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A Comparison of Methods for Computing Gravitational Potential Derivatives

L. J. GULICK

ROCKVILLE, MARYLAND
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A Comparison of Methods for Computing Gravitational Potential Derivatives

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ELECTRONIC COMPUTING DIVISION

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Contents

A Comparison of Methods for Computing Gravitational Potential Derivatives

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ABSTRACT. Computation of the derivatives of the gravitational potential of the earth using spherical harmonics requires the formation of a number of double sums. The expression of these double sums is the most important consideration for achievement of maximum efficiency. Three methods of computing the derivatives (subroutines given in Appendixes C, D, and E) are compared for efficiency, and it is found that the double sums are most efficiently computed using polar coordinates with subsequent transformation of the derivatives to rectangular coordinates by simple application of the chain rule. A subroutine using this method (Appendix C) uses less than 700 words of central memory and requires only 7.3 milliseconds for execution on the CDC 6600 computer when harmonic coefficients up to degree and order 14 are used.

Expressions for the derivatives of polar coordinates with respect to rectangular coordinates and a derivation of the recursion formulas used in computing associated Legendre polynomials are also included as Appendixes A and B, respectively.

I. INTRODUCTION

The equation for representing the gravitational potential of the earth in terms of spherical harmonics is widely given in the literature, for example, Hotine (1969). We have

$$V(\beta, \lambda, r)=\frac{GM}{r}\left\{1+\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n}P_n^m(\sin\beta)\times(C_n^m\cos m\lambda+S_n^m\sin m\lambda)\right\} \quad (1)$$

where GM is the gravitational constant times the mass of the earth, a is the equatorial radius of a reference ellipsoid, the C_n^m and S_n^m are the coefficients representing the mass distribution. P_n^m $(\sin \beta)$ is the associated Legendre polynominal of degree n and order m, and β, λ, and r are polar coordinates defining the position of a point at which the potential V is to be determined. For computational purposes the sum over n is finite with upper limit N.

The partial derivatives of V with respect to inertial rectangular coordinates X_1, X_2, X_3 represent the components of the gravitational vector, and standard forms for these partials of V are widely distributed; but treatment of these derivatives with regard to conservation of computer time on large, modern machines is rare. For instance, the simple practice of normalizing the Legendre polynomials is treated in many discussions or reports, for example, Heiskanen and Moritz (1967), but it is seldom noted that normalization may be a useless extravagance on a machine, such as the CDC 6600, with a large-word size. The CDC 6600 uses a 60-bit central memory word which retains 15 decimal digits; exhaustive testing has shown that for the Legendre polynomials up to degree 14 and order 14, at least 10 significant digits are retained for X between minus one and one. This result should not cause rejection of normalization, but it should be recognized that normalization is not automatically recommended or productive. Furthermore, even if more accuracy is desired, it should not be assumed that normalization is more efficient that the use of double precision for all or part of the computation.

In the following text, three methods of representing the first and second partials of V are presented. These are not the only methods which can be employed, but they are among the most interesting and logical and, at the same time, illustrate the variety to be found in such representations. In discussing these methods, programming efficiency will be presumed, and the discussion will therefore primarily encompass only the computational characteristics of the different mathematical methods. For possible further comparison, for illustration of the programming envisioned, and for possible use by others, we also present as separate appendixes subroutines for each of the methods discussed.

II. EXPRESSIONS FOR THE PARTIAL DERIVATIVES

Three methods of expressing the partial derivatives of V will be studied. For future reference purposes, they are given the descriptive labels: "the method of coefficient modification," "the chain-rule method," and "the function method."

A fourth method given by DeWitt (1962) and Cunningham (1970) has computational characteristics similar to those of the method of coefficient modification and is not treated separately.

A. Method of Coefficient Modification

Both the first and second partial derivatives of V may be expressed by formulas which are very near to the formula for V itself, using nine sets of modified harmonic coefficients. A full discussion of the derivation of the equations used in this method was given by Hotine (1969) where formulas for the modified coefficients are also to be found. Specifically, the formulas are [1]

$$\frac{\partial V}{\partial X_i}=\frac{GM}{r^2}\sum_{n=0}^{N}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n+1}P_{n+1}^{m}\,(\sin\beta)$$
$$\times\,(\bar{C}_{n,\,m,\,i}\cos m\lambda+\bar{S}_{n,\,m,\,i}\sin m\lambda)\quad(2)$$

and

$$\frac{\partial^2 V}{\partial X_i\partial X_j}=\frac{GM}{r^3}\sum_{n=0}^{N}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n+2}P_{n+2}^{m}\quad(\sin\beta)$$
$$\times\,(\,\bar{\bar{C}}_{n,\,m,\,i,\,j}\cos m\lambda+\,\bar{\bar{S}}_{n,\,m,\,i,\,j}\sin m\lambda\,).\quad(3)$$

Given the nine sets of modified coefficients, these are certainly the most compact forms of the equations possible. However, the coefficient sets are distinct, and the above expressions must be considered as representative of nine actual equations.

B. Chain-Rule Method

This method takes advantage of the fact that V is expressed as a function of the polar coordinates only, simplifying the expressions containing the double sums. It also admits the use of an equation for the derivative of $P_n^m(\sin\beta)$ which avoids increasing the degree.

Quite simply we have

$$\frac{\partial V}{\partial X_i}=\frac{\partial V}{\partial r}\frac{\partial r}{\partial X_i}+\frac{\partial V}{\partial\beta}\frac{\partial\beta}{\partial X_i}+\frac{\partial V}{\partial\lambda}\frac{\partial\lambda}{\partial X_i};\quad(4)$$

and using the general relation[2]

$$\frac{dP_n^m(x)}{dx}=-\frac{mxP_n^m(x)}{1-x^2}+\frac{P_n^{m+1}(x)}{(1-x)^{1/2}},\quad(5)$$

we may write

$$\frac{\partial V}{\partial\beta}=GM\sum_{n=1}^{N}\left(\frac{a^n}{r^{n+1}}\right)\sum_{m=0}^{n}\hat{P}_{n,m}\hat{C}_{n,m}\quad(6)$$

$$\frac{\partial V}{\partial\lambda}=GM\sum_{n=1}^{N}\frac{a^n}{r^{n+1}}\sum_{m=0}^{n}mP_n^m\hat{S}_{n,m}\quad(7)$$

$$\frac{\partial V}{\partial r}=-\frac{GM}{r^2}-GM\sum_{n=1}^{N}\,[(n+1)a^n/r^{n+2}]\sum_{m=0}^{n}P_n^m\hat{C}_{n,m}.\quad(8)$$

Equations for the second partials follow readily as

$$\frac{\partial^2 V}{\partial\beta^2}=GM\sum_{n=1}^{N}\frac{a^n}{r^{n+1}}\sum_{m=0}^{n}\hat{C}_{n,m}\{P_n^{m+2}$$
$$-(2m+1)\tan\beta\,P_n^{m+1}$$
$$+mP_n^m\,(m\tan^2\beta-\sec^2\beta)\}\quad(9)$$

$$\frac{\partial^2 V}{\partial\lambda^2}=-GM\sum_{n=1}^{N}\frac{a^n}{r^{n+1}}\sum_{m=0}^{n}m^2P_n^m\hat{C}_{n,m}\quad(10)$$

$$\frac{\partial^2 V}{\partial r^2}=\frac{2GM}{r^3}+GM\sum_{n=1}^{N}\frac{(n+1)(n+2)a^n}{r^{n+3}}\sum_{m=0}^{n}P_n^m\hat{C}_{n,m}\quad(11)$$

$$\frac{\partial^2 V}{\partial\beta\partial\lambda}=GM\sum_{n=1}^{N}\frac{a^n}{r^{n+1}}\sum_{m=0}^{n}m\hat{S}_{n,m}\hat{P}_{n,m}\quad(12)$$

$$\frac{\partial^2 V}{\partial\beta\partial r}=-GM\sum_{n=1}^{N}\frac{(n+1)a^n}{r^{n+2}}\sum_{m=0}^{n}\hat{P}_{n,m}\hat{C}_{n,m}\quad(13)$$

$$\frac{\partial^2 V}{\partial\lambda\partial r}=-GM\sum_{n=1}^{N}\frac{(n+1)a^n}{r^{n+2}}\sum_{m=0}^{n}mP_n^m\hat{S}_{n,m}\quad(14)$$

[1] The modified coefficients are barred in accordance with Hotine's notation. They do not indicate normalization.

[2] This relation is derived in Appendix B, Equation (8b).

where the function notation has been dropped for convenience, and we have defined for notational purposes

$$\hat{P}_{n,m} = P_n^{m-1} - m \tan\beta \, P_n^m \tag{15}$$

$$\hat{C}_{n,m} = C_n^m \cos m\lambda + S_n^m \sin m\lambda \tag{16}$$

$$\hat{S}_{n,m} = - C_n^m \sin m\lambda + S_n^m \cos m\lambda. \tag{17}$$

The partials of the polar coordinates with respect to the rectangular coordinates are given in Appendix A.

