    2 X(31,I) = X(2,I) - X(1,I)
      A(1,3) = (X(31,1) - X(31,5) - X(31,2) *X(31,4) )/(X(31,1)*X(31,4) +
    1                X(31,2)*X(31,5) )                          (Tan κ)
      A(1,3) = A(1,3)/SQR(1.0-A(1,3)**2.0)                     (Sin κ)
      S = SQR( (X(31,4)**2.0+ X(31,5)**2.0+ X(31,6)**2.0)/                (Scale)
                    (X(31,1)**2.0+ X(31,2)**2.0+ X(31,3)**2.0)

      JC= 8
   10 DO 3 J = 1,3
    3 A(2,J) = SQR(1.0-A(I,J)**2.0                  (Cosines)

      DO 4 I = 3,26
      DO 4 J = 1,3                                  (Clearing)
    4 A(I,J)= 0.0

      A(3,1) = 1.0                                  (Putting Ones in
      A(7,2) = 1.0                                   plane rotations)
      A(11,3) = 1.0

      A(4,2) = A(2,1)
      A(4,3) = A(1,1)
      A(5,2) = ( -A(1,1))                           (Transfers to form
      A(5,3) = A(2,1)                                constituent plane
      A(6,1) = A(2,2)                                rotation matrices)
      A(6,3) = ( -A(1,2))
      A(8,1) = A(1,2)
      A(8,3) = A(2,2)
      A(9,1) = A(2,3)
      A(9,2) = A(1,3)
      A(10,1) = ( -A(1,3))
      A(10,2) = A(2,3)

      IA = 12
      IB = 14
      IC = 2
      ID = 3
    5 DO 6 I = IA,IB
      DO 6 J = 1,3
      DO 6 K = 1,3
      L = K + IC
      IE = I - 6
    6 A(I,J) = A(I,J) + A(IE,K)*A(L,J)              (matrix product)

      IA = 15
      IB = 17
```

```
      IC = 11                                    (Second matrix product--
      ID = ID + 2                                 the primary 3D rotation
      GO TO ID                                    matrix)
      CONTINUE
    7 GO TO JC

    8 DO 9 J = 1,3
      I = 17 + J                                 (Formation of partial
      K = 14 + J                                  derivatives of the
      A(I,2) = ( -A(K,3))                         primary matrix)
      A(I,3) = A(K,2)
      A(24,J) = A(16,J)
      A(25,J) = ( - A(15,J))
      A(21,J) = ( - A(14,J) *A(2,3))
      A(22,J) = A(14,J) *A(1,3)
    9 A(23,J) = A(12,J)

      DO 11 I = 1,M                              (Clearing part of
      DO 11 J = 7,21                              X-table
   11 X(I,J) = 0.0

      JA = 7
      JB = 9
      IDA = 14
      SUB = 1.0                                  (Matrix product AX)
   12 DO 19 I = 1,M
      DO 13 J = JA,JB
      DO 13 K = 1,3
      L = JA + 8
   13 X(I,J) = X(I,J) + A(L,K) *X(I,K)
   19 X(I,J) = X(I,J) *SUB

   14 JA = 10
      JB = 12
      IDA = 15
      SUB = S                                    (Matrix product
      GO TO 12                                    (∂A/∂ω) X S)

   15 JA = 13
      JB = 15                                    (Matrix product
      IDA = 16                                    (∂A/∂φ) X S)
      GO TO 12

   16 JA = 16
      JA = 18                                    (Matrix product
      IDA = 17                                    (∂A/∂κ) X S)
      GO TO 12

   17 IF (X(31,19) 103,103,103                   (Test for presence or
                                                  absence of a previously
                                                  computed value)

  103 DO 18 I = 1,M
      DO 18 J = 19,21
      K = J - 15                                 (Translation terms
      L = J - 12                                  B = x - A X s)
   18 X(I,J) = X(I,K) - X(I,J) *S

      DO 20 J = 1,3
      DO 20 I = 1,M                              (Sums of translation
   20 X(31,J) = X(31,K) + X(I,K)                  terms)
```

```
      DO 21 I = 19,21
   21 X(31,I) = X(31,I) /MN                    (Means of translation
      X(31,3) = X(31,8) /N                      terms)

  102 DO 22 I = 1,7                            (Clearing normal and
      DO 22 J = 1,8                             auxiliary tables for
      C(I,J) = 0.0                              equation solution)
   22 D(I,J) = 0.0

      DO 23 I = 5,7                            (Clear 3 terms in
   23 P(I) = 0.0                                P-table)

      IA = 1                                   (Test for presence or
      DO 29 I = 1,M                             absence of control
   25 IF(X(I,IA))27,26,27                       coordinate)
   26 GO TO 29

