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U.S. DEPARTMENT OF COMMERCE  
National Oceanic and Atmospheric Administration  
National Ocean Survey

## Errors of Quadrature Connected With the Simple Layer Model of the Geopotential

KARL-RUDOLF KOCH

ROCKVILLE, MD.

December 1971

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ERRORS OF QUADRATURE  
CONNECTED WITH THE SIMPLE LAYER MODEL OF THE GEOPOTENTIAL

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ABSTRACT. When using the simple layer model of the geopotential in satellite geodesy, one has to integrate over surface elements of the earth on which the density of the layer is assumed constant. The integration is solved numerically by subdividing the elements and by assuming constant kernels of the integrals for the subdivisions. This quadrature causes errors that are investigated for a sphere with the mean earth radius. By assuming unit density for the surface layer, exact values of the gravitational potential of the sphere and its gradient can be computed and compared with the values obtained by quadrature. Different sizes of surface elements and different methods of subdivisions are investigated. The error of quadrature, of course, increases with size of the surface elements and decreases with the number of subdivisions. An efficient and accurate way of subdividing is the method presently applied in the analysis of satellite data.

1. INTRODUCTION

If the simple layer model of the geopotential is applied in satellite geodesy, the earth's gravitational potential is divided into a known part, represented by an expansion in spherical harmonics of finite degree, and into an unknown part,  $T$ , to be determined by satellite observations. The potential  $T$  is represented by the potential of a simple layer distributed over the surface of the earth. The unknown density of the layer is a

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function of the position at the earth's surface. For simple numerical evaluation, constant density is assumed for the surface elements into which the surface of the earth is divided. Hence, we obtain

$$T = \sum_{i=1}^k \chi_i \iint_{\Delta E_i} \frac{dE}{\ell} \quad (1)$$

(Koch and Morrison 1970, Morrison 1971b) where  $k$  is the number of surface elements  $\Delta E_i$ ;  $\chi_i$ , the density of  $\Delta E_i$ ; and  $\ell$ , the distance between the fixed point and the moving point.

The integral over the surface element  $\Delta E_i$  is solved numerically, so errors of quadrature arise. These errors are investigated here; Morrison (1971a) investigated their influence on satellite orbits.

## 2. DIVISION INTO SURFACE ELEMENTS

To define the surface of the earth, one may use an ellipsoid of revolution whose shape approximates that of the earth. The surface elements for the ellipsoid are formed by means of the meridians  $L = \text{const}$  and the parallels  $B = \text{const}$ ,  $B$  and  $L$  being the geographic latitude and longitude. To obtain surface elements of nearly equal size, one applies the following method. If  $s^\circ$  (e.g.,  $s^\circ = 20^\circ$ ) is the chosen side length in latitude for the surface elements, the surface of the ellipsoid is divided into strips bordered by the parallels  $B = 90^\circ$  and  $B = 90^\circ - s^\circ$ ,  $B = 90^\circ - s^\circ$  and  $B = 90^\circ - 2s^\circ$ , and so on, provided  $180^\circ/s^\circ$  is an integer. The area of the strip, which includes the Equator or is bordered by the Equator, is computed and divided by  $360^\circ/s^\circ$ . Thus, the area of a block of  $5^\circ \times 5^\circ$  at the Equator is obtained. The areas of the strips are then divided by the area of the block at the Equator; the result is rounded to the nearest integer denoted by  $j_B$ . Each strip is then divided by the meridians  $L = 0^\circ$ ,  $L = 360^\circ/j_B$ ,  $L = 2(360^\circ/j_B)$ , and so on, to obtain the surface elements  $\Delta E_i$  at the surface of the ellipsoid. The

elements  $\Delta E_i'$  are triangular shaped at the poles and rectangular elsewhere.

The area of a surface element  $\Delta E_i'$  on the ellipsoid is computed by

$$\begin{aligned} \Delta E_i' &= \int_{B_i}^{B_{i+1}} \int_{L_i}^{L_{i+1}} MN \cos B \, dB \, dL \\ &= (L_{i+1} - L_i) b^2 \int_{B_i}^{B_{i+1}} \frac{\cos B \, dB}{(1-e^2 \sin^2 B)^2} \end{aligned}$$

where M and N are the radii of curvature of the meridian and the parallel;  $B_i$ ,  $B_{i+1}$ ,  $L_i$ , and  $L_{i+1}$ , the latitudes and longitudes of the two parallels and meridians that border  $\Delta E_i'$ ; and where b is the semiminor axis of the ellipsoid, and e is its eccentricity. The integral over B can be solved by substituting  $e \sin B = \sin \psi$ . We obtain

$$\Delta E_i' = (L_{i+1} - L_i) \frac{b^2}{2} \left[ \frac{\sin B}{1-e^2 \sin^2 B} + \frac{1}{2e} \ln \frac{1+e \sin B}{1-e \sin B} \right]_{B_i}^{B_{i+1}} \quad (2)$$