C. Function Method

The gravitational potential as given in Equation (1) is expressed as a function of the spherical coordinates of a point in space at which the potential is being evaluated, whereas most allied computations are more conveniently performed in rectangular coordinates so that forces can be expressed as components in three orthogonal directions. In the two above methods this disparity of coordinates has been clearly evident, requiring many sets of harmonic coefficients for the method of coefficient modification and requiring transformation of coordinates for the chain-rule method. In the following paragraphs, we will present a method for the representation of the first and second partial derivatives of V by a single equation in each case, requiring neither further transformation of systems nor numerous sets of harmonic coefficients. Essentially, the only obstacle to the achievement of such representation is the asymmetry of the polar coordinates as functions of the rectangular coordinates. As shown in figure 1, we have

$$r = [X_1^2 + X_2^2 + X_3^2]^{1/2} \tag{18}$$

$$\sin\beta = X_3/(X_1^2 + X_2^2 + X_3^2)^{1/2} \tag{19}$$

$$\lambda = \text{Arctan}\left(\frac{v}{u}\right) \tag{20}$$

where

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{21}$$

and θ is the angle through which the (X_1, X_2) plane must be rotated for the X_1-axis to coincide with the Greenwich Meridian. It is clear that r is the only polar coordinate which is a symmetric function of the rectangular coordinates. Therefore, to obtain

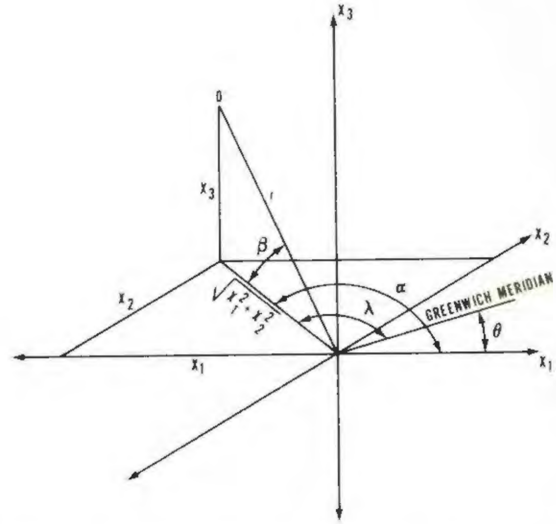

FIGURE 1. – Geocentric polar and rectangular coordinate systems.

general formulas for the partial derivatives of the functions defined in Equations (18), (19), and (20), we propose to use the following functions

$$G(i) = H(i) \cdot [\tan\beta]^{H(i)} \tag{22}$$

and

$$F(i) = \begin{Bmatrix} -1; & i=1 \\ 1; & i=2 \\ 0; & i=3 \end{Bmatrix} \tag{23}$$

where

$$H(i) = \begin{Bmatrix} 1; & i=1,\, i=2 \\ -1; & i=3 \end{Bmatrix} \tag{24}$$

Using these functions, we can write

$$\frac{\partial r}{\partial X_i} = \frac{X_i}{r} \tag{25}$$

$$\frac{\partial \sin\beta}{\partial X_i} = -\frac{X_i G(i)}{r^2} \cos\beta \tag{26}$$

$$\frac{\partial \lambda}{\partial X_i} = \frac{F(i) \sin\alpha \cos\alpha}{X_i} \tag{27}$$

where α is the sum of the angles λ and θ.

For notational purposes, it is convenient to define function U_n^m such that

$$U_n^m\,(\lambda, \beta, r) = \left(\frac{1}{r^{n+1}}\right) P_n^m\,(\sin\beta)\, \hat{C}_{n,m}(\lambda), \tag{28}$$

allowing Equation (1) to be rewritten as

$$V(\lambda, \beta, r) = \frac{GM}{r} + GM \sum_{n=1}^{N} a^n \sum_{m=0}^{n} U_n^m (\lambda, \beta, r) \tag{29}$$

and

$$\frac{\partial V}{\partial X_i} = -\frac{(GM)X_i}{r^3} + GM \sum_{n=1}^{N} a^n \sum_{m=0}^{n} \frac{\partial U_n^m(\lambda, \beta, r)}{\partial X_i}. \tag{30}$$

Clearly then, we need only exhibit a general formula for the partials of U_n^m.

Using Equations (5), (25), (26), and (27) and simply differentiating in the standard manner, we arrive at the equation

$$\frac{\partial U_n^m}{\partial X_i} = \frac{X_i \hat{C}_{n,m}}{r^{n+3}} [-G(i)\hat{P}_{n,m} - (n+1)P_n^m] + \frac{mAF(i)P_n^m \hat{S}_{n,m}}{r^{n+1}X_i} \tag{31}$$

where $A = \sin \alpha \cos \alpha$.

Using Equation (30), we can write for the second derivative

$$\frac{\partial^2 V}{\partial X_i \partial X_j} = \frac{3 \cdot GM \cdot X_i \cdot X_j}{r^5} - \frac{GM \cdot \delta_{i,j}}{r^3} - GM \sum_{n=1}^{N} a^n \sum_{m=0}^{n} \frac{\partial^2 U_n^m}{\partial X_i \partial X_j}; \tag{32}$$

and after differentiation and much manipulation, we arrive at the formula

$$\begin{aligned}\frac{\partial^2 U_n^m}{\partial X_i \partial X_j} = {} & \frac{mA\hat{S}_{n,m}}{r^{n+3}} \left\{ -\hat{P}_{n,m} \left(\frac{X_i G(i) F(j)}{X_j} + \frac{X_j G(j) F(i)}{X_i} \right) \right. \\ & \left. - (n+1) P_n^m \left(\frac{X_i F(j)}{X_j} + \frac{X_j F(i)}{X_i} \right) \right\} \\ & + \frac{mA^2 F(i) F(j) P_n^m}{r^{n+1} X_i X_j} \left\{ \hat{S}_{n,m} \left(\frac{X_1^2 - X_2^2}{X_1 X_2} \right) - m\hat{C}_{n,m} \right\} \\ & + \frac{\delta_{i,j}}{X_i} \left\{ \frac{\partial U_n^m}{\partial X_i} - \frac{2mAF(i)P_n^m \hat{S}_{n,m}}{X_i r^{n+1}} \right\} \\ & + \frac{X_i X_j \hat{C}_{n,m}}{r^{n+5}} \left\{ G^* [(m^2 - 2m) T^2 P_n^m - (T^{-1} + 2mT) P_n^{m+1} + P_n^{m+2}] \right. \\ & \left. + G^+ (n+3) \hat{P}_{n,m} + (n+1)(n+3) P_n^m \right\} \end{aligned} \tag{33}$$

where

$$T = \tan \beta \tag{34}$$

$$G^* = G(i) \cdot G(j) \tag{35}$$

$$G^+ = G(i) + G(j), \tag{36}$$

and

$\delta_{i,j}$ is the Kronecker delta.

III. COMPUTATION CHARACTERISTICS

For a comparison of computational efficiency, the methods must be examined most critically because seemingly small differences can materially affect computing time due to two immutable characteristics. Firstly, whichever method is used, it must be usable often because fitting a satellite orbit over a period of only 24 hours might require as many as 2,880 separate computations of the potential derivatives (twice for each 1-minute time step—once for the prediction cycle and once for the correction cycle in the usual Gauss-Jackson multistep method of numerical integration). Secondly, the computation of the double sum inherent in any method used is the major time consumer.

The most obvious difference in the three methods is in core-storage requirement. The method of coefficient modification requires storage for the nine sets of coefficients. Storage for all three methods is reduced by fully utilizing double-subscripted arrays for the coefficient storage. In those cases involving the chain-rule and function methods, we need only to store the original coefficients in an $(N) \times (N+1)$ array as follows:

$$C(i,j) = \begin{vmatrix} C_1^1 & S_1^1 & S_2^1 & . & . & . & S_N^1 \\ C_2^1 & C_2^2 & S_2^2 & . & . & . & S_N^2 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ C_N^1 & C_N^2 & . & . & . & C_N^N & S_N^N \end{vmatrix}$$

$$i = 1, 2, \ldots N; \quad j = 1, 2, \ldots N+1 \tag{37}$$

which provides relatively simple access according to the relations

$$C_n^m = C(n, m) \tag{38}$$

$$S_n^m = C(m, n+1), \tag{39}$$

the zonal harmonics being stored separately.

However, it must be noted that in the method of coefficient modification, the coefficients have been expanded as well as modified. It then becomes more convenient to store these sets in arrays $CI(I, J, K)$ and $CIJ(I, J, K)$ as follows:

$$CI=\left|\begin{array}{ccccccccc} \bar{C}_{1,1,k} & \bar{C}_{1,2,k} & \bar{S}_{N,N+1,k} & \bar{S}_{N,N,k} & \cdot & \cdot & \cdot & \cdot & \bar{S}_{N,1,k} \\ \bar{C}_{2,1,k} & \bar{C}_{2,2,k} & \bar{C}_{2,3,k} & \bar{S}_{N-1,N,k} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bar{S}_{2,3,k} & \bar{S}_{2,2,k} & \bar{S}_{2,1,k} \\ \bar{C}_{N,1,k} & \bar{C}_{N,2,k} & \bar{C}_{N,3,k} & \cdot & \cdot & \cdot & \bar{C}_{N,N+1,k} & \bar{S}_{1,2,k} & \bar{S}_{1,1,k} \end{array}\right| \qquad (40)$$

so that the relationship between equation constants and computer array becomes

$$\bar{C}_{n,m,k}=CI(n, m, k) \qquad (41)$$

$$\bar{S}_{n,m,k}=CI(N+1-n, N+4-m, k). \qquad (42)$$

It should be noted that the conservation of core storage in the above case slightly increases computer operations in the all-important formation of the double sums, and if the method was used in a situation wherein storage may be totally neglected, the above should be discarded. But in most instances, the trade-off is such that use of the above arrays is considered justified, and the loss of time is not material enough to affect results in a comparison with other methods.

