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Analytic Aerotriangulation

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Technical Bulletin Series

This series of Technical Bulletins was inaugurated to present primarily to the personnel of the Coast and Geodetic Survey and incidentally to others technical information related to the Bureau's scientific and technical activities. Since many of the bulletins deal with new practices and new techniques, the views expressed are those of the authors and do not necessarily represent final Bureau policy.

Technical Bulletin No. 21 constitutes a report on the present state of the development of analytic aerotriangulation and complete formulation suitable for computer programing. The system is based on relatively simple and unconventional apparatus: a point-marking stereoscope, a monocular comparator, and moderate size electronic computer. Accuracy improvement over conventional first-order stereoscopic plotting instruments has been experienced.

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Analytic Aerotriangulation

W. D. HARRIS, G. C. TEWINKEL, AND C. A. WHITTEN

U.S. Coast and Geodetic Survey

ABOUT five years ago it was decided in the Coast and Geodetic Survey to explore the possibilities of analytic aerotriangulation in an effort to improve accuracy, to augment its productive capacity in this field, to automate some of the more tedious operations, to exploit the advantages of the Bureau's computer facilities, and to avoid the procurement of additional mechanical analog plotters. To this end it was necessary to formulate various mathematical relationships, procure some special instruments, devise new computer programs and make numerous tests. Now that 25 strips of aerial photographs have been processed successfully for productive utilization, it is with a measure of confidence through experience that the system is reported here in considerable detail. Most of the goals have been realized in part, with the possible exception of automation, which is surely to follow, and many improvements are naturally expected as further experience is gained.

The development of the system has been accomplished through the cooperative effort of several organizational units and a large number of individuals within the Bureau without whom the project could not have succeeded. Capt. Lawrence W. Swanson maintained the interest, unswerving conviction, vision, encouragement, and propulsion that are always needed in this kind of development where the study extends over several years. Although formulation was done in the Photogrammetry Division, significant contributions were added and several changes made in the Electronic Computing Division, where all of the computer programing and data processing were accomplished. Even after development and during present productive work, the Photogrammetry Division furnishes digital image coordinate data to the Computing Division which provides all the processing and returns "answers" to the original division for use in subsequent map compilation or other applications. Of the Photogrammetry Division, Mr. Morton Keller assisted with the formulation and performed many sample computations with desk calculator; Mr. B. Frank Lampton did all the initial observing and data collection and aided materially in overcoming several initial difficulties with electronic equipment; Mr. Ray J. Puhl has since furnished a large amount of observational data. Miss Rosemary Riordan of the Electronic Computing Division did all the computer programing, debugging, and initial computer operations. Finally, one cannot overestimate the contribution of Dr. Hellmut H. Schmid of the Ballistic Research Laboratories, Aberdeen Proving Ground, Md., who furnished the initial basic mathematics, instructed photogrammetric personnel of the Coast and Geodetic Survey, and acted unofficially in the capacity of a willing consultant in all phases of the work.

The authors humbly and gratefully acknowledge these many contributions and the eager and unselfish cooperation of those named and many unnamed who devoted their special talents to the success of the project.

Mr. Harris is Development Assistant to the Chief of the Photogrammetry Division; Mr. Tewinkel is Photogrammetrist in the Office of Research and Development; and Mr. Whitten is Chief of the Electronic Computing Division.

INTRODUCTION

Analytic aerotriangulation is a method for accurately determining the ground positions of objects throughout a strip or block of overlapping aerial photographs, using relatively few known ground positions, by means of digital calculations based on coordinate measurements of pertinent image positions on each photograph. This method differs from the more conventional instrumental method that is based on measurements of a stereoscopic model which is perfected or solved through the use of an analog device (first-order stereoscopic plotter). The analytic method offers certain worthwhile advantages accruing from automation, digital accuracy, least-squares adjustment, and freedom from the mechanical discrepancies contributed by the plotting instrument.

In the system developed in the Coast and Geodetic Survey, emphasis has been placed on the use of relatively simple instruments of moderate cost and of ready availability. Consequently, a stereoscopic point-transfer device is utilized which is already being manufactured in series lots and which is simple in the sense that it has very few moving parts and the only critical feature is that a neat, small, round hole be placed in the photographic emulsion at the same location as indicated by the index marks with an accuracy of a few microns (1,000 microns is one millimeter; one micron is approximately 1/25,000 inch). Also a conventional comparator originally developed for astronomers is used in

distortions, every image-lens-object set of three points lies on a common straight line. If all systematic distortions of the photographs are corrected so that any remaining discrepancies are randomly distributed, then all such three-point collineations in a group of overlapping photographs should be satisfied simultaneously such that the residual discrepancies in the corrected image ***coordinates*** obey the principles of least squares.

1. DATA PROCUREMENT

Data procurement is considered here to include the three phases: aerial photography, image identification and coordinate measurement, which are discussed separately.

1.1 *Aerial Photography and Photo Preparation*

All photographs processed to date have been obtained with one of the three cameras: Wild RC-5, RC-8 or RC-9. All of them are film cameras, the first two having focal lengths of 6 inches and aviogon lenses, the last one, 3.6 inches and a superaviogon 120° lens. A fiducial mark is automatically photographed sharply in each of the four corners of the picture format (a reseau is not used). Generally the film had the Plus-X emulsion on the standard topographic base, except for tests with polyester bases. The exposed film was normally airexpressed to Washington for processing from various project sites throughout the nation. Thus a few days time elapsed between photography and processing. Diapositives were prepared usually about 30 days after development using a LogEtronic printer. Diapositives are printed on the commercial 1/4-inch superflat plates. No attempt has been made to use thinner plates.

Lens-distortion data consisted of the manufacturer's reports indicating the radial distortion on each of the two diagonals to the nearest micron together with the calibrated focal lengths to the nearest 10 microns. These data were verified by testing photographs taken of the Bureau's Ohio Calibration Area.[3]

Aerial photography was usually at 20,000 feet altitude, using the standard camera mount (no automatic stabilization) and without any glass between the camera and ground, nor any special heating arrangements around the camera.

The system is generally independent from the type of photography--any focal length, format, distortion pattern, tilt, altitude, and relief is acceptable. Obviously, the accuracy of the system depends on, among other items, a knowledge of the focal length and lens distortion pattern within close tolerances.

[3]Nine-lens calibration area by Louis J. Reed, *The Journal, Coast and Geodetic Survey*, No. 2, p. 60, 1949.

1.2 *Stereoscopic Marking of Images*

It is considered to be a basic and fundamental concept of photogrammetry that stereoscopic observation is a necessary operation in any aerotriangulation system. Eventually automatic machines may be developed to perform this function, but here the manually operated and relatively simple stereoscopic marking devise, Wild PUG-2 (fig. 2), is used. The purpose of the operation is to select and mark images which are unmistakably and accurately identified on two photographs of the same area. The images must also be identified on a third photograph in the aerotriangulation of strips of photographs, and perhaps on as many as nine for blocks of photographs.

Three categories of points are employed: pass points, control points, and photo points. Pass points are located in the usual rectangular pattern in six locations of each overlap area. Two points, a few millimeters apart, are selected in each of the six locations.

Control points are those for which survey data are available--either their positions (X, Y, or latitude and longitude (horizontal control)), or their elevations (vertical control), or both, are known from ground surveys. Ordinarily these points are not premarked except for accuracy studies, although it is recognized that the failure to premark horizontal control introduces into the system one of the largest sources of error. In lieu of premarking, a system of "substitute" stations is applied--a system in current use in instrumental aerotriangulation. Near each horizontal station the field surveyor identifies at least two natural objects which are discernible on the aerial photographs and determines their positions by short traverses including the necessary observations for geodetic azimuth. One of these substitute stations is treated during data processing as a control station and the other as a check point. Ordinarily a larger number of control points is used than is uniquely required to control the computation. The frequency of control stations is governed by the accuracy specifications for the particular project and the availability of stations already established during previous surveys. As an example, the control for a strip of photographs may consist of horizontal stations distributed roughly one in every sixth photograph and a pair of vertical stations situated near the edges of the strip in roughly every fourth photograph.

Photo points are images whose ground coordinates are wanted from the computation for any one of several applications in subsequent mapping or charting work. Presumably it is this group of points for which the aerotriangulation is performed, although pass points frequently serve this purpose.

The instrument contains a "floating" dot in the optical train and a system of prisms enabling images to be viewed in any orientation, i.e., in the normal manner, pseudoscopically, and at right angles so that y-parallaxes appear as elevational discrepancies. A fine drill can be pressed so that a small hole is formed in the emulsion of the diapositive at the same location as indicated by the floating dot. The drills are honed by the operator to yield desirable sizes. A diameter of 65 microns seems to be about the smallest that can be produced reliably, and 150 microns is needed for eventual mapping with the Kelsh Plotter. As no loss of accuracy in using the larger holes has been indicated in recent results, it may eventually be considered advisable to standardize on this size. Incidently, the diapositives cannot be stacked after marking without damaging the holes.

1.3 *Measurement of Coordinates*

The purpose of coordinate measurement is to assign x and y linear coordinates to all marked images and the four fiducial marks for each photograph. The origin of coordinates is immaterial as later any origin can be readily applied by the computer (See Section 2.11) based on the observed values at the fiducial marks. The accuracy of the measurements is important: the fidelity and stability of the machine, the straightness of the ways, the right-angle validity of the x and y carriages, the acuteness of the viewing system, human fatigue during operation, etc. An important auxilliary is the digitization of the observations so that the operator need not write down the coordinates. Instead, the coordinates are punched into paper tape which is fed directly into the computer processing phase without manual intervention.

A sample form of the digital readout is:

15 16 3 11 02135010 02281120 99.

The first seven digits comprise the point identification number, the next two eight digit numbers are the x and y coordinate values, and the final two digits are a computer program code number. Photograph numbers are conceived as consisting of only two digits, 00 to 99. The value 15 is the photograph number on which the measure is made; 16 is that of the photo whose center is nearest the point being measured; 3 is the class of point, such as control point or pass point; the next 1 is the number of the pass point location, and the second 1 is the serial number of the pass point in the location. The initial and terminal zeros in the coordinate values are dummy characters, and the remaining six digit values are number of microns from the fixed machine origin. The computer code number refers to fiducial marks and is dropped after the first computer program. The identification and code numbers are dialed on the Telecordex prior to pressing the readout switch which actuates both the dialed numbers and coordinate values. The coordinate values are also displayed at all times. Identification numbering is normally uniform from plate to plate.

Ordinarily the operator makes 5 readings on each of the four fiducial marks and 3 readings on each other point, numbering at least 18 per plate. Averages are determined by the computer and retained to the nearest 0.1 micron. Seldom does the spread between the three readings exceed 2 microns, which indicates the repeatability of the device and also may indicate that plural readings may not be necessary. This comparator model has a motor drive for only one coordinate direction for rough positioning. The reticle consists of a combination of a dot to fit just inside the drilled holes and also a system of diagonal lines to facilitate setting on the fiducial marks. Incidently, it is unnecessary to position or orient the plate precisely before observing.

Because of unavoidable backlash in a screw-type comparator, the operator must always approach a mark from the same direction and must not overrun. If overrun does occur, one must backup at least 3mm. before returning. The operator seldom glances at either the display panel nor the typewritten record during the reading of an entire plate. If he is cognizant of a mistake, a means is provided for obliterating the erroneous portion of the paper tape record. The typed record is scanned before the tape is released for data processing, and the typed form comprises a record for any subsequent references. Hand notes are added to help identify specific control points, etc.

The cathode ray tube display of the light energy passing through a drilled hole has not yet been rigidly tested, but initial observations indicate that repeatability is excellent and operator fatigue is reduced. The system seems to work equally well with drilled holes, black dots, or grid intersections.

The comparator is quite sensitive to temperature variations. Not only is the room temperature closely maintained but an off-duty heater is also provided in the instrument maintaining at all times a heat flow equal to that of the projection lamp. Otherwise 2 hours of warmup time is needed until the machine reaches a stable condition. Difficulties of this nature are detected at the end of observing a plate by reobserving the initial point.

2. DATA PROCESSING

As indicated in figure 1, data processing includes all the computer steps beginning with the

digital xy-coordinates produced by the comparator and ending with the computed X, Y, Z coordinates of points on the ground in any convenient system such as a State Plane Coordinate System, the form of the output is both printed and on IBM cards.

Data processing is considered as being composed of four stages. The ***stages*** are not identical to computer ***programs*** as, for example, the first stage is accomplished with a single program and six programs are used in the second stage. Again it is emphasized that no practical work has yet progressed beyond the Cantilever Adjustment in the second stage although Earth Curvature and the Inverse Transform have been applied occasionally omitting Resection and Block Adjustment.

2.1 *Reduction of Measured Coordinates*

The reduction of the measured coordinates includes the several steps needed beginning with the punched paper tape output of the comparator and ending with a set of image coordinates in a form suitable for entry into the perspective computation phase. The reduction includes the averaging of multiple observations, corrections for all known sources of systematic errors, and a transformation of the image coordinates into a system in which the perspective center serves as the origin.

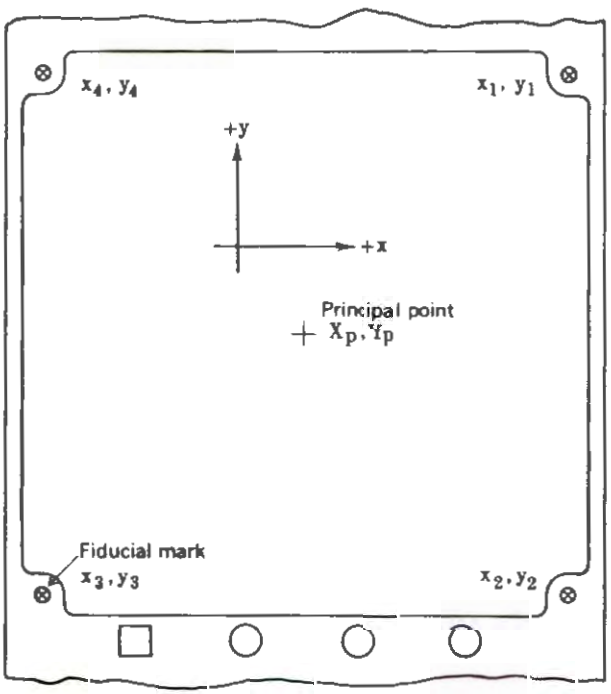

FIG. 5.—Diagram of photograph format indicating the order of numbering the corner fiducial marks.

2.11 Film Distortion Correction

Any type of plastic aerial film changes shape slightly and nonuniformly between the time the photograph is exposed and the time it is printed on glass, when any further changes are considered to be arrested. Evidence of the distortion is revealed by the comparator measurements of fiducial marks as compared to the known fixed measurements in the camera itself. The following equations[4] are used to correct for the distortion in the best manner available to date, as described more fully in appendix 1.

The corner fiducial marks are considered to be numbered clockwise from 1 to 4 (fig. 5), and corner 3 is arbitrarily selected as an interim computational origin, with the ordinate passing through corner 2. The given constant camera coordinates of the corner fiducial marks and principal point are designated as X_j, Y_j; X_p, Y_p where $Y_2 = X_3 = Y_3 = O$, $j = 1 \ldots 4$. The observed coordinates of the four corners based on the comparator coordinate system are correspondingly designated by small letters x_j, y_j.

A series of coefficients are defined:

$$u_j = x_j - x_3 \qquad v_j = y_j - y_3$$

$$d = u_2 v_4 - u_4 v_2$$

$$m = (X_2 v_4 - X_4 v_2)/d \qquad u_1' = m u_1 + n v_1$$

$$n = (X_4 u_2 - X_2 v_4)/d \qquad v_1' = p u_1 + q v_1$$

$$p = v_2 Y_4/d \qquad r = (X_1 - u_1')/u_1' v_1'$$

$$q = -u_2 Y_4/d \qquad s = (Y_1 - v_1')/u_1' v_1'$$

Once the six coefficients m, n, p, q, r, s have been determined for a given diapositive, then the observed coordinates x_i, y_i, of any image i on the photograph can be transformed into compensated coordinates x_i', y_i' through the application of the formulas:

$$x_i' = (mu_1 + nv_1)\left[1 + r(pu_1 + qv_1)\right] + x_p$$

$$y_i' = (pu_1 + qv_1)\left[1 + s(mu_1 + nv_1)\right] + y_p \quad . \qquad (3)$$

The effect of the transformation is to apply linear transformations (translation, rotation, dilation) to correct three corners and then to apply a conformal fitting to correct the fourth point without disturbing the other three. Tests so far

[4]Film distortion compensation for photogrammetric use by G. C. Tewinkel, *Coast and Geodetic Survey Technical Bulletin No. 14*, 1960.

have indicated smaller remaining residual discrepancies from film distortion than was originally anticipated, but it is nevertheless recognized that a more effective correction method is needed in order to exploit fully the accuracy potential of the analytic system. It is also recognized that some residual systematic distortions can be absorbed in the relative orientation program and attributed to orientation parameters without indicating any abnormal y-parallax residuals. The effectiveness of the correction is exhibited only by an analysis of the results of block adjustment in an overcontrolled area.

It should be pointed out that the application of Equation (3) not only corrects for film distortion but also translates the origin of the final coordinate system to the principal point of the photograph, through the presence of the terms x_p, y_p, thus furnishing the data in the form needed in subsequent operations.

2.111 *Translation to the Perspective Center*

Although the translation of the image coordinates to the perspective center (or the principal point) as the origin is not theoretically associated with film distortion, nevertheless the film-distortion computational phase offers a convenient place for accomplishing the step. This is executed through the addition of the values x_p, y_p shown in Equation (3). The two values are constants associated with the particular camera, its fiducial marks and the system adopted for numbering the marks (fig. 5).

2.12 Radial Lens Distortion

Corrections for lens distortion are applied in two steps, one for the asymmetric, elliptical or tilt effect, and the other for the normal symmetrical radial distortion. No effort is made in the present programs for the correction of tangential distortion. This is not to deny the existance of tangential distortion, but only to admit that it has not yet been determined for the cameras.

2.121 *Asymmetric Radial Lens Distortion Correction*

In all of the aerial cameras used to date, the radial lens distortions are not identical for the different radii, resulting in noticeable residual discrepancies if an average uniform distortion is assumed. Another way of visualizing the condition is that lines of equal distortion are not symmetric or circular with respect to the principal point. However, the pattern in each case closely conforms to an ellipse drifted off center and, as indicated by F.E. Washer[5], is practically identical to a small tilt of the focal plane. Consequently, a false tilt is introduced as shown in appendix 2 to correct all image points for the asymmetric effect, after which the total remaining correction for uniform radial distortion is applied.

The false tilt is composed of two parameters: a direction and a magnitude. The "upper"[6] end of the axis of the ellipse is considered to form an angle θ with the x-axis of the photograph in the sense of analytic geometry, and it is defined that $a = \sin\theta$, $b = \cos\theta$, which become constants for a camera. It is convenient first to rotate the coordinate axis for an image through this angle θ, make the asymmetric correction and then rotate back into the original photographic coordinate system.

The formulas for the initial rotation are:

$$x_\theta = ax + by \quad y_\theta = -bx + ay \tag{4}$$

where x, y are the coordinate values of an image after film distortion compensation.

Based on the analysis in the Manual of Photogrammetry[7] and utilizing the formula

$$d_x = x^2 (\sin t)/f$$

in which d is the x-component of the radial tilt displacement of an image having an abscissa x on (*the "upper" side of*) a photograph of tilt t and focal length f, it can be shown that the corrected coordinates of the image are

$$\begin{aligned} x_t &= x_\theta \left[1 + x_\theta (\sin t)/f\right] \\ y_t &= y_\theta \left[1 + x_\theta (\sin t)/f\right] \end{aligned} \tag{5}$$

The term $(\sin t)/f$ becomes a constant for a camera. The angle t is determined from an analysis of the radial distortions on four or more diagonals. The small values of for t of 10, 17, and 18 seconds, respectively, for the three cameras used to date, allows certain approximations in deriving the simplified Equations (5).