The surface of the reference ellipsoid does not coincide with the surface of the earth; thus area  $\Delta E_i'$  must be corrected. If H denotes the height of the earth's surface (consisting of the geoid undulation plus the topographic height) above the reference ellipsoid, the area of the earth's surface element  $\Delta E_i$  is computed from  $\Delta E_i'$  by

$$\begin{aligned} \Delta E_i &= \int_{B_i}^{B_{i+1}} \int_{L_i}^{L_{i+1}} (M+H)(N+H) \cos B \, dB \, dL \\ &\approx \left( 1 + \frac{2H}{R} + \frac{H^2}{R^2} \right) \Delta E_i' \end{aligned} \quad (3)$$

where H is assumed constant over  $\Delta E_i'$  and R denotes the mean radius of the earth.

If a spherical surface is used instead of an ellipsoidal reference surface, we obtain

$$\Delta E_i' = (L_{i+1} - L_i) R^2 [\sin B]_{B_i}^{B_{i+1}} \quad (4)$$

instead of eq (2).

### 3. SUBDIVISION OF THE SURFACE ELEMENTS

The integral over the surface element  $\Delta E_i$  in eq (1) is solved numerically by subdividing  $\Delta E_i$  into  $n^2$  elements  $\Delta E_{im}$  ( $m = 1, 2, \dots, n^2$ ) and replacing the distance  $\ell$  by  $\ell_{im}$  computed between the fixed point and the midpoint  $M_{im}$  of  $\Delta E_{im}$ . We thus obtain

$$\iint_{\Delta E_i} \frac{dE}{\ell} \approx \sum_{m=1}^{n^2} \frac{\Delta E_{im}}{\ell_{im}} \quad (5)$$

This quadrature causes errors dependent upon the size of the surface elements  $\Delta E_i$ , the number  $n^2$  of subdivisions  $\Delta E_{im}$ , and the definition of the midpoint  $M_{im}$  of  $\Delta E_{im}$ . To decrease the influence of the errors of quadrature in the analysis of satellite data, one sets the preliminary density values equal to zero so that the errors of quadrature enter only the variational equations for the parameter-sensitivity matrix and not the trajectory equations (Koch and Witte 1971).

In the application of the surface-layer model of the geopotential by Koch and Morrison (1970) and Koch and Witte (1971), the following method of subdividing the surface elements and defining the midpoints of the subdivisions was used. This method, which also forms the subdivisions  $\Delta E_{im}$  by parallels and meridians, shall be called method A. If  $\Delta B_i$  and  $\Delta L_i$  are the differences in latitude and longitude between the parallels and meridians that border the surface element  $\Delta E_i$ , the differences  $\Delta B_{im}$  and  $\Delta L_{im}$  in latitude and longitude between the parallels and meridians that

border the subdivisions  $\Delta E_{im}$  are found simply by

$$\Delta B_{im} = \Delta B_i/n$$

and

$$\Delta L_{im} = \Delta L_i/n .$$

(6)

The differences  $\Delta B_M$  and  $\Delta L_M$  between the latitude and longitude of the midpoint  $M_{im}$  of  $\Delta E_{im}$  and the borders are given by

$$\Delta B_M = \Delta B_{im}/2$$

and

$$\Delta L_M = \Delta L_{im}/2 .$$

(7)

Although the division of the surface of the earth into the elements  $\Delta E_i$  provides approximately equal areas for  $\Delta E_i$ , the subdivision according to method A does not lead to equal areas for the elements  $\Delta E_{im}$ . Given an element  $\Delta E_i$  subdivided into  $\Delta E_{im}$ , the elements  $\Delta E_{im}$  closer to the poles are smaller than the ones farther away. The differences between the areas of  $\Delta E_{im}$  reach a maximum for the triangular elements  $\Delta E_i$  containing one of the poles. Also, the meridian and the parallel through the midpoint  $M_{im}$  of  $\Delta E_{im}$  do not divide  $\Delta E_{im}$  into equal areas. Thus, the areas closer to the pole are smaller than the ones farther away.