Specifically, core requirements are given by the following formulas:

$$\text{No. core words (coefficient mod.)}=10N^2+57N+39 \qquad (43)$$

$$\text{No. core words (chain-rule)}=2N^2+12N+72 \qquad (44)$$

$$\text{No. core words (function)}=2N^2+8N+104. \qquad (45)$$

The core requirement variance with N is depicted graphically in figure 2.

It was previously noted that, given the nine sets of modified coefficients, the method of coefficient modification uses the most compact equations. In actual computation, we are given the nine sets because they need only be computed once in a lifetime, and programming is thus simple. But in the function method, we have compacted the number of equations and this proves to be more efficient. Furthermore, we seem to have done neither in the chain-rule method, and it proves to be the most efficient of all. The reasons for these results are examined in the following paragraphs.

First, for a given N, the method of coefficient modification requires the computation of Legendre polynomials up to degree $N+2$ and order $N+2$. The other methods require the Legendre polynomials to be computed only to degree N and order $N+2$; in practice, this means computing only to order N also because the polynomials are identically zero whenever the order exceeds the degree.

Second, and more important, the method of coefficient modification funnels more of the computation into terms used in forming the double sum; by producing nine widely disparate double sums, the

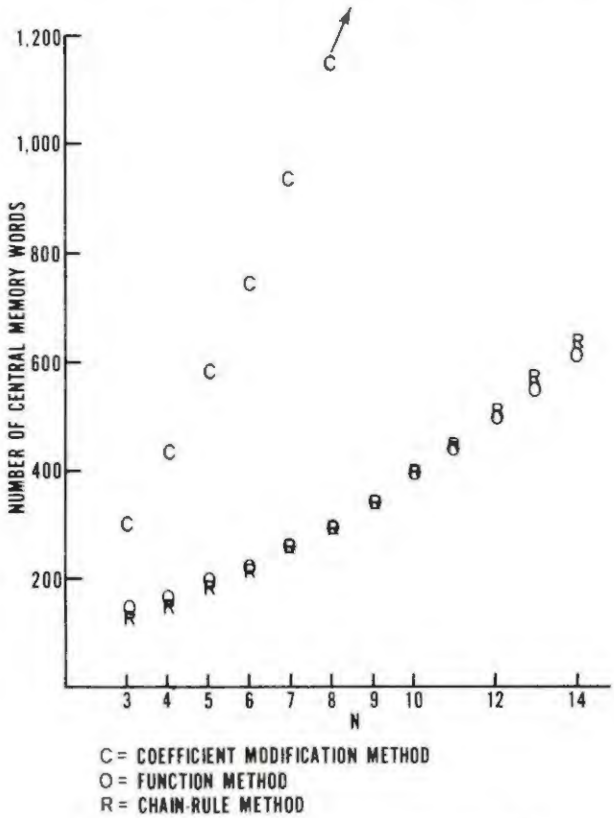

FIGURE 2. – Central memory requirement versus N.

method fails to take advantage of the fact that some double sums can be made common to more than one partial derivative. In particular, we should look at the manner in which the function method is computed because it also directly computes partials with respect to rectangular coordinates.

To use the function method in computation, we would form the 14 sums listed below, only nine of which are distinct double sums (showing differences over the innermost index, m) and providing many interrelations even among the nine. The sums are:

$$\sigma_1 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^m \hat{C}_{n,m} \tag{46}$$

$$\sigma_2 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^{m+1} \hat{C}_{n,m} \tag{47}$$

$$\sigma_3 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^{m+2} \hat{C}_{n,m} \tag{48}$$

$$\sigma_4 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m P_n^m \hat{C}_{n,m} \tag{49}$$

$$\sigma_5 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m^2 P_n^m \hat{C}_{n,m} \tag{50}$$

$$\frac{\partial^2 V}{\partial X_i \partial X_j}\left(\frac{1}{GM}\right) = \frac{A}{r^3}\left\{\left(\frac{X_i G(i) F(j)}{X_j} + \frac{X_j G(j) F(i)}{X_i}\right)(T\sigma_8 - \sigma_9) - \left(\frac{X_i F(j)}{X_j} + \frac{X_j F(i)}{X_i}\right)(\sigma_{14} + \sigma_7)\right\}$$
$$+ \frac{A^2 F(i) F(j)}{X_i X_j r}\left[\hat{X}\sigma_7 - \sigma_5\right]$$
$$+ \delta_{i,j}\left\{\frac{-G(i)\,\hat{\sigma} - \sigma_{10} - \sigma_1}{r^3} - \frac{AF(i)\,\sigma_7}{rX_i^2}\right\}$$
$$+ \frac{X_i X_j}{r^5}\left\{G^*\left[T^2(\sigma_5 - 2\sigma_4) + 2T\sigma_6 + \sigma_3 - T^{-1}\sigma_2\right]\right.$$
$$\left. + G^+\left[\sigma_{12} - T\sigma_{13} + 3\hat{\sigma}\right] + \sigma_{11} + 4\sigma_{10} + 3\sigma_1 + 3\right\} \tag{61}$$

$$\sigma_6 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m P_n^{m+1} \hat{C}_{n,m} \tag{51}$$

$$\sigma_7 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m P_n^m \hat{S}_{n,m} \tag{52}$$

$$\sigma_8 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m^2 P_n^m \hat{S}_{n,m} \tag{53}$$

$$\sigma_9 = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m P_n^{m+1} \hat{S}_{n,m} \tag{54}$$

$$\sigma_{10} = \sum_{n=1}^{N} n \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^m \hat{C}_{n,m} \tag{55}$$

$$\sigma_{11} = \sum_{n=1}^{N} n^2 \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^m \hat{C}_{n,m} \tag{56}$$

$$\sigma_{12} = \sum_{n=1}^{N} n \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^{m+1} \hat{C}_{n,m} \tag{57}$$

$$\sigma_{13} = \sum_{n=1}^{N} n \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m P_n^m \hat{C}_{n,m} \tag{58}$$

$$\sigma_{14} = \sum_{n=1}^{N} n \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} m P_n^m \hat{S}_{n,m}. \tag{59}$$

Using these sums, Equation (30) becomes

$$\frac{\partial V}{\partial X_i}\left(\frac{1}{GM}\right) = \frac{X_i}{r^3}\left[-G(i)\hat{\sigma} - \sigma_{10} - \sigma_1 - 1\right] + \left(\frac{AF(i)}{rX_i}\right)\sigma_7, \tag{60}$$

and Equation (32) becomes

where

$$\hat{\sigma} = \sigma_2 - T\sigma_4 \tag{62}$$

$$\hat{X} = \frac{X_1^2 - X_2^2}{X_1 X_2}. \tag{63}$$

At this point, it might be noted that the 14 sums given could as well be used in computing by the chain-rule method. However, Equations (6) through (14) may be programmed directly with no loss of efficiency because they already contain many like terms. The sums over m are identical in Equations (8) and (11), (7) and (14), and (6) and (13); and the sums over m in Equations (6) and (12) have similar terms as do Equations (8) and (10). In practice, the two methods—the chain-rule method and the function method—are essentially related because the functions F and G are applied to partials of the polar coordinates or to functions of polar coordinates with respect to rectangular coordinates.

As was previously noted, all three methods have been programmed and subroutines using these methods have been included as Appendixes C, D, and E, allowing direct comparisons of operating efficiency. An examination of the subroutines reveals that the number of operations for each subroutine call is given by the following set of formulas:

$$\text{No. ops. (coeff. mod.)} = (59/2)N^2 + (337/2)N + 75 \quad (64)$$

$$\text{No. ops. (chain-rule)} = (29/2)N^2 + (129/2)N + 203 \quad (65)$$

$$\text{No. ops. (function)} = 15N^2 + 71N + 320. \quad (66)$$

These formulas are graphically depicted in figure 3. The difference between Equations (65) and (66) is largely in the constant number of operations. This difference occurs because, for comparative purposes, we made no use of the symmetry of the derivatives in the function method subroutine and made complete use of it in the chain-rule method subroutine.

Figure 4 shows the results of actual test runs on a CDC 6600 of the subroutines given in the appendixes. Each subroutine was executed 1,000 times for each value of N; the times shown are thus in units of seconds per 1,000 executions. It will be noted that the results of the test comparison are in keeping with the comparison of the number of operations performed.