Then the final rotation is the inverse form of the initial one:

$$\begin{aligned} x' &= ax_t - by_t \\ y' &= bx_t + ay_t \end{aligned} \tag{6}$$

2.122 *Radial Lens Distortion Correction*

Uniform radial lens distortion is corrected along with atmospheric refraction through the use of the formulas

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[5] Prism effect, camera tipping and tangential distortion by Frances E. Washer, *Photogrammetric Engineering*, v. 23, p. 721, 1957.

[6] Manual of Photogrammetry, *American Society of Photogrammetry*, second edition, p. 321, 1952.

[7] Ibid.

$$r^2 = x^2 + y^2$$
$$x' = x[1 + (rd)/r^2 + k_1 + k_2 r^2]$$
$$y' = y[1 + (rd)/r^2 + k_1 + k_2 r^2] \qquad (7)$$

in which the k_1 and k_2 terms relate to atmospheric refraction, which is discussed later. Here our interest is confined to the term $(rd)/r^2$. The coordinates x, y are those of any image after they have been corrected for the asymmetric condition, x' and y' are the resulting corrected values, and d is the uniform radial distortion factor at radius r after the removal of the asymmetric portion and is essentially the average distortion factor along all four radial directions. (See also app. 3).

The value rd is obtained in the computer through table lookup and interpolation based on r^2 as the independent argument. Values of the product rd are supplied as constants for each camera for 150 values of r, that is, one for each millimeter ranging from the principal point to any corner. The values are determined by desk calculator from the lens distortion data.

The numerical error of these methods for distortion compensation for asymmetric and uniform radial lens distortion is in the order of 0.1 micron, which is somewhat better than the validity of the distortion data.

2.13 Atmospheric Refraction Correction

As indicated previously, the k_1 and k_2 terms in Equation (7) relate to the atmospheric refraction for near vertical photographs only: if oblique photographs are used, this method of correction will need to be revised.

Based on the tables by Axel Leiyonhufvud[8], the values of the k's have been determined by desk calculator to correct the coordinates through their radial distances in accordance with the simplified power series (Appendix 4),

$$x' = x(1 + k_1 + k_2 r^2)$$
$$y' = y(1 + k_1 + k_2 r^2) \qquad (8)$$

where r is as defined by Equation (7). The values of the k's have been determined so that a very close agreement is obtained with the tables. The photogrammetrist who submits the comparator data for computer processing is supplied with large scale graphs of the k-values in terms of the camera altitude and terrain elevation as independent arguments. He reads off the two values and enters them as constants on the record form which accompanies the coordinate data for a strip of photographs.

[8]On astronomic, photogrammetric and trigonometric refraction by Axel Leijonhufvud, Kungl. Boktryckeriet *P. A. Norstadt & Soner*, Stockholm, 1950.

Thus Equation (7) is used to compensate image coordinates both for radial lens distortion by table-lookup, and for refraction by using the two constants.

It is realized that the refraction correction is based on the assumption that the nadir point of the photograph coincides with the principal point, that is, that the tilt is zero. Thus a small error is introduced because the rms of the tilt is a little less than 1°. However, this error is probably not as great as the assumption of a standard atmosphere nor as applying no factors for surface and aerial variations in temperature, humidity and barometric pressure.

2.2 *Provisional Data for the Block Adjustment*

As indicated in the Introduction, a series of computational steps is utilized to obtain an approximate, provisional solution so that the block adjustment can be accomplished with a single iteration. The reason for the provisional solution is an economic one inasmuch as each iteration of the block adjustment involves the solution of a very large system of simultaneous equations

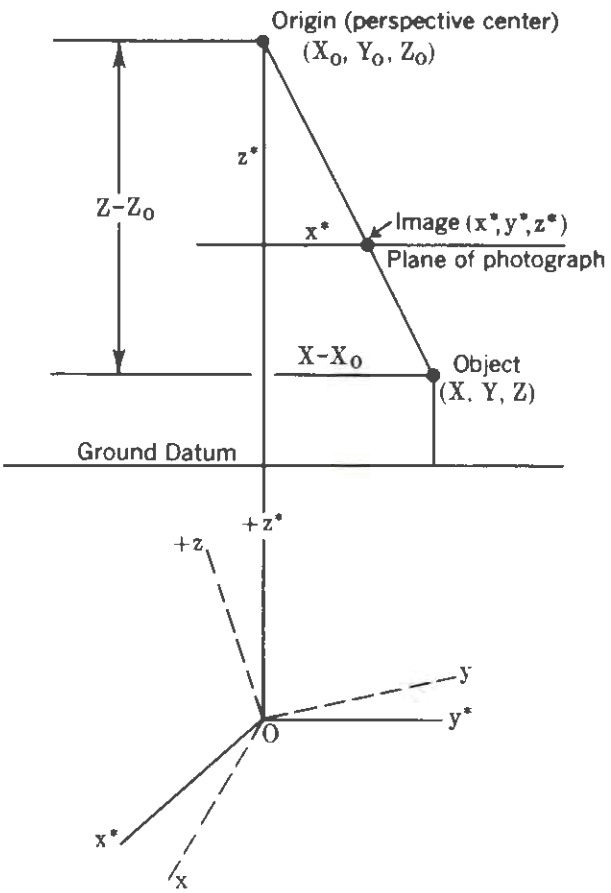

FIG. 6.—Geometry of the projective situation, colineation and rotation.

whose unknowns may number in the hundreds whereas the provisional solution is relatively easy and short.

2.21 Relative Orientation

Relative orientation is defined both here and in conventional instrumental photogrammetry as the determination of the three angular and two linear parameters that specify the attitude and position of one photograph (camera station) with respect to another (overlapping) one that shows a sufficiently large common area. Relative orientation is perhaps the most important item in this analytic system: it embodies all the basic mathematics that is peculiar to the system, is utilized again later in resection and the block adjustment, and requires the second-largest computer effort next to the block adjustment (nevertheless it is accomplished through a single IBM-650 program). It is in relative orientation that the principles of projective geometry are applied wherein the mathematics of the system may differ from that of other engineering and computational disciplines.

A classic geometric rotation of the axes in three dimensions (fig. 6) is needed in relative orientation to express the attitude of one photograph to another. Instead of using the three angles between the respective axes as in analytic geometry, a system of three sequential rotations are used, the primary one ω about a horizontal x-axis, the secondary one φ about the once rotated y-axis, and the tertiary one κ about the camera axis, as explained by G.H. Rosenfield[9] and also derived in appendix 5. The rotation equations are

$$x = x^* \cos\varphi \cos\kappa + y^*(\cos\omega \sin\kappa + \sin\omega \sin\varphi \cos\kappa) + z^*(\sin\omega \sin\kappa - \cos\omega \sin\varphi \cos\kappa)$$

$$y = x^* (-\cos\varphi \sin\kappa) + y^* (\cos\omega \cos\kappa - \sin\omega \sin\varphi \sin\kappa) + z^*(\sin\omega \cos\kappa + \cos\omega \sin\varphi \sin\kappa) \qquad (9)$$

$$z = x^* \sin\varphi + y^* (-\sin\omega \cos\varphi) + z^*(\cos\omega \cos\varphi).$$

where x, y, z are the coordinates of any image on a photograph and x^*, y^*, x^* are the corresponding coordinates in an erect, untilted (rectified) system in which the x^*, y^*, z^* axes may also be conceived as being parallel, respectively, to those of a ground survey coordinate system, X, Y, Z. The coordinate z corresponds to focal length. Equations (9) may be written in the form

$$\begin{aligned} x &= a_{11}\, x^* + a_{12}\, y^* + a_{13} z^* \\ y &= a_{21}\, x^* + a_{22}\, y^* + a_{23} z^* \\ z &= a_{31}\, x^* + a_{32}\, y^* + a_{33} z^* \end{aligned} \qquad (10)$$

In matrix notation this becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} \qquad (11)$$

The inverse notation is also useful:

$$\begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (12)$$

It is also convenient to write Equations (11) and (12), respectively, as, using $\mathbf{X}$ to include all x, y, z,

$$\begin{aligned} \mathbf{X} &= \mathbf{A}\,\mathbf{X}^* \\ \mathbf{X}^* &= \mathbf{A}^{-1}\,\mathbf{X} = \mathbf{A}^T\mathbf{X}. \end{aligned} \qquad (13)$$

in which explicitly

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos\varphi \cos\kappa & \cos\omega \sin\kappa + \sin\omega \sin\varphi \cos\kappa & \sin\omega \sin\kappa - \cos\omega \sin\varphi \cos\kappa \\ -\cos\varphi \sin\kappa & \cos\omega \cos\kappa - \sin\omega \sin\varphi \sin\kappa & \sin\omega \cos\kappa + \cos\omega \sin\varphi \sin\kappa \\ \sin\varphi & -\sin\omega \cos\varphi & \cos\omega \cos\varphi \end{bmatrix} \qquad (14)$$

It is convenient to note that the values of the nine elements of $\mathbf{A}$ are handily formed in a computer by matrix multiplication as indicated by

[9] The problem of exterior orientation in photogrammetry by George H. Rosenfield, *Photogrammetric Engineering*, v. 25 (2.7), p. 544, 1959.

Rosenfield[10] which demonstrates that A is composed of the three sequential plane rotations:

$$A = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \tag{15}$$

It is also useful later to form the product of the last two matrices first inasmuch as the order of formation is otherwise irrelevant:

$$A = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\omega\sin\varphi & -\cos\omega\sin\varphi \\ 0 & \cos\omega & \sin\omega \\ \sin\varphi & -\sin\omega\cos\varphi & \cos\omega\cos\varphi \end{bmatrix}. \tag{16}$$

As derived in appendix 6, the basic projective transformation equations are (fig. 6):

$$\frac{x}{z} = \frac{(X-X_0)\,a_{11} + (Y-Y_0)\,a_{12} + (Z-Z_0)\,a_{13}}{(X-X_0)\,a_{31} + (Y-Y_0)\,a_{32} + (Z-Z_0)\,a_{33}}$$

$$\frac{x}{z} = \frac{(X-X_0)\,a_{21} + (Y-Y_0)\,a_{22} + (Z-Z_0)\,a_{23}}{(X-X_0)\,a_{31} + (Y-Y_0)\,a_{32} + (Z-Z_0)\,a_{33}} \tag{17}$$

where X, Y, Z are the coordi ates of an object on the ground, X_0, Y_0, Z_0 are the coordinates of the camera station in the same system and x, y, z are the image coordinates, in which $z = -f$, the camera focal length. (Compare Equation (1).) By clearing fractions and transposing,

$$\begin{aligned} &x\left[(X-X_0)\sin\varphi + (Y-Y_0)(-\sin\omega\cos\varphi) + (Z-Z_0)\cos\omega\cos\varphi\right] \\ &\quad -z\left[(X-X_0)\cos\varphi\cos\kappa + (Y-Y_0)(\cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa)\right. \\ &\qquad \left. + (Z-Z\ )(\sin\omega\sin\kappa - \cos\omega\sin\varphi\cos\kappa)\right] = 0 \\ &y\left[(X-X_0)\sin\varphi + (Y-Y_0)(-\sin\omega\cos\varphi) + (Z-Z_0)\cos\omega\cos\varphi\right] \\ &\quad -z\left[(X-X_0)(-\cos\varphi\sin\kappa) + (Y-Y_0)(\cos\omega\cos\kappa - \sin\omega\sin\varphi\sin\kappa)\right. \\ &\qquad \left. + (Z-Z_0)(\sin\omega\cos\kappa + \cos\omega\sin\varphi\sin\kappa)\right] = 0\ . \end{aligned} \tag{18}$$

If A_i is defined as representing the three elements in row i of the matrix in Equation (11)

$$A_i = (a_{i1}\ a_{i2}\ a_{i3})$$

and also

$$B = \begin{bmatrix} X-X_0 \\ Y-Y_0 \\ Z-Z_0 \end{bmatrix}$$

Then Equation (17) can be expressed in determinant notation:

$$\begin{vmatrix} x & z \\ A_1 B & A_3 B \end{vmatrix} = 0, \quad \begin{vmatrix} y & z \\ A_2 B & A_3 B \end{vmatrix} = 0. \tag{19}$$

It should be noted that Equations (17), (18), and (19) are merely different forms of the same equation which expresses the condition that the image, object and perspective center are colinear. It is this condition we seek to enforce, and if perchance the condition does not exist, we wish to allow incremental corrections to the observed coordinates x, y, such that sum of the squares of the corrections is minimum.

Equation (18) is transcendental and, in the most general case, all twelve terms are considered as unknowns. Consequently a form of Newton's Method[11] is used to solve them as shown in appendix 8. This is an iterative method based on initial approximations which are quite easily obtained for all the unknowns. Experience indicates that about 95 percent of the problems require three iterations. Applying partial differentiation and rearranging the terms, using $v_x = dx$ and $v_y = dy$, the following *observation* equations can be formed (as explained in app. 8)

[10] Ibid.

[11] Handbook of engineering fundamentals by Ovid W. Eshbach, *John Wiley & Sons*, second edition, pp. 2-16, 1957.

$$A_3B\,v_x = p_{11} + p_{12}d_\omega + p_{13}d\varphi + p_{14}d\kappa - p_{15}dX_0 - p_{16}dY_0 - p_{17}dZ_0 + p_{15}dX + p_{16}dY + p_{17}dZ$$
$$A_3B\,v_y = p_{21} + p_{22}d\omega + p_{23}d\varphi + p_{24}d\kappa - p_{25}dX_0 - p_{26}dY_0 - p_{27}dZ_0 + p_{25}dX + p_{26}dY - p_{27}dZ \tag{20}$$

in which the p-coefficients are defined by Equation (21) (table 1). (See also app. 10.)

Table 1.—The basic coefficients of the observation equations (Equation (21)) used in relative orientation, resection, and block adjustment

$$p_{11} = \begin{vmatrix} x & z \\ A_1B & A_3B \end{vmatrix} \qquad p_{21} = \begin{vmatrix} y & z \\ A_2B & A_3B \end{vmatrix}$$

$$p_{12} = \begin{vmatrix} x & z \\ \frac{\partial A_1}{\partial \omega}B & \frac{\partial A_3}{\partial \omega}B \end{vmatrix} \qquad p_{22} = \begin{vmatrix} y & z \\ \frac{\partial A_2}{\partial \omega}B & \frac{\partial A_3}{\partial \omega}B \end{vmatrix}$$

$$p_{13} = \begin{vmatrix} x & z \\ \frac{\partial A_1}{\partial \varphi}B & \frac{\partial A_3}{\partial \varphi}B \end{vmatrix} \qquad p_{23} = \begin{vmatrix} y & z \\ \frac{\partial A_2}{\partial \varphi}B & \frac{\partial A_3}{\partial \varphi}B \end{vmatrix}$$

$$p_{14} = \begin{vmatrix} x & z \\ \frac{\partial A_1}{\partial \kappa}B & \frac{\partial A_3}{\partial \kappa}B \end{vmatrix} \qquad p_{24} = \begin{vmatrix} y & z \\ \frac{\partial A_2}{\partial \kappa}B & \frac{\partial A_3}{\partial \kappa}B \end{vmatrix}$$

$$p_{15} = \begin{vmatrix} x & z \\ a_{11} & a_{31} \end{vmatrix} \qquad p_{25} = \begin{vmatrix} y & z \\ a_{21} & a_{31} \end{vmatrix}$$

$$p_{16} = \begin{vmatrix} x & z \\ a_{12} & a_{32} \end{vmatrix} \qquad p_{26} = \begin{vmatrix} y & z \\ a_{22} & a_{32} \end{vmatrix}$$

$$p_{17} = \begin{vmatrix} x & z \\ a_{13} & a_{33} \end{vmatrix} \qquad p_{27} = \begin{vmatrix} y & z \\ a_{23} & a_{33} \end{vmatrix}$$

The partial derivatives of A are formed from Equation (14):

$$\frac{\partial A}{\partial \omega} = \begin{bmatrix} 0 & -\sin\omega\sin\kappa + \cos\omega\sin\varphi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa \\ 0 & -\sin\omega\cos\kappa - \cos\omega\sin\varphi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\varphi\sin\kappa \\ 0 & -\cos\omega\cos\varphi & -\sin\omega\cos\varphi \end{bmatrix} = \begin{bmatrix} 0 & -a_{13} & a_{12} \\ 0 & -a_{23} & a_{22} \\ 0 & -a_{33} & a_{32} \end{bmatrix} \tag{22}$$

$$\frac{\partial A}{\partial \varphi} = \begin{bmatrix} -\sin\varphi\cos\kappa & \sin\omega\cos\varphi\cos\kappa & -\cos\omega\cos\varphi\cos\kappa \\ \sin\varphi\sin\kappa & -\sin\omega\cos\varphi\sin\kappa & \cos\omega\cos\varphi\sin\kappa \\ \cos\varphi & \sin\omega\sin\varphi & -\cos\omega\sin\varphi \end{bmatrix} \tag{23}$$

(Note for computation that Equation (23) closely resembles Equation (16): the first two rows can be formed from the third row of Equation (16) by multiplying by ($-\cos\kappa$) and ($+\sin\kappa$), respectively, and that the third row of (23) is the first row of (16).)

$$\frac{\partial A}{\partial k} = \begin{bmatrix} -\cos\varphi\sin k & \cos\omega\cos\kappa - \sin\omega\sin\varphi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\varphi\sin\kappa \\ -\cos\varphi\cos k & -\cos\omega\sin k - \sin\omega\sin\varphi\cos k & -\sin\omega\sin k + \cos\omega\sin\varphi\cos k \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ -a_{11} & -a_{12} & -a_{13} \\ 0 & 0 & 0 \end{bmatrix} \tag{24}$$

Equation (20) is in a sense a "universal" type of formula for analytic photogrammetry. It is used here to solve three somewhat different problems: relative orientation, resection, and in the block adjustment of either a strip and a genuine block of photographs. If the approximate values of X, Y, Z are sufficiently near correct (which is the function of the Provisional Solution, fig. 1), the dX, dY, dZ terms may be neglected leaving six unknowns. In relative orientation, however, these three terms cannot be neglected, and also the term in dX_0 has no significance. It is shown presently how the three terms can be eliminated, leaving five independent unknowns.