   27 IB = IA + 7
      IC = IA + 12
      ID = IA + 15
      IE = IA + 6
      IDA = IA + 18
      JA = IA + 3                              (Coefficients of one
      JB = IA + 4                               observation equation)

      P(JA) = 0.0
      P(1) = X(I,IB)
      P(2) = X(I,IC)
      P(3) = X(I,ID)
      P(4) = X(I,IE)
      P(JB) = 1.0
      P(8) = X(I,JA) - X(31,IDA) -S*P(4)

      DO 28 I = 1,7                            (Adding contribution of
      DO 28 J = 1,8                             observation equation to
   28 C(I,J) = C(I,J) + P(I)*P(J)               normal equation system)

      IF(IA - 3)30,29,29                       (Testing for last of
                                                three coordinates of a
                                                point)

   30 IA = IA + 1                              (Instruction modification)

      GO TO 25                                 (For next coordinate)

   29 CONTINUE
      DO 33 J = 1,8
      DO 33 I = 1,7                            (Auxiliary table for
      DO 39 K = 1,6                             solution of equations)
   39 DUMMY = C(I,J) - D(I,K)*D(K,J)
      IF (I - J)31,32,32
   31 D(I,J) = DUMMY/D(I,I)
      GO TO 33
   32 D(I,J) = DUMMY
   33 CONTINUE

      DO 34 I = 1,7                            (Clearing P-table for
      P(I) = 0.0                                storage of "answers")

      DO 35 L = 1,7
```

```
      I = 8 - L                                   (Back solution: "answers")
      DO 35 J = 1,7
      DUMMY = D(I,8) - D(I,J) *P(J)
   35 P(I) = DUMMY

      DO 36 I = 1,3
      A(1,I) = A(1,I) + P(I)                      (Applying corrections
      K = I + 4                                    (modifying) the seven
      IDA = I + 18                                 parameters)
   36 X(31,IDA) = X(31,IDA) + P(K)
      S = S + P(4)

      JC = 46
      GO TO 10                                    (Step 10 follows Step 2)

   46 IF(P(1)**2.0 - 1.0E - 10)37,37,8            (Test for termination)
   37 IF(P(2)**2.0 - 1.0E - 10)38,38,8
   38 IF(P(3)**2.0 - 1.0E - 10)40,40,8

   40 DO 41 I = 1,M
      DO 41 J = 7,12                              (Clearing part of
   41 X(I,J) = 0.0                                 X-table)

      DO 42 I = 1,M
      DO 42 J = 7,9                               (New AX matrix products)
      DO 42 K = 1,3
      L = J + 8
   42 X(I,J) = X(I,J) + A(L,K) *X(I,K)

      DO 43 I = 1,M
      DO 43 J = 10,12
      IA = J - 3                                  (Residuals)
      IB = J - 9
      IC = J - 6
      IDA = J + 9
   43 X(I,J) = S*X(I,IA) - X(31,IDA) + X(I,IC)    (Residuals)

      DO 44 I = 1,M
   44 PRINT, X(I,22), X(I,10), X(I,12)            (Residuals)

      DUMMY = 0.0
      DO 45 I = 1,M
      DO 45 J = 10,12
   45 DUMMY = DUMMY + X(I,J) **2.0                (RMS)
      RMS = SQR(DUMMY/(MN + N))
      PRINT, RMS

      PUNCH, A(15,1),A(15,2),A(15,3),A(16,1),A(16,2),A(16,3), IDENT
      PUNCH, A(17,1),A(17,2),A(17,3),S,X(31,19),X(31,20),X(31,21), IDENT
      PUNCH, A(1,1),A(1,2),A(1,3),S,X(31,19),X(31,20),X(31,21), IDENT

      STOP
      END


      THREE DIMENSION TRANSFORM APPLIED                                TEWINKEL
      DIMENSION A(3,3), B(3), X(3,3)

      READ, A(1,1),A(1,2),A(1,3),A(2,1),A(2,2),A(2,3), IDENT
      READ, A(3,1),A(3,2),A(3,3),S,B,(1),B(2),B(3), IDENT
                        (Input from first program)
```

```
1 READ, X(1,1),X(1,2),X(1,3),N                    (Input model coordinates)
  X(3,1) = 0.0
  X(3,2) = 0.0
  X(3,3) = 0.0
  DO 2 J = 1,3
2 X(2,J) = (X1,J) - B(J))/S                       (Translation and dilation)

  DO 3 I = 1,3
  DO 3 J = 1,3
3 X(3,I) = X(3,I) + X(2,J)*A(J,I)                 (Rotation)

  PUNCH, N, X(3,1),X(3,2),X(3,3),IDENT

  END
```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