Since a uniform division of the surface of the earth promises to give the best results for the quadrature, a subdivision called method B is tried; this method uses equal areas for the subdivisions  $\Delta E_{im}$  and places the midpoint  $M_{im}$  of  $\Delta E_{im}$  in such a way that the meridian and parallel through  $M_{im}$  divides  $\Delta E_{im}$  into four equal areas. The difference  $\Delta L_{im}$  in longitude between two meridians bordering  $\Delta E_{im}$  and the difference  $\Delta L_M$  in longitude between the midpoint  $M_{im}$  of  $\Delta E_{im}$  and the meridians bordering

$\Delta E_{im}$  again are found as in eq (6) and (7) by

$$\Delta L_{im} = \Delta L_i / n$$

and

$$\Delta L_M = \Delta L_{im} / 2 .$$
(8)

The difference  $\Delta B_{im}$  in latitude between two parallels bordering  $\Delta E_{im}$ , however, is determined in such a way that

$$\Delta E_{im} = \frac{\Delta E_i}{n^2} .$$
(9)

Likewise, the difference  $\Delta B_M$  in latitude between the midpoint  $M_{im}$  of  $\Delta E_{im}$  and the parallels bordering  $\Delta E_{im}$  is found by the parallel through  $M_{im}$ , dividing  $\Delta E_{im}$  into equal areas. If  $B_i$  and  $B_{i+1}$  now denote latitude of the parallels bordering  $\Delta E_{im}$ , then (from eq 4, Rapp 1971) for a spherical surface we obtain

$$\Delta E_{im} = \frac{\Delta L_i R^2}{n} (\sin B_{i+1} - \sin B_i)$$
(10)

where

$$\sin B_{i+1} = \sin B_i + \frac{\Delta E_i}{n \Delta L_i R^2}$$

from which we find

$$\Delta B_{im} = B_{i+1} - B_i$$
(11)

and, correspondingly,  $\Delta B_M$ . For an ellipsoidal surface, eq (2) must be applied. However, a closed formula like (10) cannot be derived for an ellipsoidal surface;  $\sin B_{i+1}$  must be computed by successive approximation.

#### 4. SPHERICAL MODEL

For obtaining a simple computation of the errors of quadrature, a spherical surface of the earth with  $R = 6368$  km is assumed. The sphere is covered by a simple layer of unit

density so that the potential  $T$  is obtained from eq (1) and (5) by

$$T = \sum_{i=1}^k \sum_{m=1}^{n^2} \frac{\Delta E_{im}}{\ell_{im}} \quad (12)$$

with

$$\ell_{im} = [(x_M - x_F)^2 + (y_M - y_F)^2 + (z_M - z_F)^2]^{1/2} \quad (13)$$

where  $x$ ,  $y$ , and  $z$  are the coordinates of an earth-centered coordinate system in which the  $z$  axis points toward the North Pole and the  $x$  axis toward the intersection of the Greenwich Meridian with the Equator. The index  $F$  denotes the fixed point, and  $M$  denotes the midpoint of  $\Delta E_{im}$ .

The mass of a sphere covered with a simple layer of constant density may be concentrated at the center of the sphere so that, in case of unit density, the potential  $T$  can be computed from

$$T = \frac{4\pi R^2}{(x_F^2 + y_F^2 + z_F^2)^{1/2}} \quad (14)$$

Methods A and B are now applied to form the subdivisions  $\Delta E_{im}$ ; thus, potential  $T$  and its gradient can be obtained from eq (12). By comparing the results with eq (14), the errors of the quadrature are obtained.

## 5. RESULTS

The way of forming the surface elements  $\Delta E_i$  provides symmetry with respect to the Equator and to the meridian  $L = 0^\circ$  and  $L = 180^\circ$ . Hence, only fixed points with  $90^\circ > B_F > 0^\circ$  and  $180^\circ > L_F > 0^\circ$  have to be considered. The coordinates  $B_F$  and  $L_F$  of 26 fixed points are selected at random with heights of  $H_F = 800$  km and  $H_F = 1000$  km above the surface of the sphere with the mean earth radius  $R = 6368$  km. For these fixed points, one computes the derivatives  $\partial T / \partial z$  from eq (12) and compares them with eq (14) to compute the relative errors given in units of

$10^{-3}$  in tables 1 through 9. The z component of grad T has been selected since it is defined above the North Pole or South Pole where the maximum errors of quadrature are found. The results are computed with  $s^\circ = 20^\circ$  and  $s^\circ = 15^\circ$  for  $\Delta E_i$ . A side length of  $20^\circ$  for  $\Delta E_i$  has been used by Koch and Witte (1971); a side length of  $15^\circ$  will be applied in a forthcoming analysis of satellite data.