It is shown by Hellmut Schmid[12] as well as in appendix 7 that the elevation Z of an object whose

[12] An analytical treatment of the problem of triangulation by stereophotogrammetry by Hellmut H. Schmid, Report No. 961, *Ballistic Research Laboratories, Aberdeen Proving Ground*, Md., p. 14, 1955.

images appear on two photographs can be expressed in general as

$$Z = \frac{(X_0'' - X_0')\, z^{*\prime}\, z^{*\prime\prime} + Z_0' x^{*\prime}\, z^{*\prime\prime} - Z_0'' x^{*\prime\prime} z^{*\prime}}{x^{*\prime}\, z^{*\prime\prime} - x^{*\prime\prime} z^{*\prime}} \quad (25)$$

where the primes refer to the first and second photographs and the asterisks are defined by Equation (12). Once Z is evaluated,

$$X = X_0' + x^{*\prime}(Z - Z_0')/z^{*\prime}$$
$$Y = Y_0' + y^{*\prime}(Z - Z_0')/z^{*\prime} \quad (26)$$

In the relative orientation problem these equations simplify to

$$Z = \frac{Z_0\, x^* - z^*}{x^* - (x/z) z^*}, \quad X = x\, Z/z \;, \quad Y = y\, Z/z \quad (27)$$

because the first photograph is considered as untilted, the first camera station can be selected as the origin, and the abscissa of the second station can be selected as unity. As a consequence of Equations (27), the terms in dX, dY, dZ in Equation (20) can be eliminated by substitution and expressed in terms of the other unknowns $d\omega$, $d\varphi$, $d\kappa$ and dZ_0 resulting in the special observation equations for relative orientation (app. 9)

$$A_3 B v_x = p_{11} + (p_{12} + S_1 T_2)\, d\omega + (p_{13} + S_1 T_3)\, d\varphi + (p_{14} + S_1 T_4)\, d\kappa - p_{16}\, dY_0 - (p_{17} - S_1 T_7)\, dZ_0$$
$$A_3 B v_y = p_{21} + (p_{22} + S_2 T_2)\, d\omega + (p_{23} + S_2 T_3)\, d\varphi + (p_{24} + S_2 T_4)\, d\kappa - p_{26}\, dY_0 - (p_{27} - S_2 T_7)\, dZ_0 \quad (28)$$

in which

$$T_2 = \begin{vmatrix} X-1 & Z-Z_0 \\ (\partial A_1^T/\partial\omega)C & (\partial A_3^T/\partial\omega)C \end{vmatrix}$$

$$T_3 = \begin{vmatrix} X-1 & Z-Z_0 \\ (\partial A_1^T/\partial\varphi)C & (\partial A_3^T/\partial\varphi)C \end{vmatrix}$$

$$T_4 = \begin{vmatrix} X-1 & Z-Z_0 \\ (\partial A_1^T/\partial\kappa)C & (\partial A_3^T/\partial\kappa)C \end{vmatrix}$$

$$T_7 = A_1^T C (= x^*)$$

$$C = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$S_1 = (1/u_2)(p_{15}\, u_3 + p_{16}\, u_4 + p_{17})$$
$$S_2 = (1/u_2)(p_{25}\, u_3 + p_{26}\, u_4 + p_{27})$$
$$u_1 = Z\, x^* - z^* \qquad u_3 = x/z$$
$$u_2 = x^* - u_3\, z^* \qquad u_4 = y/z$$

The subscript 2 in C indicates the image coordinates on the second photograph, A^T being defined by Equations (13) and (12). Specifically

$$A_1^T C = x^* = a_{11}\, x_2 + a_{21}\, y_2 + a_{31}\, z_2 \quad (= T_7)$$
$$A_3^T C = z^* = a_{13}\, x_2 + a_{23}\, y_2 + a_{33}\, z_2$$

and the partial terms refer to elements of *columns* in Equations (21), (22), and (23). (See also app. 10.)

The pair of observation Equations (28) occurs for each image on each photograph. In the resection and aerotriangulation problems both equations apply, whereas in relative orientation only the y-equation is significant. Nevertheless both equations are utilized for programming uniformity. The problem then becomes one of solving a large number of simultaneous linear equations, applying least squares, for unique values of the five unknowns $d\omega$, $d\varphi$, $d\kappa$, dY_0, dZ_0. These are changes which need to be made in the five approximate values of the unknowns themselves, after which the computation is repeated using the corrected values. The iteration terminates when the corrections become insignificant, specifically, when each of three $d\omega$, $d\varphi$, $d\kappa$ is less than 10^{-5} radians.

Initial approximations consist of zeros for all five parameters but other values can be entered if they are considered to be more appropriate.

The output of the computation consists of (1) the values of the five parameters, (2) the values of the nine elements in the final A-matrix, (3) provisional or "model" coordinates for each object, (4) and the residuals (p_{11}, p_{21}) for each image.

The purposes of the relative orientation procedure consists of: (1) grossly mistaken data are indicated by the residuals enabling discarding and recomputing; and (2) an interrelated system of Cartesian coordinates of ground objects which can be transformed into a single continuous coordinate system for an entire strip or block of photographs.

In relative orientation, each pair of photographs is treated entirely independent from the others. Consequently no special problem occurs if one or more intermediate pairs are recomputed.

Relative orientation is arranged as a single IBM 650 program requiring about five minutes per model.

2.22 Cantilever Assembly

After the completion of the relative orientation computation for each model of a strip of pho-

tographs, the models are connected into a single continuous chain by means of successive coordinate transformations of rotation, dilation, and translation for all the objects in the models based on the coordinates of common objects in adjacent models, along with the orientation data, and without any ground control information, but referred to the first model as a reference system. It is termed "cantilever" because of the successive attachment of each model to the preceeding one. As one might expect, the assembly is affected by an accumulation of systematic errors, but these errors are adjusted as described in the next section.

If $\mathbf{R}_i$ denotes a rotation matrix for the model coordinates of the i-th model in the same sense that $\mathbf{A}$ is used in Equations (13), (12), (11), (10) and

$$\mathbf{X}' = \mathbf{R}\,\mathbf{X} \tag{29}$$

such that $\mathbf{X}$ consists of the model coordinates before rotation and $\mathbf{X}'$ those after rotation, it can be shown that

$$\mathbf{R}_i = \mathbf{R}_{i-1}\,\mathbf{A}^T_{i-1}\,. \tag{30}$$

In words, the rotation matrix for any model can be determined by forming the matrix product of that of the previous model and the transpose matrix from relative orientation. In order to get the system started, (1) the model coordinates in the first model are considered to be already in the desired system and require no further rotation; (2) those in the second model are rotated by the application Equation (29) in which $\mathbf{R} = \mathbf{A}^T$ of the first model; (3) thereafter Equation (30) applies. It may be noteworthy that $\mathbf{R}$ is a function only of angular parameters, ω, φ, κ,

A scale factor m is determined from

$$m^2 = \frac{(X_2' - X_1')^2 + (Y_2' - Y_1')^2 + (Z_2' - Z_1')^2}{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \tag{31}$$

in which X, Y, Z are the model coordinates of a selected common object and X', Y', Z' are its final coordinates in the previous model. The two objects are selected by a photogrammetrist from opposite sides of the model on the basis of favorable residual parallax values from relative orientation. All coordinates of objects in a model being attached are then changed by multiplying by the common factor m. The coordinates X_0, Y_0, Z_0 of both camera stations are also multiplied by this factor. (In the present program the initial camera station in each model is assigned arbitrary, constant, nonnegative coordinates, whereas some simplification might have been achieved by assigning zeros.)

A translation of coordinates is needed so as to form a single continuous system. The translation elements a, b, c are based on the coordinates of the camera station which is common to the two models:

$$a = X_0' - X_0\,, \quad b = Y_0' - Y_0\,, \quad c = Z_0' - Z_0 \tag{32}$$

where the primed coordinates are those of the previous model and the unprimed coordinates are those of the subject model after rotating and scaling. Then the translated coordinates of any object point are

$$X' = X + a,\ Y' = Y + b,\ Z' = Z + c \tag{33}$$

which is applied to all the object coordinates in the subject model ***and also those of the second camera station*** for use in the next model.

After the three operations of rotation, scaling, and translation are complete, those object points common to both models will obviously have two slightly different sets of coordinates. The mean value is computed and adopted as the final value, and the deviation from the mean for each of the three coordinates is printed out. The function of the deviations is to enable the detection by visual scanning of any unusually large errors which might be sufficient reason to disregard a given point in subsequent mapping applications.

2.23 Adjustment of the Cantilever Data

For the adjustment of the cantilever strip, two computer programs normally used in the adjustment of data from plotting instruments are applied without change. These were described by W.D. Harris[13 14] based on the work of R.S. Brandt[15] and C.W. Price[16] and the formulation is repeated here. The programs are relatively short and might easily be combined into a single step. The formulas provide a generally conformal transformation of the coordinates of points using quadratic and cubic terms to correct for the accumulation of systematic errors. The input of the horizontal program[17] consists of a list of cantilever coordinates (Equation (33)) for four to ten control points and also a list of the State Plane Coordinates of the same points,

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13 Aerotriangulation adjustment of instrument data by conventional methods by William D. Harris, *Coast and Geodetic Survey Technical Bulletin No. 1*, 1958.

14 Vertical adjustment of instrument aerotriangulation by computational methods by William D. Harris, *Coast and Geodetic Survey Technical Bulletin No. 10*, 1959.

15 Resume of aerial triangulation adjustment at the Army Map Service by Robert S. Brandt, *Photogrammetric Engineering*, v. 17, 1951.

16 Some analysis and adjustment methods in planimetric aerial triangulation by Charles W. Price, *Photogrammetric Engineering*, v. 19, 1953.

17 Harris, William D., see footnote 13, above.

in addition to a list of the cantilever coordinates of all the other points in the strip of photography. The vertical program[18] similarly consists of a list of coordinates of points in the two systems, including the ground elevations of the objects. The ***positions*** of bench marks are obtained from the horizontal adjustment; and the vertical program furnishes ***elevations*** of horizontal control stations, values which are usually not determined during field operations. The output consists of the State Plane Coordinates and elevations of all the points used in the strip. The coordinates are converted to geographic positions by a separate computer program.

As the programs were devised for use with plotting instrument data, some features and precautions might be unnecessary in analytic work. But the cost of reprograming, and the added work load on the programing staff, precludes unneeded reprograming for the present since the program serves adequately.

Horizontal Adjustment.—The origin of the coordinate system is preferred to be at the center of a strip because of symmetry and computer scaling, and the abscissa is assigned to the longitudual axis of the strip. To transform the centilever strip coordinates x, y, into such a system, a standard type of combined rotation and translation is introduced:

$$x' = ax - by + c, \quad y' = bx + ay + d \tag{34}$$

in which x', y' are the transformed values. The constants a, b, c, d, determined by solving (34) for two known points simultaneously, are

$$a = \frac{(x_1 - x_2)(x'_1 - x'_2) + (y_1 - y_2)(y'_1 - y'_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$b = \frac{(x_1 - x_2)(y'_1 - y'_2) - (y_1 - y_2)(x'_1 - x'_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$c = x'_1 - ax_1 + by_1, \quad d = y'_1 - bx_1 - ay_1$$

in which x_1, y_1, x_2, y_2 are the coordinates in the cantilever of points near the centers of the first and last photographs of the strip, respectively, and x'_1, y'_1, x'_2, y'_2 are the transformed coordinates of these two centers having the values

$$x'_2 = + \tfrac{1}{2} \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{1/2}$$

$$x'_1 = -x_2, \qquad y'_1 = y'_2 = 0.$$

Once a, b, c, d are determined, all the points in the strip are transformed using Equation (34).

The next part of the solution uses the same type of transformation to convert the new strip coordinates into ground coordinates, and the inverse transformation to convert the State Plane Coordinates of horizontal ground control stations into the strip coordinate system for purposes of comparison. Actually the inverse is utilized first, and then the direct transformation is used later as a final step in the procedure. This routine was considered necessary because of the use of diagonally flown photographs. The transformation is based on the State Plane Coordinates X_1, Y_1, X_2, Y_2 of two control points near the ends of the strip, together with their strip coordinates x_1, y_1, x_2, y_2. The direct transformation is

$$X = ax - by + c, \qquad Y = bx + ay + d \tag{35}$$

where the inverse is

$$x = a'X + b'Y - c, \quad y = -b'X + a'Y - d' \tag{36}$$

and the values of the coefficients are

$$a = \frac{(x_1 - x_2)(X_1 - X_2) + (y_1 - y_2)(Y_1 - Y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$b = \frac{(x_1 - x_2)(Y_1 - Y_2) - (y_1 - y_2)(X_1 - X_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{37}$$

$$c = X_1 - ax_1 + by_1, \quad d = Y_1 - bx_1 - ay_1$$

$$a' = a/(a^2 + b^2), \qquad b' = b/(a^2 + b^2)$$

$$c' = (ac + bd)/(a^2 + b^2), \quad d' = (ad - bc)/(a^2 + b^2).$$

Consequently, the constants are determined after which Equation (36) is used to transform the coordinates of all the horizontal control stations into the cantilever coordinate system. The similarity between Equations (34) and (35) is utilized advantageously in computer programing.

The cantilever coordinates x, y ordinarily differ from the transformed control coordinates x', y' for all the control points except the two that are used to determine the constants in Equations (37). A conformal polynomial transformation[19] is used to express the relationship between these differing coordinate values:

$$x' = Ax^3 + Bx^2 + (C+1)\,x - 2Dxy - Ey - F$$

$$y' = 3Ax^2y + 2Bxy + (C+1)\,y + Dx^2 + Ex + G\,. \tag{38}$$

These equations are used first with control point coordinates to determine the values of the seven unknown coefficients $A \ldots G$. Ordinarily more than four control points are used, making more than eight equations which are solved applying least squares in the same routine as

[18] Harris, William D., see footnote 14, above.

[19] Harris, William D., see footnote 13, above.

Equations (28) earlier. Then the equations are applied to determine corrected cantilever coordinates for all the other points in the strip. Next, Equation (35) is applied to transform the cantilever coordinates into the State Plane Coordinates. This is the place where the program for horizontal coordinates has terminated for all production jobs to date.

Vertical Adjustment.—The vertical adjustment follows the horizontal adjustment although it is considered that they ought to be done simultaneously. The horizontal data from the previous program is utilized. Specifically, the values after the application of Equations (38) (and not Equation (35)) are used because they are symmetrically arranged with regard to the center of the strip.

As a preliminary measure, the ground elevations are transformed into the cantilever system through scaling and translation using

$$z' = Z/g + k \tag{39}$$

whose inverse form is

$$Z = g(z' - k) \tag{40}$$

where

$$g^2 = \frac{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \tag{41}$$

the coordinates being those used to determine the constants in Equation (37), and where k is a rough constant determined by solving Equation (39) with any normal vertical control station. Once g and k are evaluated, all the control elevations in the strip are transformed using the same equation.

As with the horizontal coordinates, the vertical cantilever coordinates z at control points do not agree with transformed values z' except for the single point used in deriving k. A polynomial similar to that used in Equation (38) is used to indicate the relationship of the different values:

$$z' = z + Hx^3 + Ix^2 + Jx + Kx^2y + Lxy + My + N \tag{42}$$

which introduces seven new unknown coefficients $H...N$. Using at least seven vertical control points, Equation (42) is formed for each point where the coefficients are regarded as unknowns. Solving these as simultaneous linear equations using least squares (as for Equations (28) and (38)), unique values are obtained for the coefficients. Then with the coefficients known, Equation (42) is applied to all the other points in the strip to determine z'. The next step is to find the corrected ground elevation Z (or h) for each of the points with Equation (40).

This until now completes the provisional adjustment, yielding coordinate values which are ready to use for map compilation or other applications as survey data. The print-out includes the computed coordinates at control stations which should be identical to known correct values and allows one to scan the results to detect faulty data. Large differences are due to erroneous data. As the computer program requires but ten minutes for a long strip, little cost is involved by recomputing with any erroneous or questionable data deleted.

Where strips have been very long and involve mountainous terrain, the secant plane system of the next section has been applied. These cantilever adjustments are repeated with the different system of control data. The initial adjustments furnish sufficiently accurate elevations of horizontal control stations and the positions (which ordinarily do not result from field surveys) of vertical control stations for the secant plane transformation. The readjustments to the secant plane data fully recognized the effect of earth curvature. Obviously, the secant plane coordinates need to be transformed back into the State Plane Coordinate system for mapping uses as indicated by the Inverse Transform (fig. 1).

2.24 Transformation of Control Data to a Secant Plane System

As pointed out heretofore, the purpose of the secant plane transformation is to account for the curvature of the earth. The term "secant plane" is used instead of "geocentric" to maintain a correct nomenclature inasmuch as the classic geocentric system results in coordinate values which are too large for convenient handling in a computer and also bear little resemblence to map coordinates except in special cases. The secant plane system is a local system pertinent to the map project and in which the coordinate directions are comparable to map directions.

The formulation for the block adjustment introduces a three dimensional Cartesian coordinate system in which X and Y are comparable to horizontal grid coordinates and Z is comparable to elevation. Geographic positions and elevations of control points (data obtained from the previous section) may be transformed into such a space system in which the Z-axis is the extension of the normal to the ellipsoid of a point near the center of the mapping project. This point (the origin) may be any selected value of latitude and longitude and not necessarily a control or pass point. The XY-plane should be "secant" to the ellipsoid in order to avoid negative Z-values. The Y axis is in the plane of the meridian of the origin and thus may be considered as the north-south axis.

Formulas for this transformation are derived by rotating and translating modified geocentric coordinates. Classical geocentric coordinates[20] are

$$X = (N + h) \cos \varphi \sin \lambda$$

$$Y = (N + h) \cos \varphi \cos \lambda$$

$$Z = [N(1 - e^2) + h] \sin \varphi \qquad (43)$$

where φ and λ are the latitude and longitude of any point, h is the elevation (the geoid separation may be included with h if thought essential), and N is the length of the normal through φ, λ. (Consider λ negative in the western hemisphere).

The center of rotation of this system is the intersection of the normal and the earth's axis of rotation (Z-axis) (fig. 7). Therefore a small increment, $N_0 e \sin \varphi_0$, should be added to the Z-coordinates. N is the length of the normal for the origin and φ_0 is the latitude of the origin. Thus,

$$Z = [N(1 - e^2) + h] \sin \varphi + N_0 e^2 \sin \varphi_0 .. \qquad (44)$$

The computation of the rotation process depends on the angular relation in space of the two sets of axes. The rotated X-axis will remain in the original XY-plane and will make an angle of λ_0 with the original X-axis. The rotated Z-axis will make an angle of $(90° - \varphi)$ with the original Z-axis. The direction of the Y-axis should be reversed so that positive Y would indicate north. Also, positive X indicates east.

The nine cosines of the angles in space between the two sets of axes can be expressed as

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = \begin{bmatrix} \cos \lambda_0 & -\sin \varphi_0 \sin \lambda_0 & +\cos \varphi_0 \sin \lambda_0 \\ -\sin \lambda_0 & -\sin \varphi_0 \cos \lambda_0 & +\cos \varphi_0 \cos \lambda_0 \\ 0 & \cos \varphi_0 & \sin \varphi_0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad (45)$$

in which the subscript G is used to indicate the modified geocentric coordinates of Equations (43) and (44), and from which the inverse expansion may be stated as

$$X = X_G \cos \lambda_0 - Y_G \sin \lambda_0$$

$$Y = -X_G \sin \varphi_0 \sin \lambda_0 - Y_G \sin \varphi_0 \cos \lambda_0 + Z_G \cos \varphi$$

$$Z = X_G \cos \varphi_0 \sin \lambda_0 + Y_G \cos \varphi_0 \cos \lambda_0 + Z_G \sin \varphi. \qquad (46)$$

These Z-coordinates are too large for practical computation, being of the order of N or 6,380,000 meters ±. Therefore, the system may be translated along the Z-axis to a "secant" position. The amount of translation is a quantity slightly less than N so that all values of Z will remain positive. The X and Y values in Equation (46) do not change during the translation to this "secant" position.

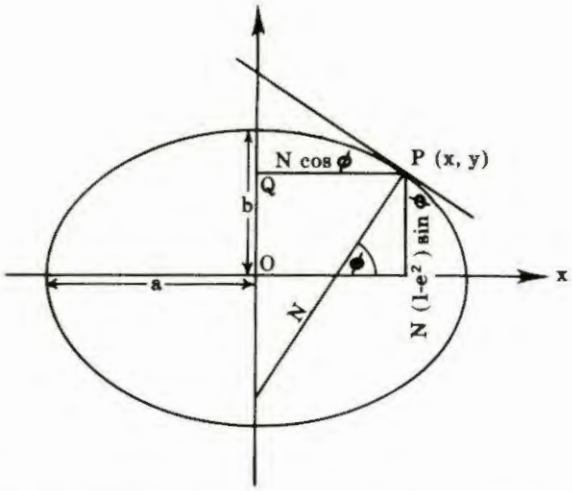

FIG. 7.—Geometric elements used in the secant plane transformation.

These X, Y, Z-coordinates are in the form required for either the readjustment described in Section 2.23 or for the next Section 2.25 on Resection.

After the block adjustment has been completed (Section 2.3.1), the final adjusted X, Y, Z's need to be transformed back to latitudes, longitudes and elevations. (Conversion to State Plane Coordinates is accomplished by a separate computer program.)