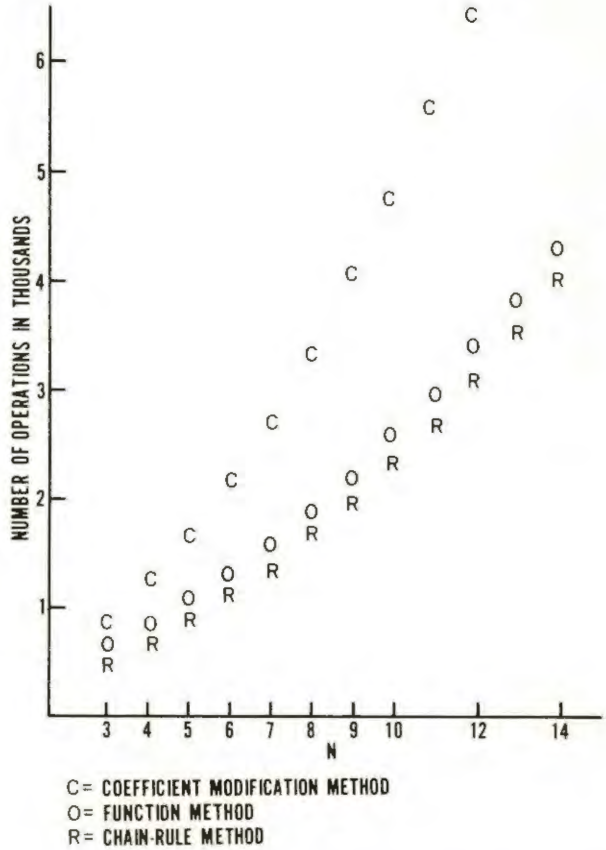

FIGURE 3.—Number of operations per subroutine call versus N.

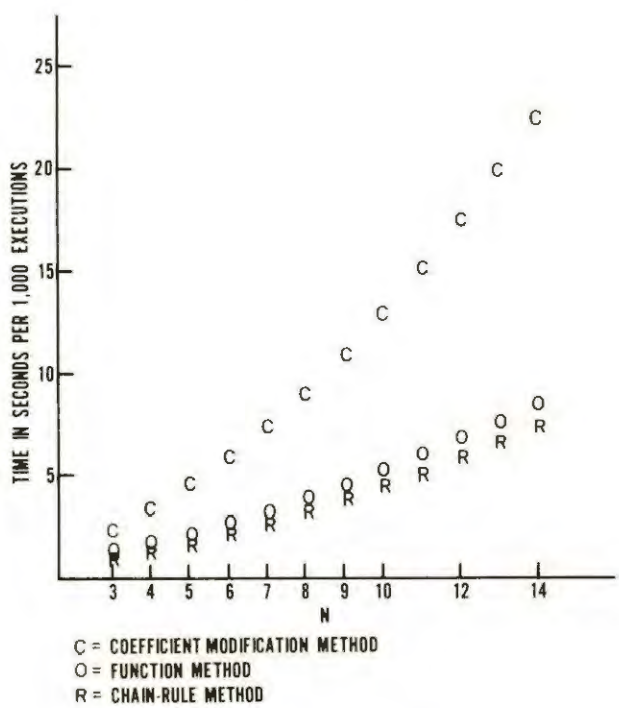

FIGURE 4.—Execution time versus N.

IV. SUMMARY

Using harmonics up to degree and order 14, computation of the partial derivatives by the chain-rule method takes 7.3 milliseconds per execution on the CDC 6600 computer. Even so, there remains the possibility that improvements can be made, but the execution time is so low that the effort is not considered worthwhile. The subroutine given in Appendix C is therefore recommended. If greater accuracy is desired, it is recommended that double precision be employed over other methods. Core memory requirements are already so low that the added storage would be unimportant, and the simplicity and generality of this method should outweigh any time considerations in such an efficient routine.

Consideration of the method of coefficient modification was included to illustrate the possible variance between methods that are useful for analysis and those that admit maximum computational efficiency. To the best of the author's knowledge, the function method is exhibited here for the first time. It is included for general interest, and it also illustrates the fact that many methods reduce ultimately to the chain-rule method.

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APPENDIX A

PARTIAL DERIVATIVES OF THE POLAR COORDINATES WITH RESPECT TO THE RECTANGULAR COORDINATES

The relationship between polar and rectangular coordinates is depicted in figure 1 in the text; the functions $F(i)$ and $G(i)$ are defined in Equations (22) and (24) in the text.

From figure 1 (text):

$$r^2 = X_1^2 + X_2^2 + X_3^2 \tag{1a}$$

$$\sin\beta = X_3/r \tag{2a}$$

$$\cos\beta = \frac{(X_1^2 + X_2^2)^{1/2}}{r} \tag{3a}$$

$$\alpha = \lambda + \theta \tag{4a}$$

$$\tan\alpha = \frac{X_2}{X_1} \tag{5a}$$

From Equation (1a):

$$2r\left(\frac{\partial r}{\partial X_i}\right) = 2X_i \tag{6a}$$

$$\frac{\partial r}{\partial X_i} = \frac{X_i}{r} \tag{7a}$$

$$\frac{\partial^2 r}{\partial X_i \partial X_j} = (r^2\delta_{i,j} - X_iX_j)/r^3 \tag{8a}$$

From Equations (2a) and (3a):

$$\cos\beta\,\frac{\partial\beta}{\partial X_1} = X_3\left[-r^{-2}\left(\frac{X_1}{r}\right)\right] = -\frac{X_1X_3}{r^3} \tag{9a}$$

$$\frac{\partial\beta}{\partial X_1} = -(X_1\tan\beta)/r^2 \tag{10a}$$

$$\frac{\partial\beta}{\partial X_2} = -(X_2\tan\beta)/r^2 \tag{11a}$$

$$\cos\beta\left(\frac{\partial\beta}{\partial X_3}\right) = \left[r - X_3\left(\frac{X_3}{r}\right)\right]/r^2$$

$$= (r^2 - X_3^2)/r^3 = (X_1^2 + X_2^2)/r^3 \tag{12a}$$

$$\frac{\partial\beta}{\partial X_3} = \frac{\cos\beta}{r} \tag{13a}$$

$$\frac{\partial\sin\beta}{\partial X_i} = -\frac{(\cos\beta\tan\beta)X_i}{r^2} \quad (i=1,2) \tag{14a}$$

$$\frac{\partial\sin\beta}{\partial X_3} = \frac{\cos^2\beta}{r} = \frac{\cos\beta\sin\beta\cot\beta}{r}$$

$$= \frac{(\cos\beta\cot\beta)X_3}{r^2} \tag{15a}$$

$$\frac{\partial\sin\beta}{\partial X_1} = -\frac{\cos\beta\,X_i\,G(i)}{r^2} \tag{16a}$$

From Equation (10a):

$$\frac{\partial^2\beta}{\partial X_1^2} = -\left[X_1(\tan\beta)(-2r^{-3})\left(\frac{X_1}{r}\right) + X_1 r^{-2}\sec^2\beta\left(\frac{\partial\beta}{X_1}\right) + (\tan\beta)\,r^{-2}\right]$$

$$= \frac{2X_1^2\tan\beta}{r^4} + \frac{X_1^2\tan\beta\sec^2\beta}{r^4} - \frac{\tan\beta}{r^2}$$

$$= \frac{\tan\beta}{r^4}\,[2X_1^2 + X_1^2\sec^2\beta - r^2]$$

$$= \frac{\tan\beta}{r^4}\left[2X_1^2 - \frac{X_2^2}{\cos^2\beta}\right] \tag{17a}$$

$$\frac{\partial^2\beta}{\partial X_2^2} = \frac{\tan\beta}{r^4}\left[2X_2^2 - \frac{X_1^2}{\cos^2\beta}\right] \tag{18a}$$

From Equation (13a):

$$\frac{\partial^2\beta}{\partial X_3^2} = r^{-2}\left[-r\sin\beta\,\frac{\partial\beta}{\partial X_3} - \cos\beta\left(\frac{X_3}{r}\right)\right]$$

$$= r^{-2}[-\sin\beta\cos\beta - \cos\beta\sin\beta]$$

$$= -\frac{2\cos\beta\sin\beta}{r^2}$$

$$= -\frac{2X_3}{r^3}\cos\beta \tag{19a}$$

From Equation (10a):

$$\frac{\partial^2\beta}{\partial X_1\partial X_2}=-\left[X_1(\tan\beta)(-2r^{-3})\frac{X_2}{r}+X_1r^{-2}\sec^2\beta\frac{\partial\beta}{\partial X_2}\right]$$

$$=\frac{X_1X_2\tan\beta}{r^4}(2+\sec^2\beta) \tag{20a}$$

$$\frac{\partial^2\beta}{\partial X_3\partial X_1}=-\frac{X_1}{r^4}\left[r^2\sec^2\beta\frac{\partial\beta}{\partial X_3}-(\tan\beta)(2r)\left(\frac{X_3}{r}\right)\right]$$

$$=-\frac{X_1}{r^4}[r\sec\beta-2X_3\tan\beta]$$

$$=\frac{X_1}{r^3\cos\beta}[2\sin^2\beta-1]$$

$$=\frac{X_1}{r^3\cos\beta}(\sin^2\beta-\cos^2\beta) \tag{21a}$$

$$\frac{\partial^2\beta}{\partial X_2\partial X_3}=\frac{X_2}{r^3\cos\beta}(\sin^2\beta-\cos^2\beta) \tag{22a}$$