Tables 1 to 3 show the relative errors of  $\partial T/\partial z$  with  $H_F = 1000$  km and  $s^\circ = 20^\circ$  for  $\Delta E_i$  subdivided by method A with  $n^2 = 1$  in table 1,  $n^2 = 4$  in table 2, and  $n^2 = 9$  in table 3. The values demonstrate the increase of accuracy with the increase of the number of subdivisions  $\Delta E_{im}$ . Although there is considerable gain with the change from one to four subdivisions, the gain is less from four to nine subdivisions. Tables 4 and 5 confirm this fact. Hence, good accuracy with little computational effort is obtained with four subdivisions  $\Delta E_{im}$ .

Tables 4 and 5 show the relative errors of quadrature with  $H_F = 1000$  km and  $s^\circ = 20^\circ$  for  $\Delta E_i$  subdivided by method B with  $n^2 = 4$  and  $n^2 = 9$ . When comparing these results with tables 2 and 3, one cannot readily judge whether the subdivision by method A or B gives better results. The same is true for the comparison between tables 6 and 8 and between 7 and 9. However, the relative error of method A for the fixed points above the pole is almost half the error of method B; therefore, method A is preferred. These smaller errors result because subdivisions  $\Delta E_{im}$  at the pole are smaller with method A than with B.

In tables 6 through 9, relative errors are given with  $s^\circ = 15^\circ$  for  $\Delta E_i$  subdivided by methods A and B. Comparisons of tables 2 and 6 and 4 and 8 show the increase of accuracy gained by use of smaller surface elements  $\Delta E_i$ . Tables 6 and 7 and 8 and 9 give the relative errors when heights of 1000 and 800 km are used for the fixed point  $H_F$  above the surface of the sphere. Although the height is only reduced by one-fifth when 800 km is used instead of 1000, the errors more than double at some fixed

points.

Since the error for a fixed point above the pole considerably exceeds that of any other fixed point, especially when method B is used, method A was applied to subdivide  $\Delta E_i$  with  $s^\circ = 15^\circ$  into nine subdivisions at the poles and into four subdivisions elsewhere. The results are given in table 10. While the accuracy increased at the pole, the results did not change for fixed points toward the Equator (cf. table 8). Therefore, the subdivisions of the surface element below the fixed point contribute mainly to the error of quadrature. This suggests using a scheme in which the subdivisions are varied for different positions of the fixed point (e.g.,  $n^2 = 9$  for the surface element below the fixed point,  $n^2 = 4$  for the surrounding elements, and  $n^2 = 1$  for the remainder).

Method A of subdividing the elements  $\Delta E_i$  gives negative errors of quadrature for fixed points above and close to the pole while method B leads to positive errors in that region. Therefore, an attempt was made to combine both methods by subdividing the elements  $\Delta E_i$  according to method A but defining the midpoints as in B. The results given in table 11 indicate that the combination method is not successful. The relative errors for fixed points close to the pole considerably exceed those of tables 6 and 8.

## 6. CONCLUSIONS

The relative errors of quadrature for the z component of grad T presented in tables 1 through 11 are obtained after summing the contribution of each surface element  $\Delta E_i$ . One may assume from the results of table 10 that the quadrature error for the surface element closest to the fixed point is the main contributor to the error given in the tables. Thus, the errors from the individual elements  $\Delta E_i$  for the three components of grad T will not surpass the maximum error of the tables found for the fixed points above the pole where the z component only is defined. The contribution of each individual element  $\Delta E_i$  to the three

components of grad T, however, enters the variational equations (Witte 1971). The relative errors of these contributions will generally be less than 1 percent and always less than 4 percent for satellites with perigees between 800 and 1000 km and for surface elements  $\Delta E_i$  (with  $s^\circ = 15^\circ$ ) divided into four subdivisions by method A. The corresponding values for elements of  $20^\circ$  sidelength are 6 percent and 3 percent. This accuracy is sufficient for the variational equations.