The direct expansion of Equation (45), after applying the constant of translation to the Z-coordinates, is

[20] Transformation of rectangular space coordinates by Erwin Schmid, *Coast and Geodetic Survey Technical Bulletin No. 15*, 1960.

$$X_G = X \cos \lambda_0 - Y \sin \varphi_0 \sin \lambda_0 + Z \cos \varphi_0 \sin \lambda_0$$

$$Y_G = -X \sin \lambda_0 - Y \sin \varphi_0 \cos \lambda_0 + Z \cos \varphi_0 \cos \lambda_0$$

$$Z_G = \qquad + Y \cos \varphi_0 \qquad + Z \sin \varphi_0 \qquad (47)$$

From Equation (43),

$$\tan \lambda = X_G / Y_G \qquad (48)$$

from which the longitude is calculated. Also from Equations (43) and (44),

$$\tan \varphi = Z_G / (X_G^2 + Y_G^2)^{1/2} \text{ approximately.} \qquad (49)$$

By an iterative process involving two iterations, the second term of Equation (44) can be corrected to be consistent with the latitude of the point being transformed rather than based on the origin, then the Z of (44) is

$$Z = (N + h) \sin \varphi \qquad (50)$$

and Equation (49) is adequate to compute an accurate latitude. The elevation h may be computed from Equation (43),

$$h = (X_G / \cos \varphi \sin \lambda) - N \quad \text{or}$$

$$h = (Y_G / \cos \varphi \sin \lambda) - N \,, \qquad (51)$$

using the equation with the larger function involving λ.

The computer program for the direct transformation may be modified to make the inverse become the similarity of operations and equations. The input and output of the two types of computation are merely reversed.

2.25 Resection

A solution of the resection problem is needed for each photograph as initial approximations for the block adjustment. However, this portion of the program is designed not only to yield the six parameters, but also to furnish the coefficients of the normal equations for the block adjustment inasmuch as the routine is a relatively short one and as this is a favorable stage for terminating one program and beginning the next one.

Resection in photogrammetry is defined as the determination of the six fundamental parameters $\omega, \varphi, \kappa, X_0, Y_0, Z_0$ of a single photograph from the given positions and elevations of at least three non-colinear points imaged on the photograph. Although most of these parameters were considered in the Section 2.21 on relative orientation, these elements were referred to different assumed and unrelated reference systems for each separate photograph. Now after Section 2.24 on each photograph are ordinarily available 18 points whose ground coordinates are known with a fair degree of accuracy expressed in a single secant plane system for the entire block of photographs. Inasmuch as all large mistakes have been detected, and as the accuracy cannot be improved appreciably at this stage, a unique solution using only three points is applicable.

A form of Equation (20) is repeated:

$$A_3 B v_x = p_{11} + p_{12} d\omega + p_{13} d\varphi + p_{14} d\kappa - p_{15} dX_0 - p_{16} dY_0 - p_{17} dZ_0$$

$$A_3 B v_y = p_{21} + p_{22} d\omega + p_{23} d\varphi + p_{24} d\kappa - p_{25} dX - p_{26} dY - p_{27} dZ_0 \,. \qquad (52)$$

These equations with the coefficients already defined by Equation (21) apply to the resection problem. The S and T terms of Equation (28) are not used because at this stage it can safely be assumed that dX, dY, dZ are insignificantly small. Thus by using an abbreviated form of the routine for relative orientation, six simultaneous linear observation equations can be formed for a photograph and solved in the same iterative manner for the unknowns as used before.

The initial approximations of the angular terms ω, φ can again be zeros, but κ and the linear terms X_0, Y_0, Z_0 are not zero. As an image near each principal point is invariably selected and carried with the others through Section 2.24, the X- and Y-coordinates of this point are satisfactory first approximations for X_0, Y_0. In the first photograph of block, Z_0 is taken initially as the reported flight altitude. For all the other photographs, the finally iterated value of the previous photograph, is used as the initial approximation.

The angle κ is approximated initially through the following analysis involving the photo coordinates x_1, y_1, x_2, y_2 of the two well-separated images and the ground coordinates X_1, Y_1, X_2, Y_2 of the two corresponding objects. The angle κ may be defined (app. 5) as the counter clockwise angle from the reference X-axis to the corresponding photo x-axis. Inasmuch as the line joining the two points in one coordinate system is identical to the line of the other system, κ is the difference in the inclinations: $\kappa = \alpha_1 - \alpha_2$ where α_1 refers to the photo system.

From trigonometry,

$$\tan \varkappa = (\tan \alpha_1 - \tan \alpha_2)/(1 + \tan \alpha_1 \tan \alpha_2).$$

But from analytic geometry

$$\tan \alpha_1 = (y_2 - y_1)/(x_2 - x_1)$$
$$\tan \alpha_2 = (Y_2 - Y_1)/(X_2 - X_1).$$

Through substitution and simplification,

$$\tan \varkappa = \frac{(y_2 - y_1)(X_2 - X_1) - (x_2 - x_1)(Y_2 - Y_1)}{(x_2 - x_1)(X_2 - X_1) + (y_2 - y_1)(Y_2 - Y_1)} \qquad (53)$$

also from trigonometry

$$\sin \varkappa = \tan \varkappa/(1 + \tan^2 \varkappa)^{1/2}$$

defining $\sin \varkappa$ which is the form needed in the computation.

Thus no new problem is involved in determining the coefficients of the six observation equations and solving them for the unknowns using the same routine as already used for Equation (28), (38), and (42). But an important principal at this point facilitates the computation: *the coefficients of the observation equations on the last iteration of resection are exactly the ones that are also required in block adjustment!* It is only necessary to include not only the three points used in resection but also all those wanted in the computation.

Then also it is not necessary to store the coefficients themselves of each observation equation, but as each equation is formed, its contribution to the normal equation system is readily computed and accumulated insofar as possible into a system of "normal" terms and "partial" normals. (However, the coefficients are punched out, one equation per card, for later use in determining the values of the residuals.) As a simple example, suppose the observation equations had only three unknowns in the form

$$ax + by + cz = d. \qquad (54)$$

Regardless of how many observation equations there were, the normal equations would consist of exactly three equations of the same form and just 12 numbers. The first term of the first normal equation would consist of what had accumulated previously plus the product aa; the second term consists of what had accumulated previously plus the product ab. In a similar way, all the terms of the normal equation system are accumulated from the contributions of each observation equation at a time, after which the equation can be "forgotten" and then the next one formed. Thus after each resection solution for a photograph, the output is not only the six parameters, but also the contributions to the normal equation system for the block adjustment.

One of the principal parts of the resection solution is then a systematic method for numbering the normal equation coefficients. This becomes somewhat involved in a pure, large block of photographs, but has been well organized in the Electronic Computing Division for the conventional geodetic problem.

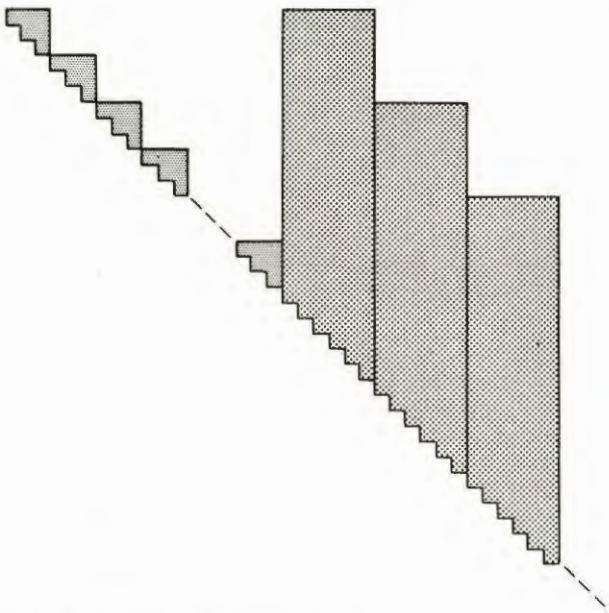

FIG. 8.—Diagram of non-zero coefficients in the system of normal equations for the block adjustment.

2.3 *Block Adjustment*

The normal equations computed in the resection procedure are very similar to those which are developed in the adjustment of area triangulation. The matrix consists of a series of disconnected sets of three unknowns each (for the ground points) and blocks of six-column data for the camera unknowns (fig. 8).

The normal equations are solved by the classical Gauss elimination with the Cholesky modification in order to reduce the computer storage requirement and to improve the numerical significance. These sets of equations are quite large but may be solved by direct methods without unreasonable effort.

An optimum number of pass points should be used to avoid developing an unduly large network. The number of unknowns in the set is equal to six times the number of photographs

plus three times the number of ground points. For example, a project with 20 photographs and an average of 5 additional ground points per additional plate would produce a network of (20 x 6) + (5 x 20 x 3) or 420 unknowns. Consequently it is planned to use only one of the two pass points observed in each of the locations. If a side overlap of 60 percent is used in a block of photographs, then the number of ground points need be only a few more than the number of photographs instead of five times the number.

The control points may be entered as observation equations and added directly to the normal equations prior to solution. If a point is a triangulation station without a known elevation, two equations $X = 0$ and $Y = 0$ are used. These may be weighted to suit the problem. For example, if a weight of 4 is used, 4^2 or 16 may be added to each respective diagonal term in the normal equations prior to solution. Similarly, $Z = 0$ may be used as the equation for elevation.

When control data are used in this manner, the X, Y, Z secant plane coordinates consistent with the control (Section 2.24) should be used in Section 2.25 for developing the respective observation equations, rather than use the X, Y, Z coordinates for the same points which resulted from the cantilever adjustment and subsequent transformation.

2.4 *The Inverse Transformation*

The inverse transformation consists of the use of Equation (46) to change the coordinate system back from that of the secant plane to a system which is pertinent to the mapping project, presumably a State Plane System for the X, Y coordinates and elevation above sea level for the Z coordinates. Obviously the final coordinate system might be any one of several systems, such as the Military Grid System, the Universal Transverse Mercator, or simply latitude, longitude and elevation (φ, λ, h). Programs had already been developed for these transformations.

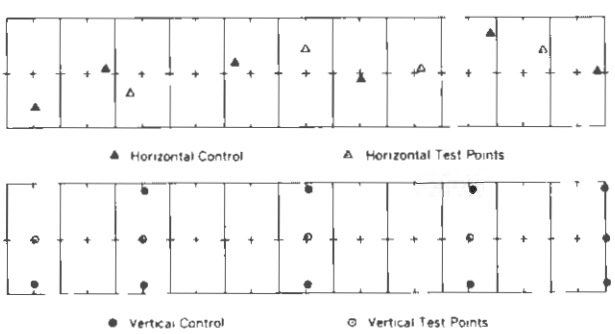

FIG. 9.—Control diagram for a portion of U.S. Route 11 temporary test area used for a comparison between analytic and instrumental methods.

3. REPORT OF RESULTS

As noted earlier, all productive results and tests completed prior to 1962 terminated with the adjustment of the cantilever strip (Section 2.23) whereas the block adjustment is only in the stage of preliminary testing. However, 25 strips of various lengths and various locations in the United States have been satisfactorily aerotriangulated in this manner for productive work alongside of conventional methods using first-order plotting (analog) instruments.

3.1 *Accuracy Tests*

Each computer program was tested initially with fictitious data as it was completed, but the most interesting part of the development of analytic aerotriangulation was the step by step testing with real photographs and the comparison of results with the rather well-perfected instrumental method. A comparison was made on 11 models of a strip of overcontrolled 1:40,000 scale photographs (fig. 9), which was flown specially for the evaluation of the analytic method. The upper diagram shows the distribution of horizontal ground control stations all of which were premarked with temporary photographic target panels. The six solid triangles indicate the control points that were used for a least squares adjustment of both instrumental and analytic aerotriangulations, and the open triangles the points that were withheld from the adjustment but included in the accuracy evaluation. The lower diagram shows the distribution of vertical points. These points, though not premarked, were established at sites where the terrain was flat. The same diapositives were used for both methods with the following results. The rms errors for the analytic method were 1.6 feet (12 μ (microns) at photo scale) for horizontal position, and 1.4 feet for elevation, while the instrumental method gave rms errors of 6.4 feet for position and 3.5 feet for elevation. The analytic method was used on another strip of photographs of the same area to compare the elevations obtained for the 69 common pass points between the strips: the rms elevation difference was 2.4 feet for the two independent analytic aerotriangulations, whereas, it was 6.9 and 7.4 feet between the instrumental and two analytic solutions. The maximum observed difference for the pass points was 6.6 feet (51μ) between the two analytic strips and 16 and 22 feet between the instrumental and the analytic strips.

As mentioned earlier, the Coast and Geodetic Survey method uses a third-degree conformal adjustment for the cantilever strip to obtain a better fit to ground control points and refined camera-orientation parameters for the final block adjustment. The adjustment curves shown in

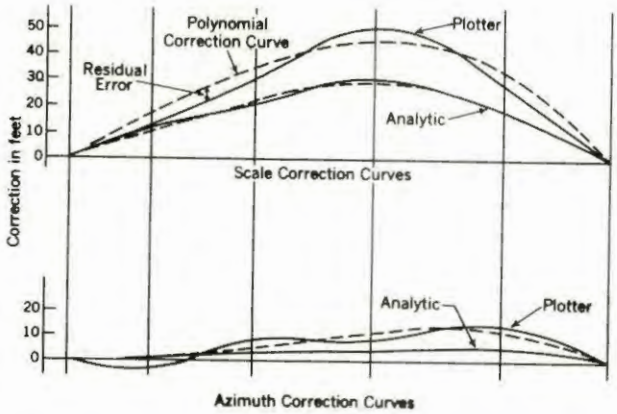

FIG. 10—Scale and azimuth correction curves for comparative test of analytic and instrumental strip aerotriangulations on area shown in figure 9.

figure 10 were obtained for the analytic and instrumental aerotriangulations in the test just described. The upper diagram shows the required scale correction for each method as a solid line. The dashed lines show the third-degree corrections that were applied by the least-squares curve-fitting adjustment and the separation between the solid and dashed lines shows the residual errors after adjustment. The lower diagram shows the same curves for the azimuth of the strip. The increased precision of the analytic method is indicated by the reduced amplitudes and smoothness of the respective curves. Notice also that the residual errors in azimuth are smaller, even to the point of practical nonexistence. The plotter curves are typical of our experience over the years and consequently an effort is always made to plan flight lines and control surveys so that a minimum of four well-spaced control points occur in each strip aerotriangulation.

The real test of analytic aerotriangulation was made by extending the number of models

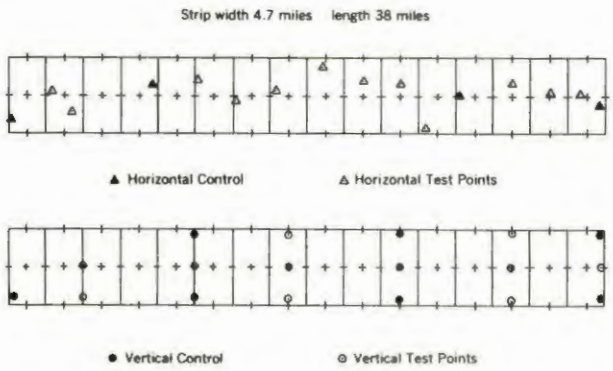

FIG. 11.—Control diagrams for a portion of U.S. Route 11 test area used for the comparison of two independent analytic aerotriangulations.

from 11 to 16 (fig. 11), thereby including 7 bands of vertical ground control points so that alternate bands could be withheld to serve as midspan and midstrip check points. At the same time, all but four of the horizontal control points were withheld. This arrangement of control permitted the use of the third-degree adjustment curves and gave a span of 8 models or 19 miles between points of known position and a span of 5 1/2 models or 12 miles between points of known elevation. Two separate sets of 1:40,000 scale photographs made with a 6-inch camera on a low distortion polyester aerial film were used on this test to determine the repeatability of results, and to compare the positions and elevations of hundreds of pass points obtained by two independent aerotriangulations. The rms horizontal error, based on all 15 control points, was 3.2 feet (24 μ) for one strip and 4.4 feet (33 μ) for the other. The corresponding vertical errors were 1.6 and 3.1 feet. Expressing these accuracies as fractions of the flight altitude, the standard error of position for 8 model spans was 1/5000 of the flight altitude, and the standard error of elevation for 6 model spans was 1/8000 of the flight altitude. From the cartographer's point of view, the accuracy of these aerotriangulations is equivalent to having all model pass points located and leveled by high-order field surveys. That is, with any but the most precise stereoscopic plotting instruments, working at scales up to five times the scale of the photography, the resulting improvement of relative and absolute orientation could not be set on the instrument and, even if it could, the additional accuracy would be lost in drafting. It was surprising to find that, prior to the nonlinear adjustment, the accumulated scale error at the center of the 38 mile strip amounted to 40 feet or 300 microns at plate scale even though the standard y-parallax within the models was only 5 microns and the standard error of tie between models was only 16 microns. It is expected that the use of a block adjustment may increase the y-parallaxes and reduce intermodel tie and accumulated scale errors.

In addition to the absolute accuracy evaluation of these two strips, all pass points were intertransferred between the two sets of photographs so that the agreement between independent aerotriangulations could be determined for points along the edges of the strip. The rms differences in pass point position and elevation determination between the two aerotriangulations were 3.5 feet (27 μ) and 3.8 feet respectively, and the maximum observed differences were 8 feet (61 μ) in position and 12 feet in elevation. The block adjustment of these two strips is awaited with interest to see how much more improvement can be made

with our present knowledge of systematic errors, such as lens and film distortion.

In another series of tests, the permanently targeted Ohio test area (fig. 12) was photographed with several stable base aerial films and with both wide-angle and the superwide-angle cameras. By selecting a photographic scale of 1:24,000, it was possible to obtain a short strip of five photographs on which the model pass-point areas contained premarked horizontal and vertical control points. The stereo-model limits, shown by the four rectangles, are representative of all strips used for camera and film comparisons. The 65 triangles show the distribution of premarked geodetic points of known position and elevation, and the crosses shown unmarked areas where the terrain elevation is known. The eight solid triangles indicate the three-dimensional control that was used for the adjustment of the strip.

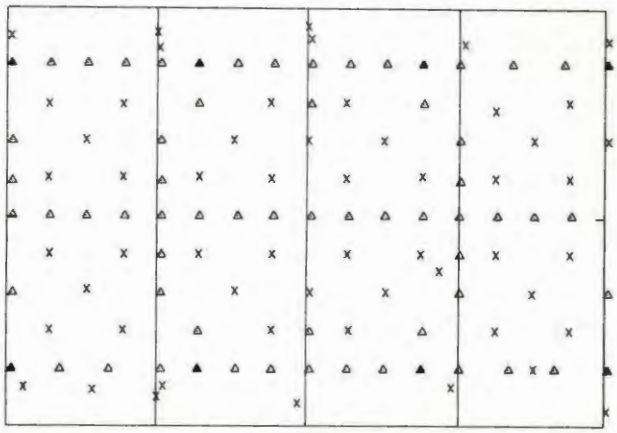

FIG. 12.—Control diagrams of the C&GS Ohio Camera Calibration Area used for comparisons of wide and super-wide angle cameras and several aerial films. The triangles show the distribution of premanently marked control points and the crosses indicate points of known elevation.