From Equation (4a):

$$\frac{\partial\alpha}{\partial X_i}=\frac{\partial(\lambda+\theta)}{\partial X_i}=\frac{\partial\lambda}{\partial X_i} \tag{23a}$$

From Equation (5a):

$$\sec^2\alpha\left(\frac{\partial\alpha}{\partial X_1}\right)=-\frac{X_2}{X_1^2} \tag{24a}$$

$$\sec^2\alpha\left(\frac{\partial\alpha}{\partial X_2}\right)=\frac{1}{X_1} \tag{25a}$$

$$\frac{\partial\alpha}{\partial X_3}\equiv 0 \tag{26a}$$

$$\frac{\partial\lambda}{\partial X_1}=-\cos^2\alpha\left(\frac{X_2}{X_1^2}\right) \tag{27a}$$

$$\frac{\partial\lambda}{\partial X_2}=\frac{\cos^2\alpha}{X_2} \tag{28a}$$

$$\frac{\partial\lambda}{\partial X_3}\equiv 0 \tag{29a}$$

$$\frac{\partial\lambda}{\partial X_i}=\cos^2\alpha\frac{F(i)}{X_i}\left(\frac{X_2}{X_1}\right)$$

$$=\cos^2\alpha\frac{F(i)}{X_i}\tan\alpha=\frac{F(i)}{X_i}(\sin\alpha\cos\alpha) \tag{30a}$$

From Equation (30a):

$$\frac{\partial^2\lambda}{\partial X_i\partial X_j}=\frac{\partial}{\partial X_i}\left[\frac{F(j)\sin\alpha\cos\alpha}{X_j}\right]$$

$$=\frac{F(j)}{X_j^2}\left\{X_j\left[\cos^2\alpha\left(\frac{\partial\alpha}{\partial X_i}\right)-\sin^2\alpha\left(\frac{\partial\alpha}{\partial X_i}\right)\right]-\delta_{i,j}\sin\alpha\cos\alpha\right\}$$

$$=\frac{F(j)}{X_j}(\cos^2\alpha-\sin^2\alpha)\left(\frac{F(i)\cos\alpha\sin\alpha}{X_i}\right)-\frac{F(j)\sin\alpha\cos\alpha}{X_j^2}\delta_{i,j}$$

$$=\frac{F(i)F(j)}{X_iX_j}(\cos\alpha\sin\alpha)\left(\frac{X_1^2-X_2^2}{X_1^2+X_2^2}\right)\left(\frac{X_1X_2}{X_1X_2}\right)-\frac{F(j)\sin\alpha\cos\alpha}{X_j^2}\delta_{i,j}$$

$$=\frac{F(i)F(j)}{X_iX_j}(\cos^2\alpha\sin^2\alpha)\left(\frac{X_1^2-X_2^2}{X_1X_2}\right)-\frac{F(j)\sin\alpha\cos\alpha}{X_j^2}\delta_{i,j} \tag{31a}$$

or:

$$\frac{\partial^2\lambda}{\partial X_1^2}=\frac{2\sin\alpha\cos\alpha}{X_1^2+X_2^2} \tag{32a}$$

$$\frac{\partial^2\lambda}{\partial X_2^2}=-\frac{2\sin\alpha\cos\alpha}{X_1^2+X_2^2} \tag{33a}$$

$$\frac{\partial^2\lambda}{\partial X_1\partial X_2}=\frac{\sin^2\alpha-\cos^2\alpha}{X_1^2+X_2^2} \tag{34a}$$

$$\frac{\partial^2\lambda}{\partial X_i\partial X_j}\equiv 0;\quad i=3\quad\text{or}\quad j=3 \tag{35a}$$

APPENDIX B

COMPUTATION OF THE LEGENDRE POLYNOMIALS

Use of a recursion formula can be one of the best means of reducing computation time; it is standard procedure to use such a formula in computing the Legendre polynomials. We have also used such a formula for computation of the values sin $(m\lambda)$ and cos $(m\lambda)$ which should also be standard procedure because sine and cosine routines are relatively inefficient.

A standard formula has been used to compute $P_n^m(x)$ for the case $m=0$, as given below in Equation (1b).

$$nP_n(x)+(n-1)P_{n-2}(x)-(2n-1)xP_{n-1}(x)=0. \tag{1b}$$

This relation is discussed by Hobson (1931) (p. 32) and Whittaker and Watson (1927) (p. 308). In both cases a proof is given, and it is noted that various proofs exist.

The formula used for the case $m\neq 0$ (the associated polynomials) is also standard, but the derivation follows readily from the definition of the associated polynomials, and we therefore include the derivation for illustrative purposes.
By definition,

$$P_n^m(x)=\frac{(1-x^2)^{m/2}}{2^n(n)!}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n, \tag{2b}$$

but

$$\begin{aligned}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n&=\frac{d^{n+m-1}}{dx^{n+m-1}}2nx(x^2-1)^{n-1}\\&=2n\left[\frac{d^{n+m-2}}{dx^{n+m-2}}(x^2-1)^{n-1}\right.\\&\qquad\left.+2(n-1)\frac{d^{n+m-2}}{dx^{n+m-2}}x^2(x^2-1)^{n-2}\right]\end{aligned} \tag{3b}$$

and

$$\begin{aligned}\frac{d^{n+m-2}}{dx^{n+m-2}}x^2(x^2-1)^{n-2}&=\frac{d^{n+m-2}}{dx^{n+m-2}}(x^2-1)^{n-2}\\&\quad+\frac{d^{n+m-2}}{dx^{n+m-2}}(x^2-1)^{n-1}\end{aligned} \tag{4b}$$

so that Equation (3b) becomes

$$\begin{aligned}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n&=2n(2n-1)\frac{d^{n+m-2}}{dx^{n+m-2}}(x^2-1)^{n-1}\\&\quad+2^2n(n-1)\frac{d^{n+m-2}}{dx^{n+m-2}}(x^2-1)^{n-2}\end{aligned} \tag{5b}$$

and Equation (2b) becomes

$$\begin{aligned}P_n^m(x)&=\frac{(2n-1)(1-x^2)^{m/2}}{2^{n-1}(n-1)!}\frac{d^{n+m-2}}{dx^{n+m-2}}(x^2-1)^{n-1}\\&\quad+\frac{(1-x^2)^{m/2}}{2^{n-2}(n-2)!}\frac{d^{n+m-2}}{dx^{n+m-2}}(x^2-1)^{n-2}\end{aligned}$$

or

$$P_n^m(x)=(2n-1)(1-x^2)^{1/2}P_{n-1}^{m-1}(x)+P_{n-2}^m(x). \tag{6b}$$

Programming Equations (1b) and (6b) is clearly quite simple, and this subroutine is included below. It should be noted that the angle beta is always in the range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ so that

$$(1-x^2)^{1/2}=|\cos\beta|=\cos\beta. \tag{7b}$$

The form we have used for expressing the derivative of the Legendre polynomials (Equation (5), text) is valid for all $m\geqslant 0$, and is derived as follows:

$$\begin{aligned}\frac{dP_n^m(x)}{dx}&=\frac{d}{dx}\left((1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\right)\\&=(1-x)^{m/2}\frac{d^{m+1}}{dx^{m+1}}P_n(x)\\&\quad-\frac{mx(1-x^2)^{m/2}}{(1-x^2)}\frac{d^m}{dx^m}P_n(x)\\&=\frac{P_n^{m+1}(x)}{(1-x^2)^{1/2}}-\frac{mxP_n^m(x)}{(1-x^2)}.\end{aligned} \tag{8b}$$

As noted previously, we feel that for the majority of cases normalization is not indicated. An example of the accuracy inherent in today's large-word computers is given by comparison runs of the routine presented in the following appendixes. These comparisons indicated agreement of results to at least nine significant figures, and it will be noted that in the program contained in Appendix D, virtually no concession was made to accuracy. Furthermore, in debuggiug these routines, we ran tests on a GE 400 time-sharing computer system and obtained results accurate to at least five significant figures, even though we entered as data the associated polynomials and the harmonic coefficients rounded to six or seven significant figures.

```
      SUBROUTINE SPNM (NMAX,S,C,P)
C
C
C                                      M
C     THIS SUBROUTINE COMPUTES P (S) FOR N UP TO NMAX.
C                                      N
C
C
C     THE ARGUMENT C IS THE SQUARE ROOT OF ONE MINUS S SQUARED
C
      DIMENSION P(NMAX+1,NMAX+3)
C
      NT2 = NT + 2
      DO 10 I = 1,NT
      DO 5 J = 1,NT2
      P(I,J) = 0.
    5 CONTINUE
   10 CONTINUE
C
      P(1,1) = 1.
      P(2,1) = S
      P(2,2) = C
      P(3,1) = 1.5 * S**2  -  .5
      P(3,2) = 3.0 * S * C
      P(3,3) = 3.0 * C**2
C
      IF ( NT .LT. 4 )  RETURN
C
      DO 20 N = 4,NT
      XN = FLOAT(N-1)
      N1 = N - 1
      N2 = N - 2
      TN = 2.*XN - 1.
      T = TN * C
C
      P(N,1) =   ( TN*S*P(N1,1) - (XN-1.)*P(N2,1) ) / XN
      P(N,N1) =  T*P(N1,N2)
      P(N,N) =   T*P(N1,N1)
C
      DO 15 M = 2,N2
      P(N,M) = P(N2,M) + T*P(N1,M-1)
   15 CONTINUE
C
   20 CONTINUE
C
      RETURN
      END
```

APPENDIX C

A FORTRAN SUBROUTINE FOR THE CHAIN-RULE METHOD

```
      SUBROUTINE  DVDXB
C
C     THIS SUBROUTINE USES THE CHAIN RULE TO COMPUTE FIRST AND SECOND
C     PARTIAL DERIVATIVES OF GRAVITATIONAL POTENTIAL.  FORMULAS USED ARE:
C
C
C
C                         NMAX                 N
C
C                        *****               *****
C                         *                   *
C                          *     N   N+1       *      M      M                 M
C     V = GM/R + GM         *   (A  /R     )    *    P   ( C  COS(M*L)  +  S  SIN(M*L) )
C                          *                   *      N      N                 N
C                        *****               *****
C
C                         N=1                 M=0
C
C
C       M          M+1               M
C     DP   /DB  = P      -   M*TAN(B)*P
C       N          N                 N
C
C
C     DR/DX   =  X   / R  FOR I=1,2,3
C          I      I
C
C
C                        2     2                                2     2
C     DL/DX   =  -X   / ( X  +  X   ) ;  DL/DX   =  X   / ( X  +  X   ) ; DL/DX  = 0
C          1       2      1     2             2      1      1     2             3
C
C
C                              2
C     DB/DX   =  -X  *TAN(B)/R    FOR I=1,2  ;  DB/DX   =  COS(B)/R
C          I      I                                  3
C
C
C     OTHER FORMULAS FOLLOW FROM APPLICATION OF THE CHAIN RULE AND/OR
C     REPEATED DIFFERENTIATION.
C
C
      DIMENSION
     *
     *  CC(NMAX,NMAX+1), CML(NMAX), D(3), DB(3), DDV(3,3), DL(3), DR(3),
     *
     *  DV(3), D2(3), D2B(3,3), D2L(3,3), D2R(3,3), P(NMAX+1,NMAX+3),
     *
```

```
     * SML(NMAX), TM(NMAX), TMS(NMAX), TM1(NMAX), XJ(NMAX), XM2(NMAX),
     *
     * X(3), DVDX(3), D2VDX2(3,3)
     *
      COMMON /COEFF/ XJ,CC,A,GM,NMAX
      COMMON /DERIV/ DVDX,D2VDX2
      COMMON /COORD/ X,THETA
C
C
      X2 = X(1)**2
      Y2 = X(2)**2
      Z2 = X(3)*X(3)
      XY2 = X2 + Y2
      XTY = X(1)*X(2)
      XY4 = XY2**2
      R2 = XY2 + X(3)**2
      R = SQRT(R2)
      R3 = R2 * R
      XY = SQRT(XY2)
      T = X(3) / XY
      S = X(3) / R
      C = XY / R
      S2 = S**2
      T2 = T**2
      C2 = C**2
      SC2 = S2 - C2
      SE = 1. / C2
      SINT = SIN(THETA)
      COST = COS(THETA)
      NM1 = NMAX + 1
      CALL SPNM (NM1,S,C,P)
C
      RK = A / R
      RN = ( RK / R ) * GM
      SML(1) = ( -X(1)*SINT + X(2)*COST ) / XY
      CML(1) = (  X(1)*COST + X(2)*SINT ) / XY
      TM(1) = T
      TM1(1) = 3.*T
      XM2(1) = 1.
      TMS(1) = T2 - SE
      TT = 2.*T
C
      DO 30 I = 1,3
      DV(I) = 0.
      DO 20 J = 1,3
      DDV(I,J) = 0.
   20 CONTINUE
   30 CONTINUE
```

```
C
      XN1  = 1.
      XN2  = 2.
      XN   = 0.
      X2N  = -1.
C
      DO 100 N = 1,NMAX
C
      NN = N + 1
      XN = XN + 1.
      XN1  = XN1 + 1.
      XN2  = XN2 + 1.
      X2N  = X2N + 2.
C
      IF ( N .EQ. 1 )  GO TO 35
      RN = RN * RK
      SML(N)  = SML(N-1)*CML(1)  + CML(N-1)*SML(1)
      CML(N)  = CML(N-1)*CML(1)  - SML(N-1)*SML(1)
      TM(N)  = TM(N-1)  + T
      TM1(N)  = TM1(N-1)  + TT
      XM2(N)  = XM2(N-1)  + X2N
      TMS(N)  = TMS(N-1)  + X2N*T2 - SE
C
   35 RN2 = ( XN1*RN )  / R
      RN3 = ( XN2*RN2)/ R
C
      D(1)  = -XJ(N)  * P(NN,2)
      D(2)  = 0.
      D(3)  = -XJ(N)  * P(NN,1)
      D2(1)  = XJ(N)  * ( T*P(NN,2) - P(NN,3) )
      D2(2)  = 0.
      D2(3)  = 0.
C
   40 DO 50 M = 1,N
      M1 = M + 1
      M2 = M + 2
      M3 = M + 3
      XM = M
C
      CSO  = CC(N,M)*CML(M)  + CC(M,NN)*SML(M)
      CSI  =-CC(N,M)*SML(M)  + CC(M,NN)*CML(M)
      CSI  = CSI * XM
      CPO  = CSO * P(NN,M1)
      PHAT  = P(NN,M2) - TM(M)*P(NN,M1)
C
      D(1)  = D(1)  + CSO*PHAT
      D(2)  = D(2)  + CSI*P(NN,M1)
      D(3)  = D(3)  + CPO
```

```
C
      D2(1) = D2(1) + CSO*(P(NN,M3)-TM1(M)*P(NN,M2)+TMS(M)*P(NN,M1))
      D2(2) = D2(2) + XM2(M)*CPO
      D2(3) = D2(3) + CSI*PHAT
C
   50 CONTINUE
C
      DV(1) = DV(1) + RN*D(1)
      DV(2) = DV(2) + RN*D(2)
      DV(3) = DV(3) - RN2*D(3)
C
      DDV(1,1) = DDV(1,1) + RN * D2(1)
      DDV(2,2) = DDV(2,2) - RN * D2(2)
      DDV(3,3) = DDV(3,3) + RN3* D(3)
      DDV(1,2) = DDV(1,2) + RN * D2(3)
      DDV(1,3) = DDV(1,3) - RN2* D(1)
      DDV(2,3) = DDV(2,3) - RN2* D(2)
C
  100 CONTINUE
C
      DB(1) = - ( X(1)*T ) / R2
      DB(2) = - ( X(2)*T ) / R2
      DB(3) = C / R
      D2B(3,3) = -2.*C*X(3)/R3
      TR4 = ( T / R3 ) / R
      D2B(1,1) =  ( 2.*X2 - Y2/C2 ) * TR4
      D2B(2,2) =  ( 2.*Y2 - X2/C2 ) * TR4
      D2B(1,2) =  ( ( XTY*T*(SE+2.) ) / R3 ) / R
      D2B(2,3) = ( X(2)*SC2 ) / ( R3*C )
      D2B(1,3) = ( X(1)*SC2 ) / ( R3*C )
C
      DR(1) = X(1) / R
      DR(2) = X(2) / R
      DR(3) = X(3) / R
      D2R(1,1) = (R2 - X2) / R3
      D2R(2,2) = (R2 - Y2) / R3
      D2R(3,3) = (R2 - Z2) / R3
      D2R(1,2) = - XTY / R3
      D2R(1,3) = - X(1)*X(3) / R3
      D2R(2,3) = - X(2)*X(3) / R3
C
      DL(1) = - X(2) / XY2
      DL(2) =   X(1) / XY2
      D2L(1,1) =  ( 2.*XTY ) / XY4
      D2L(2,2) =  -D2L(1,1)
      D2L(1,2) =  ( Y2 - X2 ) / XY4
C
      DV(3) = DV(3) - GM/R2
```

```
      DDV(3,3) = DDV(3,3) + 2.0*GM/R3
C
      DVDX(1) =  DV(1) * DB(1)  +  DV(2) * DL(1)  +  DV(3) * DR(1)
      DVDX(2) =  DV(1) * DB(2)  +  DV(2) * DL(2)  +  DV(3) * DR(2)
      DVDX(3) =  DV(1) * DB(3)                    +  DV(3) * DR(3)
C
      SBX = DDV(1,1)*DB(1) + DDV(1,2)*DL(1) + DDV(1,3)*DR(1)
      SLX = DDV(1,2)*DB(1) + DDV(2,2)*DL(1) + DDV(2,3)*DR(1)
      SRX = DDV(1,3)*DB(1) + DDV(2,3)*DL(1) + DDV(3,3)*DR(1)
      SBY = DDV(1,1)*DB(2) + DDV(1,2)*DL(2) + DDV(1,3)*DR(2)
      SLY = DDV(1,2)*DB(2) + DDV(2,2)*DL(2) + DDV(2,3)*DR(2)
      SRY = DDV(1,3)*DB(2) + DDV(2,3)*DL(2) + DDV(3,3)*DR(2)
      SBZ = DDV(1,1)*DB(3)                  + DDV(1,3)*DR(3)
      SLZ = DDV(1,2)*DB(3)                  + DDV(2,3)*DR(3)
      SRZ = DDV(1,3)*DB(3)                  + DDV(3,3)*DR(3)
C
      D2VDX2(1,1) = DV(1)*D2B(1,1) +  DB(1)*SBX  +
     *              DV(2)*D2L(1,1) +  DL(1)*SLX  +
     *              DV(3)*D2R(1,1) +  DR(1)*SRX
      D2VDX2(2,2) = DV(1)*D2B(2,2) +  DB(2)*SBY  +
     *              DV(2)*D2L(2,2) +  DL(2)*SLY  +
     *              DV(3)*D2R(2,2) +  DR(2)*SRY
      D2VDX2(1,2) = DV(1)*D2B(1,2) +  DB(1)*SBY  +
     *              DV(2)*D2L(1,2) +  DL(1)*SLY  +
     *              DV(3)*D2R(1,2) +  DR(1)*SRY
      D2VDX2(3,3) = DV(1)*D2B(3,3) +  DB(3)*SBZ  +
     *              DV(3)*D2R(3,3) +  DR(3)*SRZ
      D2VDX2(1,3) = DV(1)*D2B(1,3) +  DB(1)*SBZ  +
     *                                DL(1)*SLZ  +
     *              DV(3)*D2R(1,3) +  DR(1)*SRZ
      D2VDX2(2,3) = DV(1)*D2B(2,3) +  DB(2)*SBZ  +
     *                                DL(2)*SLZ  +
     *              DV(3)*D2R(2,3) +  DR(2)*SRZ
      D2VDX2(3,1) = D2VDX2(1,3)
      D2VDX2(3,2) = D2VDX2(2,3)
      D2VDX2(2,1) = D2VDX2(1,2)
      RETURN
      END
```