Data reduction is not complete on this airborne comparison of camera-film combinations, but results to date (table 2) indicate that both film and lens distortions are now well enough determined and compensated to enable their use with confidence as to their effect on accuracy. The first two columns show the step by step comparison of the two cameras using the same film, and the last two columns the comparison of two types of film bases in the same camera. The only significant difference in the size of y-parallax residuals was the low 4.8 microns obtained in relative orientation with the wide angle camera. This figure, which was obtained from clusters of four points at each of the pass point areas, increased to 6.3 microns when all 300 points in the strip were included. The difference in the base-height ratios of the cameras explains the reversal of position and elevation precisions indicated by the intermodel ties, except for the topographic film base in the RC-9 superwide angle camera which give essentially the same results as the polyester base in the wide angle camera. This will probably be explainable when data reduction is complete on the wide-angle camera with topographic base film. The only significantly different value in the final adjusted ground position and elevation errors is the 37.7 microns or 2.9 feet for vertical points in the superwide-angle-camera-topo-base combination, which is twice that of any other

Table 2.—Results of Ohio Camera Calibration tests of two cameras and two film bases

FILM—CAMERA COMBINATIONS

	RC-8 camera polyester base	RC-9 camera polyester base	RC-9 camera topo base
	microns	*microns*	*microns*
RMS residual y-parallax:			
72 Relative orientation points	4.8	7.6	7.5
300 Photogrammetric points	6.3	7.3	7.6
RMS intermodel tie residuals:			
36 Horizontal positions	7.2	11.5	6.7
36 Elevations	12.9	7.8	10.4
RMS errors after adjustment			
Horizontal control points	16.3(1.3 ft.)	15.4(1.2 ft.)	18.2(1.4 ft.)
Vertical control points	15.3(1.2 ft.)	18.2(1.4 ft.)	37.7(2.9 ft.)
RMS errors after block adjustment	?	?	?

combination. This is directly related to the intermodel tie discrepancy above it, and will also probably be resolved when data reduction is complete. Again the results of the block adjustment on this test series is awaited with particular interest because the residual plate-coordinate errors may indicate the existance of any residual systematic distortions.

It may be significant that the best accuracy attained was of the order of 15 microns which is considered to be composed largely of residual film distortion discrepancies.

3.2 *Film Distortion Studies*

While the accuracies already attained with the analytic method are pleasing indeed, the staff is now more determined than ever to reduce the residual systematic errors of the photograph. The largest of these is probably the residual film distortion that remains after mathematical restoration of the fiducial marks. An aversion to the use of glass negatives or pressure plate reseaus in the aerial camera has lead to an investigation of the metric characteristics of the aerial film after processing, drying and storage. For this investigation, a calibrated grid plate was contact-printed on several types of film along with aerial photography so that the grid exposures would be treated as ordinary aerial photography. (Incidentally, through the use of these grid exposures, one of our older developing machines was found to be contributing measurably to film distortion.) At regular time intervals, the grid exposures are printed on glass plates and measured with the precision comparator. The resulting plate coordinates are first treated with a regular scale change to determine the residual distortion which exists in the model formed in the stereoplotter during map compilation, then they are treated with the analytic nonaffine transformation (section 2.11) to determine the residuals that are propagated through the strip or block.

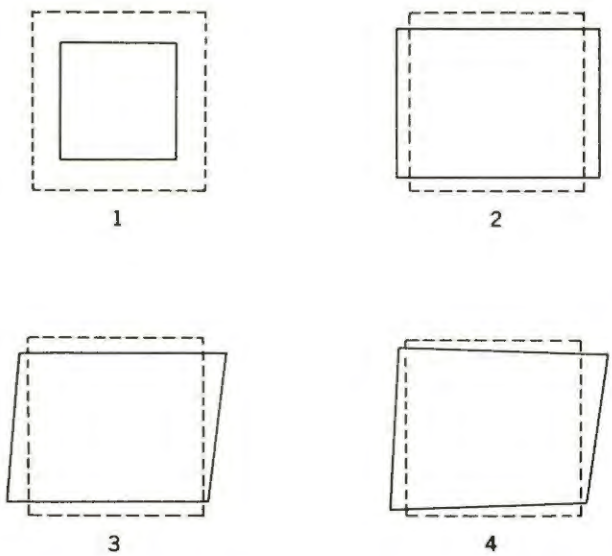

FIG. 13.—Diagram of film distortion illustrating (1) overall shrinkage, (2) differential shrinkage, (3) angular distortion, and (4) random distortion.

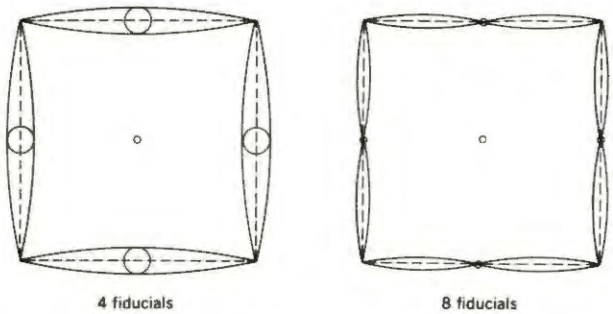

FIG. 14.—Diagrams showing residual film distortions after numerical compensation using four and eight fiducial marks.

The first diagram in figure 13 illustrates regular shrinkage which is readily compensated by ratio printing or by the principal distance adjustment on a plotting instrument. This results in an averaging of the differential distortion as shown in the second diagram. A closer look at the residuals at this point reveals that the original square figure has not become a true rectangle, but is more like a parallelogram as shown in the third diagram. Finally, a very close examination shows the parallelogram to be actually an irregular quadrilateral, something like diagram four. This is the figure that is compensated in the analytic distortion compensation program (2.11) which restores the fiducial marks to their calibrated positions.

Through the use of the grid exposures, it has been possible to investigate the effectiveness of the analytic compensation at points between the fiducial marks. As one might expect, the residual distortion, although quite small, was nonlinear. This type of distortion is shown in a general way in the first diagram of figure 14. Although analysis has not progressed sufficiently to draw final conclusions regarding the adequacy of film negatives for aerotriangulation or the required density of film distortion control points, it is apparent that films having smaller regular shrinkage do not necessarily have smaller residual distortion after mathematical distortion compensation. It is also apparent that the center point defined by the intersection of lines connecting the fiducial marks is determined with only a little more precision than the midpoints of the sides. Nevertheless, it is recommended that aerial cartographic cameras used for analytic photogrammetry be equipped with at least

eight fiducial marks located at the four corners and the midpoints of the sides. Furthermore, these marks should reproduce on the negative with sufficient legibility to enable determination of their plate coordinates within 3 microns. The reason for eight marks is twofold: first, as shown in the second diagram of figure 14, the residuals are reduced to one quarter of their magnitude if the span between marks is reduced one-half; and secondly, eight fiducial marks provide local control of film distortion near the six normal pass-point areas, so that the larger residual discrepancies are confined to intermediate model areas whereas only the smaller discrepancies are propagated in aerotriangulation.

3.3 *Control Point Identification*

If there is any source of error greater than film distortion, it is control point identification. Analytic aerotriangulation precision has made the need for premarking horizontal control points much more apparent. In fact, it now seems that even the most expert field men can seldom, if ever, find nearby objects suitable for use as substitute control points for analytic aerotriangulations which have standard errors as small as 25 to 50 microns at plate scale. Inasmuch as premarking is not always practicable, present requirements for the selection of substitute stations state that unless the object is very small and symmetrical in shape, the contrast between it and its background must be low, and its reflectivity must be such that the resulting image-density will occur in the middle gray tones to minimize image-spread in the emulsion. Furthermore, at least two substitute points must be established for each horizontal control station. In spite of these precautions, the residual errors between the pairs of adjacent substitute control points are frequently greater than the ***maximum*** errors obtained in premarked test area aerotriangulations. It is therefore concluded, that when maximum accuracy is desired, horizontal control points must be premarked with symmetrical photographic targets.

Appendixes

APPENDIX 1

DERIVATION OF FORMULAS FOR FILM DISTORTION COMPENSATION

Formulas were stated in Section 2.11 for correcting the measured coordinates of images for the effect of the irregular distortion of the aerial film based on the dislocations of the fiducial marks in the four corners. The fiducial marks are projected onto the film in the camera at the instant the aerial photograph is exposed. A glass plate photograph exposed in the camera in the laboratory serves as the master specimen on which are measured the fiducial coordinates that become the fixed camera constants on which the compensation formulas are based. It is assured that any ***uniform*** expansion caused by the thermal difference between the environments at flight altitude and in the laboratory has an insignificant effect as indicated by E. D. Sewell[21] inasmuch as the change is equivalent to a small error in focal length.

[21]Investigation of the magnitude of errors in ground positions caused by errors in the focal length or in the principal distance of the aerial camera by Eldon D. Sewell, *Syracuse University Press*, 1940.

The symbols X_j, Y_j, $j = 1 \ldots 4$, denote the measured constant fiducial coordinates for a given camera and X_p, Y_p are the coordinates of the principal point in the same coordinate system as determined from the camera calibration data. As indicated in Section 2.1.1, the mark in one corner (Corner No. 3) is arbitrarily selected as the origin of the coordinate system, and the X-axis is selected so as to pass through another corner (Corner 2). (See fig. 5.) The terms x_i, y_i, $i = 1 \ldots n$, denote the coordinates of any point on a photograph as measured on the comparator, and x_j, y_j are the corresponding coordinates of the fiducial marks where the coordinate system is that of the comparator and is random with respect to the fiducial marks. The purpose of this routine is to determine the constant coefficients of an orderly transformation that will convert x_i, y_i image coordinates into corrected x_i', y_i' values such that the observed x_j, y_j values at the corners are converted exactly into the corresponding X_j, Y_j fixed constant values at all four of the corners for every photograph, and

finally to translate the origin of the coordinate system to the principal point. The general scheme is to apply linear transformations insofar as possible to achieve the desired agreement at three corners, and then to apply a nonlinear scheme for the fourth corner.

The first step is to make $x_3 = y_3 = X_3 = Y_3 = 0$ (translation) by subtracting x_3 from each abscissa and y_3 from each ordinate, calling the new coordinates u_i, v_i:

$$u_i = x_i - x_3 \qquad v_i = y_i - y_3 . \tag{1.1}$$

The second step consists of (1) a rotation of the coordinate axes about Corner 3 so that the u-axis passes through Corner 2 ($v_2' = 0$), (2) an adjustment of the angle between the u and v axes so that the v-axis passes through Corner 4 ($u_4' = 0$) and (3) changes in the scales along the u and v axes so that the new value $u_2' = X_2$ and $v_4' = Y_4$. All four of these conditions can be satisfied through the use of the Helmert transformation

$$u' = mu - nv \qquad v' = pu - qv \tag{1.2}$$

which is linear. By substituting the known coordinates at Corners 2 and 4 into Equation (1.2), two pairs of simultaneous linear equations are formed in the unknowns m, n, p, q:

$$\begin{cases} X_2 = u_2\, m + v_2\, n \\ X_4 = u_4\, m + v_4\, n \end{cases} \qquad \begin{cases} Y_2 = u_2\, p + v_2\, q \\ Y_4 = u_4\, p + v_4\, q . \end{cases} \tag{1.3}$$

If these are solved for the unknowns, recognizing that $Y_2 = 0$ and defining d as the determinant

$$d = \begin{vmatrix} u_2 & v_2 \\ u_4 & v_4 \end{vmatrix} ,$$

$$m = (X_2 v_4 - X_4 v_2)/d \qquad p = v_2 Y_4/d$$

$$n = (X_4 u_2 - X_2 v_4)/d \qquad q = -u_2 Y_4/d . \tag{1.4}$$

Based on these known values for m, n, p, q, Equation (1.2) can then be applied to Corner 1:

$$u_1' = mu_1 - nv_1 \qquad v_1' = pu_1 - nv_1 . \tag{1.5}$$

If the film distortion is linear, $u_1' = X_1$, $v_1' = Y_1$, but normally they will not be equal except for glass plates. It is noteworthy that all operations thus far have been affine or linear rather than conformal, that all the affine schemes for fitting at Corner 1 are exhausted, and that any further attempt at fitting must be nonlinear. The following equations have been selected for making this "warping":

$$u^* = u' + r\, u'\, v' \qquad v^* = v' + s\, u'\, v' . \tag{1.6}$$

If r and s are solved for in terms of Corner 1,

$$r = (X_1 - u_1')/u_1' v_1' , \quad s = (Y_1 - v_1')/u_1' v_1' \tag{1.7}$$

It may be obvious that Equation (1.6) applies a maximum correction at Corner 1 and that the correction diminishes linearly to zero along both of the coordinate axes so that the corrected coordinates at the first three corners are not disturbed.

Finally, if Equations (1.2) are substituted into (1.6), and the coordinates of the principal point applied so as to translate the origin from Corner 3 to the photo center, the new compensated coordinates are

$$x_i' = (mu_i + nv_i)\,[1 + r(pu_i + qv_i)] - X_p$$

$$y_i' = (pu_i + nv_i)\,[1 + x(mu_i + nv_i)] - Y_p \tag{1.8}$$

which were stated in Section 2.11 and which follow Equations (1.1), (1.4), and (1.7) in application.

APPENDIX 2

DERIVATION OF FORMULAS FOR ASYMMETRIC LENS DISTORTION

An idealistic method for adjusting image coordinates for the systematic effects of lens distortion might consist of a table lookup system in which the corrections in x and in y are stored in the computer for, say, each millimeter square over all the photographic area. In such a system it would be possible to correct for asymmetric, radial, and also tangential distortion accurately and in a single operation. However, the data storage for this amounts to about 100,000 three-digit words each with a decimal point and an algebraic sign.

An alternative approximate scheme is used herein. To avoid this storage problem, this scheme has the characteristic of leaving a residual error, which is made sufficiently small so as to be of little or no practical consequence. In this alternative scheme, radial distortion is corrected by means of a table-lookup involving 150 stored values, a simple second-degree correction is used for the asymmetric characteristic, and tangential distortion is considered to be nonexistant (based mainly on a lack of data at this time).

As indicated in Section 2.121, the asymmetric distortion is corrected by first rotating the coordinate axes through the use of the formulas

$$x_c = ax + by \qquad y_c = -bx + ay, \tag{2.1}$$

then correcting the coordinates by

$$x_t = x_c (1 + c\, x_c) \quad y_t = y_c (1 + c\, x_c) \qquad (2.2)$$

and finally rotating back into the original coordinate system using the inverse of Equation (2.1)

$$x' = ax_t - by_t \quad y' = bx_t + ay_t \qquad (2.3)$$

We shall proceed to show how the constants a, b, c are determined and to justify the use of these formulas.

Figure 15 is a graphic representation of the total radial distortion along each of the four semidiagonals of a photograph as reported for an aerial camera. It is noteworthy that the values differ along the four directions, and it is this difference which is referred to as asymmetric distortion. Here, an average correction is affected through radial distortion compensation discussed in appendix 3, and the remainder by the present analysis.

Figure 16 shows the ***differences*** from the average with the negative values (as along the *D*-radial) plotted as positive values. It should be expected that the *B* graph should be identical to the *D*, and the *A* to the *C*. The fact that they are not identical is partly explained by unavoidable observational errors which occur in the practical determination of the distortion values. We approximate the *B-D* and *A-C* envelopes by the two expressions

$$d = c_1\, r^2 \quad \text{and} \quad d = c_2\, r^2 \qquad (2.4)$$

to which we limit ourselves. Here $c = (\sin t)/f$, but one need not be concerned with the terms t or f. It is sufficient to recognize that this is quite similar to a tilting effect on the photo plane as shown by F. E. Washer[22] and that it behaves like tilt as depicted by L. W. Swanson[23].

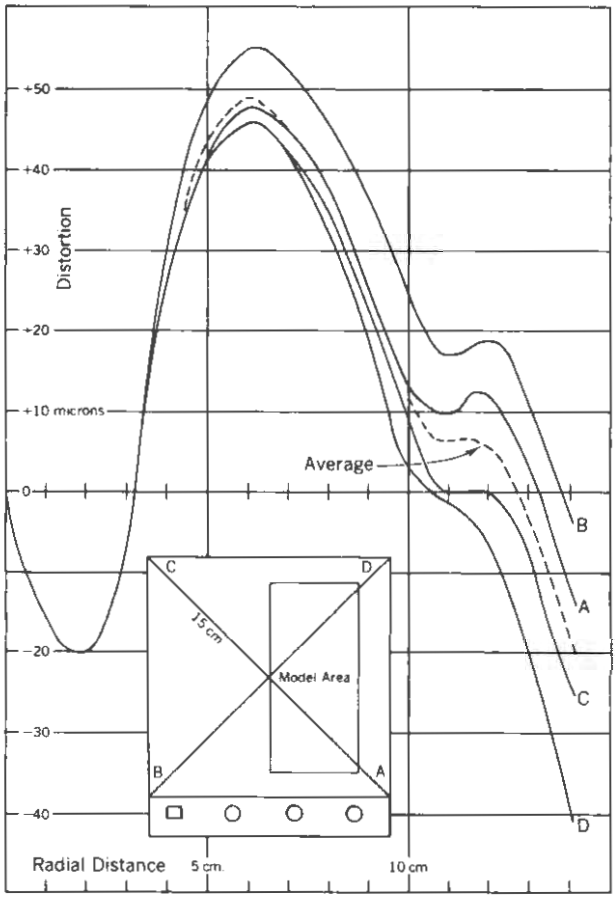

FIG. 15.—Radial lens distortion along the four semi-diagonals of the aerial photograph.

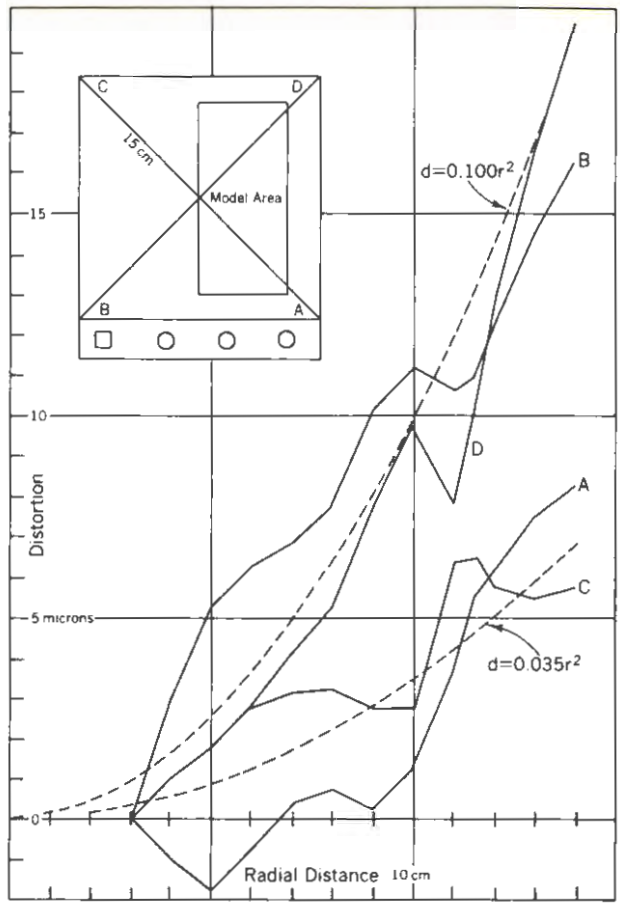

FIG. 16.—Asymmetric radial distortion on the four semi-diagonals derived by subtracting the average from the total distortion at each radius and plotting negative values in the positive direction.

22Washer, Frances E., see footnote 5, above.

23Topographic Manual by Lawrence W. Swanson, *Coast and Geodetic Survey Special Publication No. 249*, Part II, p. 162, 1949.

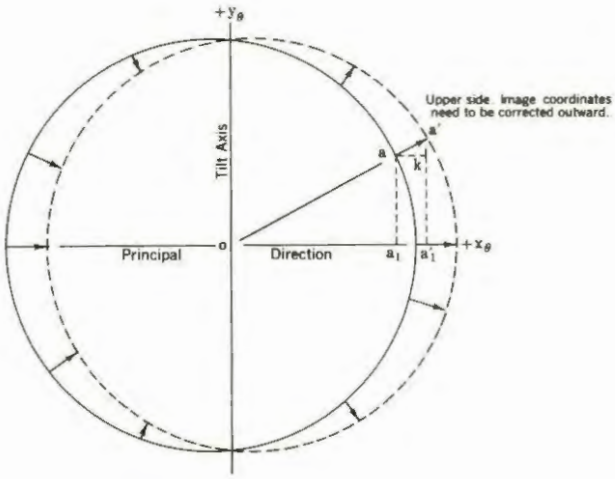

FIG. 17.—Graphic depiction of asymmetric lens distortion.

Specifically, if a plane is tilted, points in a perspective system on the upper side are displaced inward toward the center, points on the lower side are displaced outward, and any corrective measures are in the opposite directions. Points equidistant from the center on opposite sides are not displaced equally, but if the inclination is small, the inequality is insignificant and in this application the inclinations encountered are very, very small. Moreover, the displacement is zero for points along the tilt axis through the center.

It was also indicated by L. W. Swanson[24] that points not along the principal direction can be considered as though they are also in a *special* principal direction whose inclination is simply but systematically smaller than that along the actual principal direction. In figure 17, the point a on the upper side needs to be corrected to the location a'. Let the coordinates of a be x_θ, y_θ and those for a' be x_t, y_t. In the right triangle $a\ a'\ k$,

$$ak = a_1\ a_1' = x_\theta^2\ (\sin t)/\ f = c\ x_\theta^2\ . \qquad (2.5)$$

From the proportionality of the sides in the similar triangles $a\ k\ a'$ and $o\ a_1\ a_1$,

$$a'k/ak = a\ a_1/oa_1 = y_\theta/x_\theta$$

$$a'k = (cx_\theta^2)\ y_\theta/x_\theta = cx_\theta\ y_\theta . \qquad (2.6)$$

Then the corrected coordinates of a' are

$$x_t = oa_1 + ak = x_\theta + cx_\theta^2 = x_\theta\ (1 + cx_\theta)$$

$$y_t = a_1'\ a' + a'k = y_\theta + cx_\theta y_\theta = y_\theta\ (1 + cx_\theta . \qquad (2.7)$$

[24] Ibid.

Inasmuch as it is unlikely that the principal direction coincides with the ficucial x-axis determined in appendix 1, a rotation is implied where Equations (2.1) and (2.3) are standard rotation formulas in which a is the cosine of the rotation angle and b is the sine. Also involved in the analysis is an angle of 45° inasmuch as distortions are normally observed and reported with reference to the corner diagonals as indicated in figure 15. We consider an angle of rotation $(\theta + n\ \pi/4)$ which transforms x, y coordinates into appropriate x_θ, y_θ values.

With reference to figure 16, consider the two second-degree curves (Equations (2.4)) which are approximations to the observed envelopes of values along each pair of semidiagonals. How does one determine the values of c_1 and c_2? Presumably the best answer is find an appropriate value based on successive trials together with one's best judgment. In aerotriangulation one is particularly interested in the distortion correction in the region from 9 to 12 cm. as this is where significant points ordinarily lie. The small number of observations and the relatively large disparity of values scarcely justify a least squares approach. Thus, values $c_1 = 0.100$, $c_2 = 0.035$ seem to be appropriate values in this example (d is measured in microns, r in centimeters). Yet these values indicate that some residual discrepancies of nearly 3 microns may exist if the observations are valid.

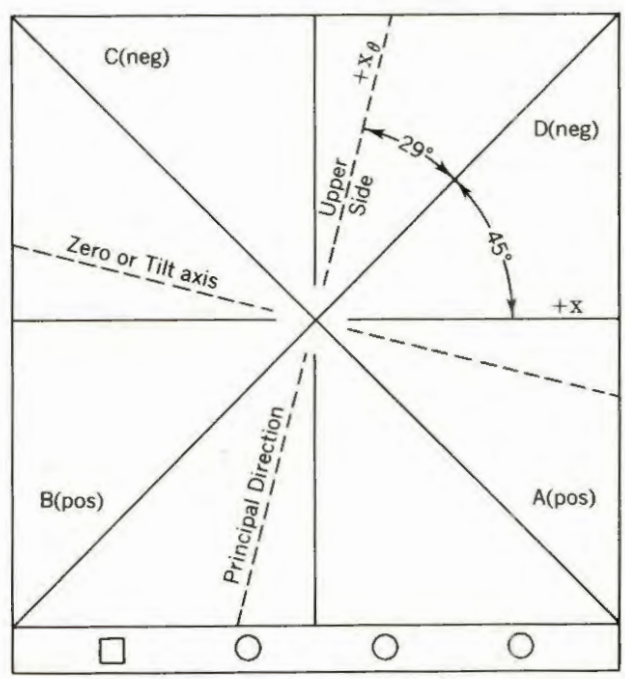

FIG. 18.—Diagram of a photograph indicating the analysis of the angle of rotation from the $+x$-axis to the $+x_\theta$-axis.

Now that c_1 and c_2 may be evaluated, it is proposed that

$$c = (c_1^2 + c_2^2)^{1/2}$$
$$\tan \theta = c_2/c_1$$
$$a = \cos(\theta + n\pi/4)$$
$$b = \sin(\theta + n\pi/4) \qquad (2.8)$$

Thus, in the example,

$$c = 0.106, \ \theta = 29.3^\circ$$

The direction quadrant for rotation is determined by inspection from the relative displacements along the separate semidiagonals. Thus, the upper end of the inclined principal direction (fig. 18) is in the general direction of the D-diagonal, (which is defined as the positive end of the x_θ-axis) because the greatest negative distortion must occur between the diagonal C and D (which are both negative) at an angle of 29.3° from the D-diagonal. This selection is verified by the fact that the zero tilt axis (perpendicular to the principal direction) occurs between the diagonals where the distortions change sign. Thus the analytic rotation angle from the x-axis to the principal direction is simply, from the figure, 29.3° + 45° = 74.3°, and

$$a = \cos 74.3^\circ = +0.271$$
$$b = \sin 74.3^\circ = +0.963.$$

The logic of this formulation is included on page 341 of the Manual[25], on pages 162 ff., by L. W. Swanson[26], and is not repeated here.

APPENDIX 3

SYMMETRICAL RADIAL LENS DISTORTION

The word *symmetrical* applied here to lens distortion implies that the distortion is considered to be identical along all radii. In the present application, the distortion is the average value along four semidiagonals where systematic departures from the average are accounted for as shown in appendix 2.

In figure 19, d is the positive distortion which has displaced an image a' at radius r to its observed position a. The observed coordinates of a are x, y and the desired corrected coordinates of a' are x', y'. Through the proportionality of corresponding sides in the similar triangles $aa'b$ and Oac,

$$\overline{a'b} = xd/r, \quad \overline{ab} = yd/r.$$

From the figure,

$$x' = x - \overline{a'b} = x - xd/r = x(1 - d/r)$$
$$y' = y - \overline{ab} = y - yd/r = y(1 - d/r), \qquad (3.1)$$

and also,

$$r^2 = x^2 + y^2.$$

Obviously it is sufficient in making a desired correction to know merely the value of the ratio d/r. However, the ratio is undefined if r is zero, which creates an awkward condition. To avoid this, one might tabulate simply the values of d and then perform the division operation as needed later. Better yet, inasmuch as r^2 is needed also for other purposes and r is not needed, the values of the product dr are entered into the computer, later to be divided by r^2 as needed, which avoids computing the square root of r^2. Consequently the values of dr are prepared by desk calculator for each millimeter of r for each aerial camera as input data to the computer for table lookup and interpolation. Correpondingly, the independent argument in the table is also r^2 instead of r. A portion of one of the tables is included as an example (table 3).

APPENDIX 4

DERIVATION OF THE CORRECTION FOR ATMOSPHERIC REFRACTION

Except for an image at the nadir, atmospheric refraction causes photographic images to be dis-

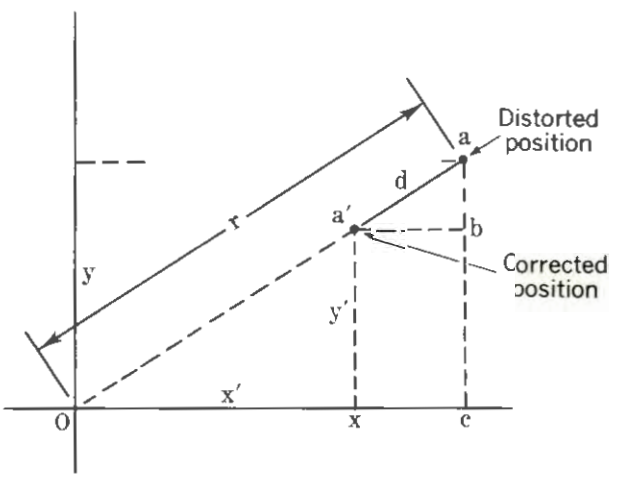

FIG. 18.—Diagram showing the analysis of radial lens distortion.

[25] Manual of Photogrammetry, see footnote 6, above.
[26] Swanson, Lawrence W., see footnote 23, above.

Table 3.—Sample of tabular values stored in computer for correcting radial lens distortion

Radial distance r	Average radial distortion d	Distortion tabular values for computer	
		Square of radial distance r^2	Distortion factor dr x10^{-6} (per sq. m.)
cm.	*micron*	*m. x 10^{-4}*	
1	- 16.0	1	+0.1600
2	- 20.0	4	+0.4000
3	- 6.0	9	+0.1800
4	+28.0	16	- 1.1200
5	+43.9	25	- 2.1950
6	+49.2	36	- 2.9520
7	+45.5	49	- 3.1850
8	+37.4	64	- 2.9920
9	+23.8	81	- 2.3220
10	+11.6	100	- 1.1600
11	+7.0	121	- 0.7700
12	+5.8	144	- 0.6960
13	- 3.3	169	+0.4290
14	- 19.5	196	+2.7300
15	- 38.4	225	+5.7600

placed outward radially from the nadir point in increasing amounts depending on the distance from the nadir to the image. To correct image coordinates it is therefore necessary to reduce their values. The basic cause of the refraction is explained by Snell's Law inasmuch as the atmosphere, through which the light is transmitted from the ground to the aerial camera, is composed of a gas whose density normally decreases with altitude due to the differing compressional mass of air above and, to a lesser extent, increases with altitude due to the decrease in air temperature with altitude.

A standard atmosphere, called the I.C.A.N. atmosphere, is assumed as explained by Axel Leijonhufvud[27]. No attempt is made here to account for temperature, humidity, or pressure variations which may arise from meteorological or geographical considerations. It is also assumed that nearly vertical photographs are used where the principal point is sufficiently near the nadir so that no significant error arises from using the principal point instead of the nadir.

As indicated previously, the refraction corrections are based on the publication by A. Lei-

[27] Leijonhufvud, Axel, see footnote 8, above.

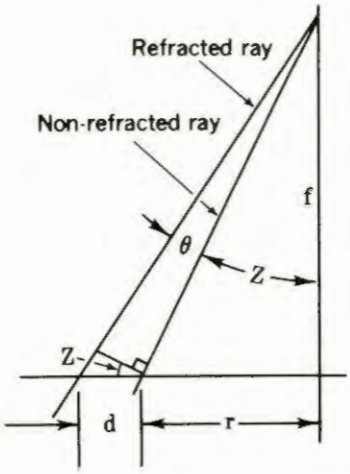

FIG. 20.—Diagram showing the relative linear displacement d cased by atmospheric refraction to the angle θ: $d = f \tan \theta / \cos^2 Z$.

Table 4.—Angular distortion θ in centesimal seconds due to atmospheric refraction at varying angles Z from the vertical (portion of table by Axel Leijonhufvud[a])

Flight altitude	Angle Z from the vertical in centesimal degrees												
(*km.*)	5	10	15	20	25	30	35	40	45	50	55	60	65
	Ground Elevation—1.5 km.												
3.0	0.8	1.7	2.5	3.4	4.4	5.4	6.5	7.7	9.1	10.6	12.4	14.6	17.3
6.0	2.1	4.3	6.6	8.9	11.3	13.9	16.7	19.8	23.3	27.3	32.0	37.6	44.5
9.0	3.0	6.1	9.3	12.6	16.0	19.7	23.7	28.1	33.1	38.7	45.3	53.3	63.2

[a] On astronomic, photogrammetric and trigonometric refraction, 1950.

jonhufvud[28]. Table 4 is a sample of a portion of one of the published tables. As depicted by the table, the angular refractive error is stated in centesimal seconds, the zenith angle of the refractive ray is given in centesimal degrees, and the airplane and terrain heights are given in kilometers. In order to express the tables in units more directly applicable to the specific problem, the reference tables were converted into table 5 which states the distortion values in microns based on a camera focal length of 153 mm., and states the linear distances in feet.

Figure 20 indicates the angles θ and Z stated in the reference and also the corresponding displacement d utilized herein. From the geometry of the figure it is evident that

$$d = f \tan \theta / \cos^2 Z$$
$$r = f \tan Z \qquad (4.1)$$

[28] Ibid.

Inasmuch as the largest value of θ is less than one minute of arc, $\theta = \tan \theta$ essentially, and

$$d = f \theta / \cos^2 Z. \qquad (4.2)$$

As f is taken as $1.53 \times 10^5 \mu$ and as one centesimal second is equal to 1.5708×10^{-6} radians,

$$d = 0.24033 \ \theta / \cos^2 Z \qquad (4.3)$$

in which d is in microns and θ is in centesimal seconds. For example, the value $\theta = 38.7$ and $Z = 50^C = 45°$ appears in Table 4 for a flight altitude of 9.0 km. and a ground elevation of 1.5 km. Applying Equations (4.1) and (4.3),

$$d = 0.24033 \times 38.7/\cos^2 45° = 18.6\mu$$
$$r = 15.3 \tan 45° = 15.3 \text{ cm.}$$

which occurs in table 5 for the ground elevation of 4,920 feet = 1.5 km.

Table 5.—Radial linear image distortion d *in microns due to atmospheric refraction for a focal length of 15.3 cm. as converted from Axel Leijonhufvud[a] tables*

Flight altitude	Radial distance r from photo center to image										Coefficients*	
	1.2	2.4	3.7	5.0	6.3	7.8	9.4	11.1	13.1	15.3	$k_1 \times 10^{-1}$	$k_2 \times 10^{-5}$
feet	*cm.*	*cm.*	*cm.*	*cm.*	*cm.*	*cm.*	*cm.*	*cm.*	*cm.*	*cm.*		
	Ground Elevation—0 feet											
9,842	0.4	0.9	1.4	1.9	2.6	3.4	4.5	5.9	7.9	10.7	3.4	1.53
19,684	0.7	1.5	2.3	3.3	4.4	5.9	7.7	10.1	13.5	18.3	6.1	2.50
29,257	0.9	1.9	3.0	4.2	5.7	7.5	9.9	13.0	17.3	23.4	7.7	3.23
	Ground Elevation—1,640 feet											
9,842	0.3	0.7	1.1	1.6	2.1	2.8	3.7	4.9	6.4	8.8	2.8	1.25
19,685	0.7	1.3	2.1	3.0	4.0	5.3	6.9	9.1	12.2	15.4	5.4	2.30
29,527	0.9	1.8	2.8	3.9	5.3	7.0	9.2	12.0	16.0	21.7	7.2	2.99
	Ground Elevation—3,280 feet											
9,842	0.3	0.6	0.9	1.3	1.7	2.2	2.9	3.9	5.1	6.9	2.2	0.99
19,685	0.6	1.2	1.9	2.7	3.6	4.8	6.3	8.2	10.9	14.5	4.8	2.08
29,527	0.8	1.6	2.6	3.6	4.9	6.5	8.5	11.2	14.9	20.1	6.7	2.76
	Ground Elevation—4,920 feet											
9,842	0.2	0.4	0.6	0.8	1.2	1.6	2.2	2.8	3.8	5.1	1.6	0.74
19,685	0.5	1.1	1.7	2.4	3.2	4.2	5.5	7.3	9.7	13.1	4.2	1.87
29,527	0.7	1.5	2.4	3.4	4.5	6.0	7.8	10.3	13.8	18.6	6.1	2.59

[a] On astronomic, photogrammetric and trigonometric refraction, 1950.
* The coefficients are scaled so that both r and d are expressed in meters instead of centimeters and microns.

Distortion values for any other focal length are directly proportional to these values which are based on 15.3 cm. where the radial distances r must also be changed to agree with the differing values of the angle Z. However, the latter can be accomplished by simply relabelling the columns in table 5 with the appropriate values of r.

As a practical method for applying the Leijonhufvud data, a power series

$$d = k_1 \; r + k_2 \; r^3 \tag{4.4}$$

is adopted where d, k, r are indicated in table 5 and where the equation serves as an approximation of the table of d values relative to r for any given values of terrain height and flight altitude. The values of the coefficients k were computed by desk calculator using the following scheme.

a. The value of k_1 is determined using Equation (4.4) by substituting the d and r tabulator values at the ***smallest*** radius (1.2 cm.) and assuming $k_2 r^3$ is zero.

b. The value of k_2 is found by substituting into Equation (4.4) at the ***largest*** radius (15.3 cm.) utilizing the value found in k_1.

c. Other values for d are computed using k_1 and k_2 and compared to the table as a check.

The rms discrepancy between the computed and tabular values was less than a micron.

The values of the two k's are plotted on large-scale graphs similar to figure 21 so that the values for any given set of photographs can be ascertained and entered onto a form for submission to the data processing unit.

By applying the same analysis used in Equation (3.1), Equation (4.4) can be applied in the form of a factor as shown by Equation (8) where the k's have negative values. The added correction for radial lens distortion results in Equation (7).

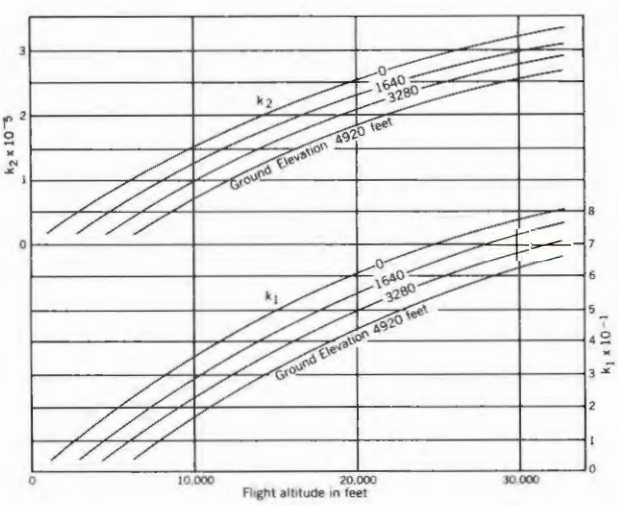

FIG. 21.—Graph of the two coefficients for the compensation of atmospheric refraction.

APPENDIX 5

AN ALGEBRAIC DERIVATION OF THE ROTATION FORMULAS

An algebraic derivation of Equations (9) and (14) is presented although the G. H. Rosenfield[29] reference may be sufficient.

An xyz-coordinate system based on the photo fiducial axes is used to identify the position of an image on a photograph. An auxiliary $x^*y^*z^*$-system is introduced, being comparable to the xyz-system, having the same origin, and differing in angular orientation only. The x^*, y^*, z^* axes are selected parallel, respectively, to the X, Y, Z axes of the local geocentric or other rectangular ground coordinate system in which $+Z$ is upward, and X and Y form a right handed sequence, such that if $+X$ is eastward, then $+Y$ is northward.

A sequence of rotations is considered (fig. 22) through which can be expressed $x^*y^*z^*$-coordinates in terms of the xyz-values, and vice versa. The first rotation (fig. 22a) ω or x-tilt is about the x^*-axis considered counterclockwise positive as viewed from the plus end of the x^*-axis, which corresponds with textbook notation in plane analytic geometry. (Rosenfield[30] considers the rotation clockwise as viewed from the origin, which is physically identical.) The second rotation φ or y-tilt is about the inclined $y^{*\prime}$-axis, and the third rotation κ or swing is about the twice rotated z^*-axis, which is the photograph perpendicular (lens axis), or the z-axis.