APPENDIX D

A FORTRAN SUBROUTINE FOR THE FUNCTION METHOD

```
      SUBROUTINE DVDXC
C
C     THIS SUBROUTINE COMPUTES FIRST AND SECOND PARTIAL DERIVATIVES CF
C     GRAVITATIONAL POTENTIAL ACCORDING TO THE FORMULAS:
C
C
C
C                                           NMAX       N
C
C                                          *****    *****
C                                          *        *
C                             -3           *     N   *      M
C     DV/DX   = -GM*X  *R       +  GM      *    A    *    DU  /DX
C          I         I                     *        *       N    I
C                                          *****    *****
C
C                                           N=1      M=0
C
C     AND:
C
C                                                       NMAX     N
C
C                                                      *****   *****
C                                                      *       *
C       2      2                 5           3         *    N  *    2 M
C     D V/DX  DX   = GM( 3X X /R   -  KD  /R   +       *   A   *   D U   /DX DX   )
C           I   J           I J         IJ             *       *       N     I  J
C                                                      *****   *****
C
C                                                       N=1     M=0
C
C     WHERE:
C
C       M                N+3          ^                                ^      N+1
C     DU  /DX   =   (X   /R    )*[ G(I)*PC  -  (N+1)*PC  ] + [ M*PS*F(I)  ]/R
C       N    I        I
C
C
C      2 M             N+3   [   ^           ^          ^
C     D U  /DX DX   = (M/R    ) [ -PS*(X *G(I)*F(J) + X *G(J)*F(I))
C        N    I  J            [       I                J
C
C
C                          ^          ^        ]   [     ^      ^      N+1] [     ^        ]
C         -(N+1)*PS*(X  *F(J)  +  X  *F(I))]  +  [  M*F(I)*F(J)/R      ] [PS*X  - M*PC]
C                     I            J           ]   [                     ] [             ]
```

```
C
C                                   [    M           ^        N+1   ]
C           + (KD  /X ) * [ DU /DX   - (2*M*PS*F(I)/R      ) ]
C                IJ   I       [   N    I                        ]
C
C
C                       N+5       [  *       2              -1
C           +   (X *X /R      ) * [ G  ( M*T *(M-2)*PC - (T   + 2*M*T)*P C + P C )
C                  I  J           [                                    1      2
C
C                                   +         ^                          ]
C                                  + G *(N+3)*PC  +   (N+1)*(N+3)*PC  ]
C                                                                        ]
C
C               M                M
C     C =    C  COS(M*LAMBDA)  + S  SIN(M*LAMBDA)
C               N                N
C
C               M                M
C     S =   -C  SIN(M*LAMBDA)  + S  COS(M*LAMBDA)
C               N                N
C
C             M                   M+1                     M+2
C     P = P  (SIN(BETA))   ;  P   = P   (SIN(BETA))   ;  P  = P   (SIN(BETA))
C             N                1     N                      N    N
C     ^
C     P =   P  - M*T*P    ;  T = TAN(BETA)
C            1
C     ^
C     F(I) =  SIN(ALPHA)*COS(ALPHA)*F(I)/X   =  DLAMBDA/DX
C                                         I              I
C      *             +                                   H(I)
C     G  =  G(I)*G(J)    ;  G  =  G(I)+G(J)    ;  G(I) =  H(I)*T
C
C     F(I) = -1; I=1 , 1; I=2 , 0; I=3
C     H(I) = 1; I=1,2  , -1; I=3
C     KD   IS THE KRONECKER DELTA
C       IJ
```

```
C
C
      DIMENSION
     *
     *  AKD(3), CC(NMAX,NMAX+1), CML(NMAX), DVDX(3), D2VDX2(3,3),
     *
     *  F(3), FI(3), F1(3,3), F2(3,3), F3(3,3), G(3), GKD(3),
     *
     *  GPLUS(3,3), GSTAR(3,3), P(NMAX+1,NMAX+3), S(14), SM(9),
     *
     *  SML(NMAX), X(3), XIJ(3,3), XI3(3), XJ(NMAX)
     *
      COMMON /COEFF/ XJ,CC,A,GM,NMAX
      COMMON /DERIV/ DVDX,D2VDX2
      COMMON /COORD/ X,THETA
      COMMON /VERCC/ F
C
      DO 20 I = 1,3
      DO 10 J = 1,3
      XIJ(I,J) = X(I)*X(J)
   10 CONTINUE
   20 CONTINUE
C
      XY2  = XIJ(1,1) + XIJ(2,2)
      R2   = XY2 + XIJ(3,3)
      XY1  = SQRT(XY2)
      R    = SQRT(R2)
      SB   = X(3)/R
      CB   = XY1/R
      TB   = X(3)/XY1
      TB2  = TB*TB
      R3   = R2*R
      R5   = R3*R2
      G(1)= TB
      G(2)= TB
      G(3)= -1./TB
      SINT= SIN(THETA)
      COST= COS(THETA)
      SAC = XIJ(1,2)/XY2
      SAC2= (SAC*SAC)/R
      SAC = SAC/R
      SML(1) = ( -X(1)*SINT + X(2)*COST ) / XY1
      CML(1) = (  X(1)*COST + X(2)*SINT ) / XY1
      XH =  ( XIJ(1,1) - XIJ(2,2) ) / XIJ(1,2)
C
      DO 30 I = 1,3
      FI(I) = F(I)/X(I)
   30 CONTINUE
```

```
C
      NM1 = NMAX + 1
      CALL SPNM (NM1,SB,CB,P)
C
      DO 50 I = 1,3
      XI3(I) = X(I)/R3
      AKD(I) = SAC*FI(I)
      DO 40 J = 1,3
      F1(I,J) = SAC * ( X(J)*FI(I) + X(I)*FI(J) ) / R2
      F2(I,J) = SAC2*F(I)*F(J)/XIJ(I,J)
      F3(I,J) = SAC * ( X(J)*G(J)*FI(I) + X(I)*G(I)*FI(J) ) / R2
      GSTAR(I,J) = G(I)*G(J)
      GPLUS(I,J) = G(I) + G(J)
      XIJ(I,J) = XIJ(I,J)/R5
   40 CONTINUE
   50 CONTINUE
      AORN = 1.
      AOR  = A/R
      DO 60 I = 1,14
      S(I) = 0.
   60 CONTINUE
C
      DO 200 N = 1,NMAX
      N1 = N + 1
      XN = N
      AORN = AORN*AOR
C
      IF ( N .EQ. 1 )  GO TO 65
      SML(N) = SML(N-1)*CML(1) + CML(N-1)*SML(1)
      CML(N) = CML(N-1)*CML(1) - SML(N-1)*SML(1)
   65 CONTINUE
      DO 80 I = 4,9
      SM(I) = 0.
   80 CONTINUE
C
      SM(1) = P(N1,1)*XJ(N)
      SM(2) = P(N1,2)*XJ(N)
      SM(3) = P(N1,3)*XJ(N)
C
      DO 100 M = 1,N
      XM = M
      M1 = M + 1
      M2 = M + 2
      M3 = M + 3
C
      CSO =  CC(N,M)*CML(M) + CC(M,N1)*SML(M)
      CSI = -CC(N,M)*SML(M) + CC(M,N1)*CML(M)
      CSI = CSI*XM
```

```
      COM = CSO*XM
C
      SM(1) = SM(1) + P(N1,M1)*CSO
      SM(2) = SM(2) + P(N1,M2)*CSO
      SM(3) = SM(3) + P(N1,M3)*CSO
      CPOM  = P(N1,M1)*COM
      SM(4) = SM(4) + CPOM
      SM(5) = SM(5) + XM*CPOM
      SM(6) = SM(6) + P(N1,M2)*COM
      CPIM  = P(N1,M1)*CSI
      SM(7) = SM(7) + CPIM
      SM(8) = SM(8) + XM*CPIM
      SM(9) = SM(9) + P(N1,M2)*CSI
C
  100 CONTINUE
C
      AORNN = XN*AORN
C
      DO 120 I = 1,9
      S(I) = S(I) + SM(I)*AORN
  120 CONTINUE
      S(10) = S(10) + SM(1)*AORNN
      S(11) = S(11) + XN*SM(1)*AORNN
      S(12) = S(12) + SM(2)*AORNN
      S(13) = S(13) + SM(4)*AORNN
      S(14) = S(14) + SM(7)*AORNN
C
  200 CONTINUE
C
      PHAT = S(2) - TB*S(4)
      SKD = S(10) + S(1)
C
      S1 = S(14) + S(7)
      S2 = XM*S(7) - S(5)
      S3 = TB2*(S(5) - 2.*S(4)) - 2.*TB*S(6) + S(3) - S(2)/TB
      S4 = S(12) - TB*S(13) + 3.*PHAT
      S5 = S(11) + 4.*S(10) + 3.*S(1) + 3.
      S6 = TB*S(8) - S(9)
C
      DO 220 I = 1,3
      AKD(I) = AKD(I)*S(7)
      GKD(I) = -G(I)*PHAT - SKD - 1.
      DVOX(I) = GM*( XI3(I)*GKD(I) + AKD(I) )
C
      DO 210 J = 1,3
      D2VDX2(I,J) = ( F3(I,J)*S6 - F1(I,J)*S1 + F2(I,J)*S2 +
     *
     *                XIJ(I,J)*(GSTAR(I,J)*S3 + GPLUS(I,J)*S4 + S5) )*GM
```

```
C
  210 CONTINUE
      D2VDX2(I,I) = D2VDX2(I,I) + (GKD(I)/R3 - AKD(I)/X(I) ) * GM
  220 CONTINUE
C
      RETURN
      END
```