By plane analytic geometry the new $x^{*\prime}$, $y^{*\prime}$, $z^{*\prime}$ coordinates of an image point P are expressed in terms of the original x^*, y^*, z^* coordinates for the first ω rotation:

$$y^{*\prime} = y^* \cos\omega + z^* \sin\omega$$
$$z^{*\prime} = -y^* \sin\omega + z^* \cos\omega. \tag{5.1}$$

The x^* coordinates are unchanged. The second rotation φ is strictly related to this once-rotated $x^*y^{*\prime}z^{*\prime}$-system inasmuch as this rotation is about the $y^{*\prime}$ axis as shown in figure 22b. Applying the same rotational formulas to the $x^*z^{*\prime}$-system,

$$z = z^{*\prime} \cos\varphi + x^* \sin\varphi$$
$$x^{*\prime} = -z^{*\prime} \sin\varphi + x^* \cos\varphi. \tag{5.2}$$

[29] Rosenfield, George H., see footnote 9, above.
[30] Ibid.

The y^* coordinates are not changed. The expression for $z^{*\prime}$ in Equation (5.1) is substituted into Equation (5.2):

$$z = -y^* \sin\omega \cos\varphi + z^* \cos\omega \cos\varphi + x^* \sin\varphi$$

$$x^{*\prime} = y^* \sin\omega \sin\varphi - z^* \cos\omega \sin\varphi + x^* \cos\varphi. \tag{5.3}$$

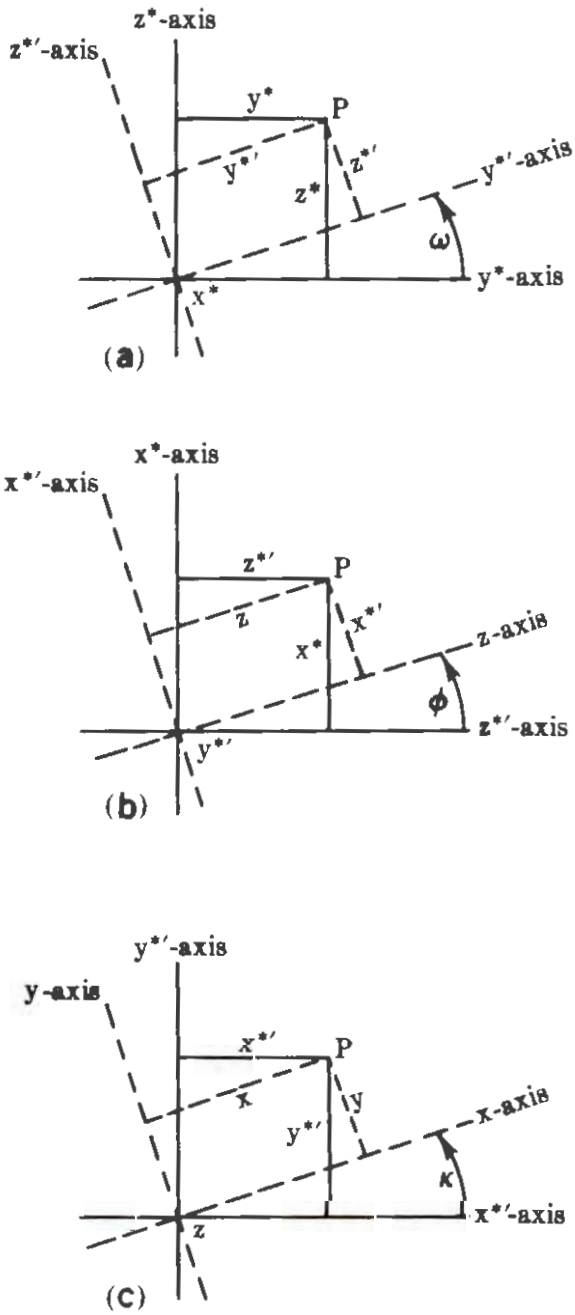

FIG. 22.—Diagram showing the three sequential plane rotations for camera orientation.

The final κ-rotation is about the z-axis as shown in figure 22c. Applying the same rotation formulas:

$$x = x^{*\prime} \cos\kappa + y^{*\prime} \sin\kappa$$

$$y = -x^{*\prime} \sin\kappa + y^{*\prime} \cos\kappa. \tag{5.4}$$

The z^* coordinate is unchanged. The expression for $x^{*\prime}$ from Equation (5.3) and $y^{*\prime}$ from Equation (5.1) are substituted into Equation (5.4):

$$x = x^* \cos\kappa \cos\varphi + y^* \sin\omega \sin\varphi \cos\kappa - z^* \cos\omega \sin\varphi \cos\kappa + y^* \cos\omega \sin\kappa + z^* \sin\omega \sin\kappa$$

$$y = -x^* \cos\varphi \sin\kappa - y^* \sin\omega \sin\varphi \sin\kappa + z^* \cos\omega \sin\varphi \sin\kappa + y^* \cos\omega \cos\kappa + z^* \sin\omega \cos\kappa.$$

Factoring,

$$x = x^* \cos\varphi \cos\kappa + y^* (\cos\omega \sin\kappa + \sin\omega \sin\varphi \cos\kappa) + z^* (\sin\omega \sin\kappa - \cos\omega \sin\varphi \cos\kappa)$$

$$y = x^* (-\cos\varphi \sin\kappa) + y^* (\cos\omega \cos\kappa - \sin\omega \sin\varphi \sin\kappa) + z^* (\sin\omega \cos\kappa + \cos\omega \sin\varphi \sin\kappa)$$

$$z = x^* \sin\varphi + y^* (-\sin\omega \cos\varphi) + z^* (\cos\omega \cos\varphi),$$

which is Equation (9) shown in the main text.

APPENDIX 6

THE PROJECTIVE TRANSFORMATION

Equation (17) is derived from the upper part of figure 6, utilizing Equation (10). It is evident from the similar triangles in the figure that

$$(X-X_o)/(Z-Z_o) = x^*/z^*$$

from which

$$x^* = (X-X_o)\ z^*/(Z-Z_o). \tag{6.1}$$

Similarly, by viewing the figure from the side,

$$y^* = (Y-Y_o)\ z^*/(Z-Z_o), \tag{6.2}$$

and by algebraic identity

$$z^* = (Z-Z_o)\ z^*/(Z-Z_o). \tag{6.3}$$

If Equations (6.1), (6.2), and (6.3) are substituted into Equation (10),

$$x = a_{11}\,(X-X_0)\,z^*/(Z-Z_0) + a_{12}\,(Y-Y_0)\,z^*/(Z-Z_0) + a_{13}\,(Z-Z_0)\,z^*/(Z-Z_0), \text{ etc.}$$

Recognizing the common factor $z^*/(Z-Z_0)$,

$$\begin{aligned} x &= \left[a_{11}\,(X-X_0) + a_{12}\,(Y-Y_0) + a_{13}\,(Z-Z_0)\right]\; z^*/(Z-Z_0) \\ y &= \left[a_{21}\,(X-X_0) + a_{22}\,(Y-Y_0) + a_{23}\,(Z-Z_0)\right]\; z^*/(Z-Z_0) \\ z &= \left[a_{31}\,(X-X_0) + a_{32}\,(Y-Y_0) + a_{33}\,(Z-Z_0)\right]\; z^*/(Z-Z_0). \end{aligned}$$

If the expressions for x and y are each divided by that for z, Equation (17) is derived.

APPENDIX 7

THE ELEVATION OF AN OBJECT FROM ITS IMAGES ON TWO OVERLAPPING PHOTOGRAPHS

The derivation of Equation (25) is repeated here although it is included in the reference.

Consider an object, recalling that its coordinates are represented simply by X, Y, Z, and solve Equation (6.1) for X:

$$X = X_0 + x^*\,(Z-Z_0)/z^*. \tag{7.1}$$

If the values pertaining to the first of two overlapping photographs are indicated by single prime marks (′) and those for the second by a double prime marks (″), Equation (7.1) can be expressed for both photographs:

$$\begin{aligned} X &= X_0' + x^{*\prime}\,(Z-Z_0')/z^{*\prime} \\ X &= X_0'' + x^{*\prime\prime}\,(Z-Z_0'')/z^{*\prime\prime}\ . \end{aligned} \tag{7.2}$$

If the second of Equations (7.2) is subtracted from the first,

$$\left[X_0' + x^{*\prime}\,(Z-Z_0')/z^{*\prime}\right] - \left[X_0'' + x^{*\prime\prime}\,(Z-Z_0'')\right] = 0$$

which, if solved for Z, results in Equation (25) of the text.

APPENDIX 8

THE COEFFICIENTS FOR THE OBSERVATION EQUATIONS

Equations (20) and (21) stated the formulation of the coefficients for the observation equations. Equations (19) (and the equivalent expressions (17) and (18)) specified the colineation condition on which the solution is based. Only the first one of Equations (19) is discussed here as the second one is analogous:

$$\begin{vmatrix} x & z \\ \mathrm{A_1B} & \mathrm{A_3B} \end{vmatrix} = 0. \tag{8.1}$$

Here x is the only observed quantity, z is the fixed constant focal length of the camera, and the **AB**-terms are composed of several elementary angular and linear geometric parameters associated with the attitude and position of the camera: $\omega, \ldots Z_0$.

Equation (8.1) pertains to an image point whose coordinates are x, y, z on a photograph. The equation is valid for as many images as there are independent parameters. But, because x is an observed quantity, if the number of points exceeds the number of parameters, Equation (8.1) cannot be valid for them all, and any solution will necessarily yield residual values v:

$$v = \begin{vmatrix} x & z \\ \mathrm{A_1B} & \mathrm{A_3B} \end{vmatrix}. \tag{8.2}$$

Presumably this relation can be solved for any number of image points so that the sizes of the v's conform to the principles of least squares--provided the **AB**-terms are properly evaluated.

Now let us consider the values of the **AB** terms. Although Equations (18) and (19) indicate formulas for **AB** in terms of the parameters $\omega \ldots Z_0$, the parameters themselves are actually unknown. Estimated approximate values are readily available, but they are not sufficiently accurate. It becomes convenient to use Equation (8.2), not only to distribute v's but also to improve the estimates of these parameters. Presumably the v's could be found in a very direct manner; but to improve also the estimates, three or more iterations (successive approximations) will be needed to arrive at a satisfactory solution for both problems.

If the right side of Equation (8.2) is represented by F when the approximate parameters are used in determining the **AB**'s, then

$$v = F + dF \tag{8.3}$$

would be a valid statement in which dF is the necessary correction to F. But dF can be expressed through the application of partial differentials of calculus in terms of all of the elementary photo parameters:

$$dF = \partial F/\partial\omega\, d\omega + \partial F/\partial\varphi\, d\varphi + \partial F/\partial\kappa\, d\kappa - \partial F/\partial X_0\, dX_0 - \partial F/\partial Y_0\, dY_0 - \partial F/\partial Z_0\, dZ_0 + \partial F/\partial X\, dX + \partial F/\partial Y\, dY + \partial F/\partial Z\, dZ. \quad (8.4)$$

In Equation (20), $p_{11} = F$, $p_{12} = \partial F/\partial\omega$, etc. By recalling that F represents the determinant of Equation (8.1), the derivation of Equation (21) is fully indicated. It may be helpful in visualizing the procedure to point out that the determinant is completely expanded in Equation (18); to express a partial derivative one assumes that only the one variable at a time is a true variable and all the others are considered as constants; and to comment that the determinant notation in Equation (21) is used to demonstrate the orderliness of formation of the coefficients as an aid in programming the computation.

The procedure consists of solving the set of simultaneous linear Equations (20) for corrections $d\omega$, etc., adding these corrections to the trial values ω, etc., and repeating. It is significant that with each iteration $d\omega$, etc., and dF are smaller than in the previous iteration, and that the iterative procedure can be continued until dF is any desired small value, specifically of insignificant size relative to v.

APPENDIX 9

SPECIAL FORM OF OBSERVATION EQUATIONS FOR RELATIVE ORIENTATION

Equation (28) of the main text states an alternative form of Equation (20) for the problem of relative orientation where the terms dX, dY, dZ are eliminated by substituting appropriate amounts of the other variables $d\omega$, $d\varphi$, $d\kappa$, dZ_0. An explanation is now presented.

Equations (27) are recalled introducing the u-terms for the sake of convenience:

$$z = u_1/u_2,\ X = u_3\ Z,\ Y = u_4\ Z$$
$$u_1 = Z_0 x^* - z^* \qquad u_3 = x/z$$
$$u_2 = x^* - u_3\ z^* \qquad u_4 = y/z \quad (9.1)$$

Alternate expressions are desired for dX, dY, dZ. Inasmuch as X and Y are expressed in terms of Z, and

$$dX = u_3\ dZ \quad \text{and} \quad dY = u_4\ dZ \quad (9.2)$$

it is logical to consider dZ specifically. Applying differential calculus to $Z = u_2/u_1$,

$$dZ = d(u_1/u_2) = (u_2\, du_1 - u_1\, du_2)/u_2^2 = (1/u_2)(du_1 - Z\, du_2). \quad (9.3)$$

But

$$du_1 = d(Z_0 x^* - z^*) = x^* dZ_0 + Z_0\, dx^* - dz^* \quad (9.4)$$

$$du_2 = d(x^* - u_3\ z^*) = dx^* - u_3\ dz^*. \quad (9.5)$$

By substitution into Equation (9.3), noting that $u_3\ Z = X$,

$$dZ = (1/u_2)\left[x^*\ dZ_0 - (Z - Z_0)\ dx^* + (X - 1)\ dz^*\right] \quad (9.6)$$

It is pertinent to express dx^* and dz^* in terms of $d\omega$, $d\varphi$, $d\kappa$. Referring to Equations (12) and (14),

$$x^* = a_{11}\ x + a_{21}\ y + a_{31}\ z = x \cos\varphi \cos\kappa + y(-\cos\varphi \sin\kappa) + z \sin\varphi$$

$$z^* = a_{13}\ x + a_{23}\ y + a_{33}\ z = x(\sin\omega \sin\kappa - \cos\omega \sin\varphi \cos\kappa) + y(\sin\omega \cos\kappa + \cos\omega \sin\varphi \sin\kappa) + z \cos\omega \cos\varphi. \quad (9.7)$$

Therefore

$$dx^* = 0\, d\omega + \left[x(-\sin\varphi \cos\kappa) + y(\sin\varphi \sin\kappa) + z \cos\varphi\right] d\varphi + \left[x(-\cos\varphi \sin\kappa) + y(-\cos\varphi \cos\kappa)\right] d\kappa.$$

Applying the notations used in Equations (13), (19), (22), (23), (24),

$$dx^* = (\partial \mathbf{A}_1^T/\partial\omega)\ \mathbf{C}\ d\omega + (\partial \mathbf{A}_1^T/\partial\varphi)\ \mathbf{C}\ d\varphi + (\partial \mathbf{A}_1^T/\partial\kappa)\ \mathbf{C}\ d\kappa \quad (9.8a)$$

where $\mathbf{C}$ is defined in terms of the image coordinates of the second photograph:

$$\mathbf{C} = \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix}$$

Similarly

$$dz^* = (\partial A_3^T/\partial\omega)\ C\ d\omega + (\partial A_3^T/\partial\varphi)\ C\ d\varphi + (\partial A_3^T/\partial\kappa)\ C\ d\kappa. \quad (9.8b)$$

A_i^T is used to indicate the three elements in the i-th row of the transform matrix Equation (12). Consequently, $\partial A_i^T/\partial\omega$ indicates the elements in the i-th *column* of Equation (22), which is not in the transformed arrangement.

By substituting Equation (9.8) into (9.6), collecting terms, and employing a determinant notation,

$$dZ = (1/u_2)\left[x^*\ dZ_0 + \begin{vmatrix} X-1 & Z-Z_0 \\ (\partial A_1^T/\partial\omega)C & (\partial A_3^T/\partial\omega)C \end{vmatrix} d\omega + \begin{vmatrix} X-1 & Z-Z_0 \\ (\partial A_1^T/\partial\varphi)C & (\partial A_3^T/\partial\varphi)C \end{vmatrix} d\varphi + \begin{vmatrix} X-1 & Z-Z_0 \\ (\partial A_1^T/\partial\kappa)C & (\partial A_3^T/\partial\kappa)C \end{vmatrix} d\kappa\right] \quad (9.9)$$

If the coefficients are symbolized by T,

$$dZ = (1/u_2)\ (T_2\ d\omega + T_3\ d\varphi + T_4\ d\kappa + T_7\ dZ_0 \quad (9.10)$$

where the definition of the T's is obvious.

Now let us recall the last three terms of Equation (20):

$$\ldots + p_{15}\ dX + p_{16}\ dY + p_{17}\ dZ$$
$$\ldots + p_{25}\ dX + p_{26}\ dY + p_{27}\ dZ. \quad (9.11)$$

Substituting Equations (9.2),

$$\ldots + (p_{15}\ u_3 + p_{16}\ u_4 + p_{17})\ dZ$$
$$\ldots + (p_{25}\ u_3 + p_{26}\ u_4 + p_{27})\ dZ. \quad (9.12)$$

Let S be defined as

$$S_1 = (1/u_2)\ (p_{15}\ u_3 + p_{16}\ u_4 + p_{17})$$
$$S_2 = (1/u_2)\ (p_{25}\ u_3 + p_{26}\ u_4 + p_{27}). \quad (9.13)$$

Then Equation (9.12) can be expressed as

$$\ldots S_1 T_2\ d\omega + S_1 T_3 d\varphi + S_1 T_4 d\kappa + S_1 T_7\ dZ_0$$
$$\ldots S_2 T_2\ d\omega + S_2 T_3 d\varphi + S_2 T_4 d\kappa + S_2 T_7\ dZ_0. \quad (9.14)$$

If (9.14) is substituted into Equation (20) for the terms in dX, dY, dZ, Equation (28) is the result, which completes the explanation.

APPENDIX 10

COMPUTER FORMULATION

The following notes on the computation of the coefficients of the observation equations may be helpful to a programer. Actually some of these ideas were proposed after the programing had been completed in a different manner, whence all of these ideas have not been tested. It should be obvious that these suggestions constitute but one of many appropriate arrangements of the computation.

This discussion commences with the introduction of the 6 approximate photo parameters X_0, Y_0, Z_0, ω, φ, κ together with the coordinates x, y, z of corresponding images on 2 photographs and terminates with the evaluation of the 14 coefficients $p_{11} \ldots p_{27}$ (Equation (21)) of the observation equations including intermediate data consisting of (1) the orientation matrix for the photograph (Equation (14)), (2) the "rectified" coordinates of the image on the second photograph (Equation (12)) and (3) the "model" coordinates of the corresponding object (Equation (27)).

It is assumed that data can be stored in the computer in table form in such a manner that a value can be identified by a table designation and one or two subscripts (A (I,J)) as in the Fortran program language. Five such tables (fig. 23) are utilized and they are designated generally by algebraic terms already introduced by equations in the text.

The first, or A-table, is discussed at some length (fig. 24).

The first two rows are input data consisting initially of approximate values, and later as corrected values. Ordinarily, with the exception of $X_0 = 1$, they will always be zero initially for the usual vertical photography. If, however, a constant angle φ or ω (as in convergent photography) or κ, (as in crabbed photographs), is present, the *sine* of the nominal value is entered. Accordingly, the computation determines the correction to the *sine* of the angle rather than the angle itself. In correcting the terms $\varphi + d\varphi$, etc., the two are added as though they were in radians, even though they are both sines, which is a valid approximation inasmuch as the corrections are relatively small, and also any significant nonvalidity merely causes an additional iteration having no effect on the final results.

Row 3 consists of the cosines of the three angles computed from, for example, $\cos\omega = (1 - \sin^2\omega)^{1/2}$.

Rows 4 to 12 are filled by data transfer in accordance with Equation (15).

Rows 13 to 15 are formed by matrix multiplication of the terms in Rows 7, 8, 9 and Rows 4, 5, 6, as indicated in Equations (16) and (15).

To conserve storage space, Rows 4 to 15 are now reused. Rows 4, 5, 6 contain the rotation matrix that indicates the orientation of the second photograph to the first (Equation (14)). It is formed by the matrix multiplication of Rows 10, 11, 12 and Rows 14, 15, 16. Rows 10, 11, 12 (Equation (23)) are formed as follows: Row 10 consists of the terms of Row 15 each multiplied by $(-\cos \kappa)$ which is in Row 3; Row 11 consists of the terms of Row 15 each multiplied by $(\sin \kappa)$ which is in Row 2; and Row 12 is identical to Row 13. After the above is completed, Rows 7, 8, 9, and 13, 14, 15 (Equations (22) and (24)) are formed by data transfer from Rows 4, 5, 6.

This completes the preliminary computations associated with the *photograph*, and these data are used next with each pair of photo image coordinates.

Rows 16 and 17 are input data relative to an image: Row 16 consists of the coordinates on the first photograph. Row 17 contains the corresponding values on the second photograph, and the latter are referred to as the vector **C** in Equation (28) and appendix Equation (9.8). These data can be read into the computer from a single card input, or can be handled in any of several other ways.

Rows 18-21 (Equation (12)) are formed by matrix multiplication of Rows 4 through 15 (by columns) and Row 17.

Row 22 is formed by formula (Equation (27)) from Row 18 and Z_0, from Row 1, filling the U-table (Equation (28)) in the process.

Row 23 (Equation (19)) is the vector **B** formed by the indicated subtractions.

Rows 24-27 are formed by matrix multiplication from Rows 4-15 (by rows) and Row 23 as indicated by Equation (21), which completes the A-table for one image.

The next step is the formation of the terms in the P-table. As indicated by Equation (21), through the application of subscripts only two different formulas are required for the 14 terms.

For the resection and block adjustment applications (Equation (52), this completes the coef-

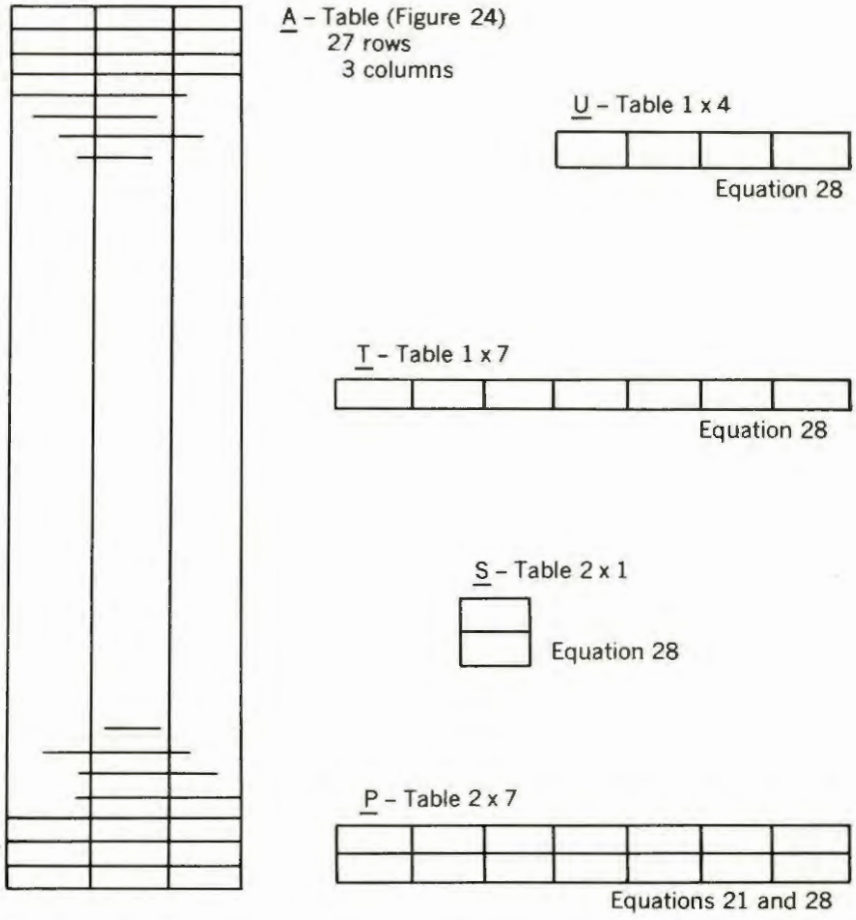

FIG. 23.—Five tables of values suggested for computer storage.

ficient computation, but for the relative orientation problem, some of the terms of the P-table require alteration (Equation (28)) for which the T and S tables are utilized. Values T_2, T_3, T_4 can be formed using the same formula; T_7 can be transferred from the A-table ($T(7) = A(18, 1)$); and $T_1 = T_5 = T_6 = 0$. The two S-terms are formed from a single formula. Then the alterations in Equation (28) can be formed from the P, S, and T tables using a single formula.

$$P(I, J) = P(I, J) + S(I) * T(J).$$

This completes the coefficient formation except for a step not otherwise indicated for storing data of the coefficients so that the P-table can be used for the next image. This is done by forming the coefficients of the normal equations and storing them elsewhere. The latter coefficients can be accumulated through the use of a simple loop formula ($D(I, J) = D(I, J) + P(I) * P(J)$) for each of the two independent rows of the P-table, after which new values for Rows 16 and 17 of the A-Table can be read in for the next image, and the procedure repeated.

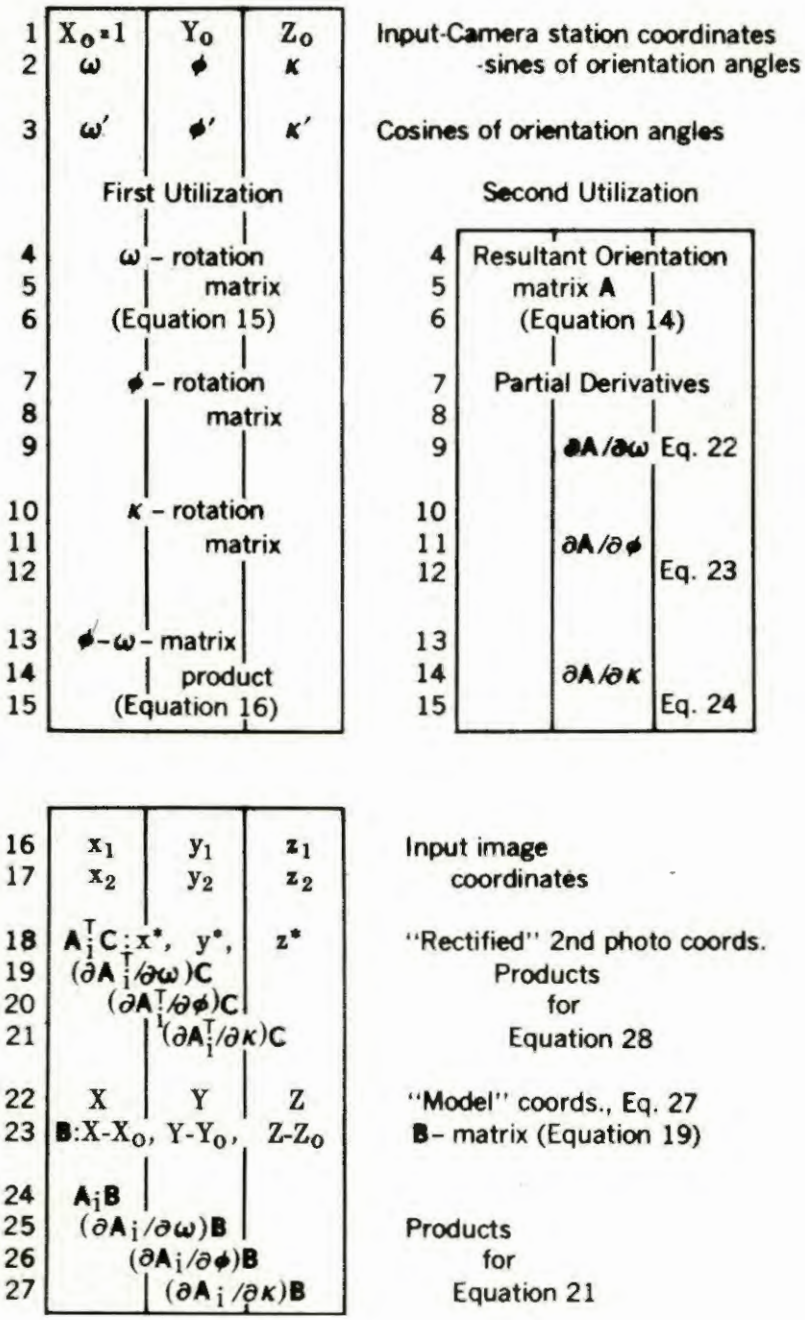

FIG. 24.—Details of the A-table.

APPENDIX 11

NOTES ON THE ADJUSTMENT OF CANTILEVER DATA

The constants $a \ldots$ in Equations (34) and (37) may require derivation although for those familiar with matrix algebra no explanation may be needed. Consider first Equation (34):

$$ax - by + c = x' \quad bx + ay + d = y' \tag{11.1}$$

which represent a plane orthogonal rotation combined with translation in analytic geometry. Suppose the transformed coordinates x', y' are known for two distinct points. Then both the original coordinates x_1 y_1, x_2 y_2 and the transformed values x_1' y_1', x_2' y_2' can be considered as known values with a, b, c, d as unknowns:

$$x_1\, a - y_1 b + c = x_1'$$
$$x_2\, a - y_2 b + c = x_2' \tag{11.2}$$
$$y_1\, a + x_1 b + d = y_1'$$
$$y_2\, a + x_2 b + d = y_2' \tag{11.3}$$

If the second equation of (11.2) is subtracted from the first, and similarly for (11.3), two simultaneous linear equations are formed in terms of the two unknowns a and b:

$$\begin{cases} (x_1 - x_2)\, a - (y_2 - y_1)\, b = (x_1' - x_2') \\ (y_1 - x_2)\, a + (x_2 - x_1)\, b = (y_1' - y_2') \end{cases} \tag{11.4}$$

If these equations are solved for a and b by any method, of which Cramer's Rule is very convenient,

$$a = \frac{(x_1 - x_2)(x_1' - x_2') + (y_1 - y_2)(y_1' - y_2')}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$b = \frac{(x_1 - x_2)(y_1' - y_2') - (y_1 - y_2)(x_1' - x_2')}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and substitution into either one of Equations (11.2) gives c and into (11.3) gives d:

$$c = x_1' - ax_1 + by_1$$
$$d = y_1' - ay_1 - bx_1 .$$

Equations (37) also state formulas for the terms a', b', c', d' in terms of a, b, c, d, based on Equation (36) which is said to be the inverse form of Equation (35). One way for defining the primed coefficients is as follows. Equations (11.1) are rearranged in the form

$$ax - by = (X - c)$$
$$bx + ay = (Y - d). \tag{11.5}$$

Then Equation (11.5) is solved for x, y in exactly the same manner as (11.4) was solved for a, b:

$$x = \frac{a(X - c)}{a^2 + b^2} + \frac{b(Y - d)}{a^2 + b^2}$$

$$y = \frac{a(Y - d)}{a^2 + b^2} - \frac{b(X - c)}{a^2 + b^2} \text{ , or}$$

$$x = \frac{a}{a^2 + b^2} X + \frac{b}{a^2 + b^2} Y - \frac{ac + bd}{a^2 + b^2}$$

$$y = \frac{-b}{a^2 + b^2} X + \frac{a}{a^2 + b^2} Y - \frac{ad - bc}{a^2 + b^2} \text{ , or}$$

$$x = a'X + b'Y - c'$$

$$y = b'X + a'Y - d' \tag{11.6}$$

which completes the definition of the primed coefficients.

An alternate way for defining the primed coefficients refers them to the eight given coordinates as used in Equation (11.2). Accepting the form of Equation (36) of the main text:

$$x_1 = a_1'\, X_1 + b'\, Y_1 - c'$$
$$x_2 = a'\, X_2 + b'\, Y_2 - c'$$
$$y_1 = a'Y_1 - b'X_1 - d'$$
$$y_2 = a'Y_2 - b'X_2 - d'. \tag{11.7}$$

Subtracting, and rearranging,

$$(X_1 - X_2)\, a' + (Y_1 - Y_2)\, b' = (x_1 - x_2)$$
$$(Y_1 - Y_2)\, a' - (X_1 - X_2)\, b' = (y_1 - y_2). \tag{11.8}$$

Solving for a' and b'

$$a' = \frac{-(x_1 - x_2)(X_1 - X_2) - (y_1 - y_2)(Y_1 - Y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$b' = \frac{-(x_1 - x_2)(Y_1 - Y_2) + (y_1 - y_2)(X_1 - X_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and substituting back into (11.7),

$$c' = a'X_1 + b'Y_1 - x_1$$
$$d' = a'Y_1 - b'X_1 - y_1 . \tag{11.9}$$

It is stated without further demonstration that

the values of corresponding coefficients in Equations (11.6) and (11.9) are identical.

Equations (38) and (42) of the text are discussed next. These are quasi-empirical, conformal, polynomial expressions which have been developed to compensate for the accumulation of discrepancies resulting from analogical aerotriangulation instruments based on the work of R. S. Brandt[31], C. W. Price[32], J. M. Zarzycki[33], A. L. Nowicki and C. J. Born[34], and others. The equations tend to treat the problem somewhat analogous to the bending of a beam in which the fibers on one side are considered to be stretched and compressed on the other. As the references indicate, complete agreement is lacking relative to the precise formation of the individual terms of the polynomials, for example, as to whether the degree should be cubic or quadratic. However, the experience gained in the Bureau indicated that cubic curves invariable fit the error distribution better than did quadratic curves. As the errors of analytic aerotriangulation were expected to be milder than analogic, these equations were adopted without change inasmuch as it seems unlikely that the higher degree corrections can be detrimental. In fact the cubic effects have definitely proved to be significantly advantageous.

APPENDIX 12

SOLUTION OF NORMAL EQUATIONS BY GAUSS-CHOLESKY ELIMINATION METHOD

When the block adjustment described in the general text is used, a large set of normal equations must be solved. There may be hundreds or even thousands of unknowns in the matrix. The Cholesky modification of the classical Gaussian elimination method offers several advantages when applied to electronic computers. Inasmuch as no other Coast and Geodetic Survey publication contains an adequate example, a brief description of the method with a general solution is published here.

Andre Louis Cholesky, a colonel in the French Artillery during World War I, proposed this modification sometimes referred to as the square root method. H. F. Rainsford[35] gives an excellent description of the method in his recent text.

The observation equations referred to in the general text of this bulletin may be written in the form,

$$ax + by + cz - l = v.$$

In this general equation, the coefficients a, b, and c are determined to sufficient accuracy to furnish the significance needed in the unknowns. Careful attention must be given to the size of these coefficients. If possible they should be near unity. The decimal place of the unknowns should be adopted so that this balance can be accomplished. In photogrammetric computations it is preferable to use fixed point arithmetic based upon proper "computer scale" rather than use floating point techniques. Normal equations are formed in the customary manner from a set of observation equations. These normal equations are of the following general form:

$$\begin{array}{ll} \text{I} & [aa]\,x + [ab]\,y + [ac]\,z - [al] = 0 \\ \text{II} & [ab]\,x + [bb]\,y + [bc]\,z - [bl] = 0 \\ \text{III} & [ac]\,x + [bc]\,y + [cc]\,z - [cl] = 0 \end{array}$$

An illustration of the use of the Gauss-Doolittle elimination technique appears in Special Publication 237, Manual of Geodetic Astronomy, on pages 138-139. The following example (table 6) gives a comparison of the Gauss-Dolittle with the Gauss-Cholesky methods. In the Gauss-Doolittle method, lines 1, 2, 5, 6, 10, and 11 must be preserved for the complete solution. In the Gauss-Cholesky method, line 2 may be substituted or stored in the computer area reserved for line 1. The same is true with line 6 to line 5, and line 11 to line 10. From this, it is obvious that the storage requirement for the Gauss-Cholesky method is one-half of that for the Gauss-Doolittle method. Also, in the Gauss-Cholesky method, the coefficients appearing in lines 2, 6, and 11 are the geometric means of corresponding numbers appearing in lines 1-2, 5-6, 10-11, of the Gauss-Doolittle. When the geometric means are multiplied against each other there is an improvement with respect to round-off errors. In the Cholesky modification, the diagonal term, which is generally the dominant term in normal equations, is reduced in size if greater than 1 and increased in size if less than 1. This fact also contributes to an improved accuracy.

The computing routine is straightforward and can be applied quite readily to any electronic computer having adequate storage. The problem is simplified considerably if the computer storage is large so that the number of input-output cycles can be kept to a minimum.

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31 Brandt, Robert S., see footnote 15, above.

32 Price, Charles W., see footnote 16, above.

33 Some theoretical and practical problems in photogrammetric bridging by J. M. Zarzycki, *Photogrammetric Engineering*, v. 24, 1955.

34 Improved stereotriangulation adjustments with electronic computers by A. L. Nowicki and C. J. Born, *Photogrammetric Engineering*, v. 26, 1960.

35 Survey adjustments and least squares by Hume F. Rainsford, *Frederick Ungar Publishing Co.*, p. 74, 1958.

Table 6.—Comparison of Doolittle and Cholesky techniques for solving a set of normal equations

	Gauss-Doolittle				Gauss-Cholesky			
	x	y	z	l	x	y	z	l
1.	$+[aa]$	$+[ab]$	$+[ac]$	$-[al]$	$+[aa]$	$+[ab]$	$+[ac]$	$-[al]$
2.	$x =$	$-\frac{[ab]}{[aa]}$	$-\frac{[ac]}{[aa]}$	$-\frac{[al]}{[aa]}$	$+[aa]^{1/2}$	$+\frac{[ab]}{[aa]^{1/2}}$	$+\frac{[ac]}{[aa]^{1/2}}$	$-\frac{[al]}{[aa]^{1/2}}$
3.		$+[bb]$	$+[bc]$	$-[bl]$		$+[bb]$	$+[bc]$	$-[bl]$
4.		$-\frac{[ab]}{[aa]}[ab]$	$-\frac{[ab]}{[aa]}[ac]$	$+\frac{[ab]}{[aa]}[al]$		$-\frac{[ab]^2}{[aa]}$	$-\frac{[ab]\ [ac]}{[aa]}$	$+\frac{[ab]\ [al]}{[aa]}$
5.		$+[bb_1]$	$+[bc_1]$	$-[bl_1]$		$+[bb_1]$	$+[bc_1]$	$-[bl_1]$
6.		$y =$	$-\frac{[bc_1]}{[bb_1]}$	$+\frac{[bl_1]}{[bb_1]}$		$[bb_1]^{1/2}$	$+\frac{[bc_1]}{[bb_1]^{1/2}}$	$-\frac{[bl_1]}{[bb_1]^{1/2}}$
7.			$[cc]$	$-[cl]$			$[cc]$	$-[cl]$
8.			$-\frac{[ac]}{[aa]}[ac]$	$+\frac{[ac]}{[aa]}[al]$			$-\frac{[ac]^2}{[aa]}$	$+\frac{[ac]\ [al]}{[aa]}$
9.			$-\frac{[bc_1]}{[bb_1]}[bc_1]$	$-\frac{[bc_1]}{[bb_1]}[bl_1]$			$-\frac{[bc_1]^2}{[bb_1]}$	$+\frac{[bc_1]\ [bl_1]}{[bb_1]}$
10.			$[cc_2]$	$-[cl_2]$			$[cc_2]$	$-[cl_2]$
11.			$z =$	$+\frac{[cl_2]}{[cc_2]}$			$*\ [cc_2]^{1/2}\ z$	$-\frac{[cl_2]}{[cc_2]^{1/2}}$

*Line 11 may be expressed as $[cc_2]^{1/2}\ z - \frac{[cl_2]}{[cc_2]^{1/2}} = 0$ or $z = +\frac{[cl_2]}{[cc_2]}$

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