APPENDIX E

A FORTRAN SUBROUTINE FOR THE METHOD OF COEFFICIENT MODIFICATION

```
      SUBROUTINE  DVDXA
C
C     THIS SUBROUTINE COMPUTES FIRST AND SECOND PARTIAL DERIVATIVES OF
C     GRAVITATIONAL POTENTIAL ACCORDING TO FORMULAS DISCUSSED BY MARTIN
C     HOTINE IN THE BOOK  -MATHEMATICAL GEODESY-  ( U. S. DEPT OF
C     COMMERCE PUB. 528:51(021) )  PAGES 153 TO 186.
C
C     IN PARTICULAR THE FORMULAS ARE:
C
C
C                          NMAX          N+1
C
C                          *****         *****
C                    -2    *          N  *      M          M                M
C DV/DX   = GM*R            *    (A/R)      *    P    *( CI   COS(M*L) + SI   SIN(M*L) )
C      I                   *             *       N+1       N                N
C                          *****         *****
C
C                          N=0           M=0
C
C     AND:
C
C                              NMAX          N+2
C
C                              *****         *****
C  2      2               -3   *          N  *      M          M                 M
C D V/DX     = GM*R             *    (A/R)      *    P    *( CIJ  COS(M*L) + SIJ  SIN(M*L) )
C        I,J                   *             *       N+2       N                 N
C                              *****         *****
C
C                              N=0           M=0
C
C
C
C     THE LEGENDRE POLYNOMIALS ARE COMPUTED BY SUBROUTINE SPNM DESCRIBED
C     ELSEWHERE.
C     THE RELATION BETWEEN VARIABLES AND CONSTANTS LISTED IN THE FORMULAS
C     ABOVE AND THE ARRAYS USED IN THE SUBROUTINE ARE:
C
C      M
C     P ( SIN(BETA) ) = P(N+1,M+1)
C      N
C
C
C       L             L
C     CI   = CI(K,L,I) ;  SI   = CI(NMAX+1-K,NMAX+4-L,I) FOR: K=1,2,...NMAX
C       K              K                                     L=1,2,...NMAX+1
C                                                            I=1,2,3
```

```
C
C            0                                         0
C     CI     = XJ(K,I)   FOR K=1,2,...NMAX    WITH CI   BUILT IN THE ROUTINE
C       K                                              0
C
C
C          L              L
C     CIJ    = CIJ(K,L,S(I,J)) ; SIJ    = CIJ(NMAX+1-K,NMAX+6-L,S(I,J))
C         K                  K
C
C     FOR: K=1,2,...NMAX        WHERE: S(I,J) = 1; I=1,J=1     4; I=1,J=2
C          L=1,2,...NMAX+2                      2; I=2,J=2     5; I=2,J=3
C          I=1,2,3                              3; I=3,J=3     6; I=1,J=3
C          J=1,2,3                              WITH S(J,I) = S(I,J)
C
C
C          0                                            0
C     CIJ    = XIJ(K,S(I,J))   FOR K=1,2,...NMAX    WITH CIJ   BUILT IN
C         K                                             0
C
C
C             K
C     (A/R)     = AORN(K)
C
C     COS(K*LAMBDA) = CML(K)   ;   SIN(K*LAMBDA) = SML(K)   FOR K=1,2,...NMAX+2
C
C
      DIMENSION
     *
     * XJ(NMAX,3), XIJ(NMAX,6), CI(NMAX,NMAX+3,3), CIJ(NMAX,NMAX+5,6),
     *
     * P(NMAX+3,NMAX+3), CML(NMAX+2), SML(NMAX+2), X(3), DVDX(3),
     *
     * D2VDX2(3,3), AORN(NMAX+2), DI(3), DIJ(6)
     *
C
      COMMON /COEFF/ XJ,XIJ,CI,CIJ,A,GM,NMAX
      COMMON /DERIV/ DVDX,D2VDX2
      COMMON /COORD/ X,THETA
C
      XY2  = X(1)**2 + X(2)**2
      R2   = XY2 + X(3)**2
      R    = SQRT(R2)
      R3 = R*R2
      XY   = SQRT(XY2)
      SINB= X(3)/R
      COSB=  XY /R
```

```
      SINT= SIN(THETA)
      COST= CCS(THETA)
      NM1 = NMAX + 1
      NM2 = NMAX + 2
      NM3 = NMAX + 3
      NM4 = NMAX + 4
      NM6 = NMAX + 6
      CALL SPNM (NM3,SINB,COSB,P)
C
      AORN(1) = A/R
      GMOR2 = GM / R2
      GMOR3 = GM / R3
      SML(1) = ( -X(1)*SINT + X(2)*COST ) / XY
      CML(1) = (  X(1)*COST + X(2)*SINT ) / XY
      DO 10 I = 2,NM2
      AORN(I) = AORN(I-1)*AORN(1)
      SML(I)  = SML(I-1)*CML(1) + CML(I-1)*SML(1)
      CML(I)  = CML(I-1)*CML(1) + SML(I-1)*SML(1)
   10 CONTINUE
C
      DVDX(1) = - COSB*CML(1)
      DVDX(2) = - COSB*SML(1)
      DVDX(3) = - SINB
      D2VDX2(1,1) = 0.5*P(3,3)*CML(2) - P(3,1)
      D2VDX2(2,2) =-0.5*P(3,3)*CML(2) - P(3,1)
      D2VDX2(3,3) = 2.0*P(3,1)
      D2VDX2(1,2) = 0.5*P(3,3)*SML(2)
      D2VDX2(2,3) =     P(3,2)*SML(1)
      D2VDX2(1,3) =     P(3,2)*CML(1)
      DO 15 I = 3,NM3
      DO 12 J = 2,NM3
      P(I,J) = P(I,J) * CML(J-1)
      PS(I,J) = P(I,J) * SML(J-1)
   12 CONTINUE
   15 CONTINUE
C
      DO 100 N = 1,NMAX
C
      N1 = N + 1
      N2 = N + 2
      N3 = N + 3
      DO 20 I = 1,3
      DI(I) = P(N2,1) * XJ(N,I)
   20 CONTINUE
C
      DO 30 I = 1,6
      DIJ(I) = P(N3,1) * XIJ(N,I)
   30 CONTINUE
```


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(Continued from inside front cover.)

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