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Table 1. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method A,  $s^\circ = 20^\circ$ ,  $n^2 = 1$ , and  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	-70.9				
$80^\circ$	34.4	-87.5	-20.3	20.8	-280.0
$70^\circ$	107.0	72.6	69.5	91.0	28.6
$50^\circ$	100.0	85.8	87.6	95.6	94.2
$40^\circ$	54.4	1.9	33.2	49.8	56.6
$10^\circ$	102.0	28.7	203.0	-108.0	282.0

Table 2. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method A,  $s^\circ = 20^\circ$ ,  $n^2 = 4$ ,  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	-44.9				
$80^\circ$	10.4	-11.9	-6.1	5.0	10.4
$70^\circ$	24.2	-15.3	-3.2	14.8	24.2
$50^\circ$	11.3	2.5	10.2	4.3	11.3
$40^\circ$	11.6	4.2	7.5	10.6	11.6
$10^\circ$	10.1	13.0	21.6	17.0	10.1

Table 3. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method A,  $s^\circ = 20^\circ$ ,  $n^2 = 9$ ,  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	-18.5				
$80^\circ$	-2.2	-3.4	-4.6	-3.4	-4.6
$70^\circ$	6.8	0.5	-6.2	-0.5	-6.2
$50^\circ$	1.7	0.9	0.3	1.0	0.4
$40^\circ$	0.7	-0.2	0.0	0.5	0.7
$10^\circ$	1.3	0.4	1.4	0.7	1.9

Table 4. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method B,  $s^\circ = 20^\circ$ ,  $n^2 = 4$ ,  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	75.5				
$80^\circ$	14.5	-11.5	-4.8	8.2	14.5
$70^\circ$	34.8	-19.7	-2.3	23.4	34.8
$50^\circ$	5.9	-5.2	5.0	-3.7	5.9
$40^\circ$	13.3	6.7	9.6	12.5	13.3
$10^\circ$	1.2	4.1	12.7	8.1	1.2

Table 5. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method B,  $s^\circ = 20^\circ$ ,  $n^2 = 9$ ,  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	43.4				
$80^\circ$	2.9	1.8	0.6	1.8	0.6
$70^\circ$	8.8	1.6	-6.2	1.6	-6.2
$50^\circ$	-1.6	-2.4	-3.1	-2.3	-3.0
$40^\circ$	0.0	-1.0	-0.7	-0.3	0.5
$10^\circ$	-3.0	-3.9	-2.8	-3.6	-2.4

Table 6. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method A,  $s^\circ = 15^\circ$ ,  $n^2 = 4$ ,  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	-24.4				
$80^\circ$	3.8	-13.5	-9.0	-0.4	3.8
$70^\circ$	2.6	-2.6	-0.3	0.3	2.6
$50^\circ$	0.9	-1.9	-1.2	0.2	0.9
$40^\circ$	2.2	1.8	0.6	1.2	2.2
$10^\circ$	5.8	6.1	4.8	4.6	5.8

Table 7. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method A,  $s^\circ = 15^\circ$ ,  $n^2 = 4$ ,  $H_F = 800$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	-37.9				
$80^\circ$	15.0	-30.1	-17.8	4.7	15.0
$70^\circ$	4.9	-4.8	1.4	-1.1	4.9
$50^\circ$	1.7	-9.3	-6.6	-1.0	1.7
$40^\circ$	6.1	4.7	0.1	2.4	6.1
$10^\circ$	18.6	19.3	15.9	15.6	18.6

Table 8. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method B,  $s^\circ = 15^\circ$ ,  $n^2 = 4$ ,  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	39.9				
$80^\circ$	17.2	2.0	5.9	13.5	17.2
$70^\circ$	-1.6	-9.6	-6.3	-4.6	-1.6
$50^\circ$	2.0	-0.6	0.0	1.4	2.0
$40^\circ$	-1.5	-1.9	-3.3	-2.6	-1.5
$10^\circ$	3.2	3.5	2.1	2.0	3.2

Table 9. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method B,  $s^\circ = 15^\circ$ ,  $n^2 = 4$ ,  $H_F = 800$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	62.9				
$80^\circ$	28.1	-8.7	1.1	19.5	28.1
$70^\circ$	-5.1	-20.1	-11.3	-13.4	-5.1
$50^\circ$	6.0	-4.0	-1.5	3.5	6.0
$40^\circ$	0.7	-0.9	-5.9	-3.3	0.7
$10^\circ$	15.8	16.6	13.1	12.8	15.8

Table 10. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for method B,  $s^\circ = 15^\circ$ ,  $n = 4$ , for polar triangles  
 $n = 9$ ,  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	25.9				
$80^\circ$	13.2	12.4	11.6	12.4	11.6
$70^\circ$	-4.4	-6.5	-5.5	-6.5	-5.5
$50^\circ$	2.1	-0.6	0.0	1.4	2.1
$40^\circ$	-1.5	-1.9	-3.3	-2.6	-1.5
$10^\circ$	3.2	3.5	2.1	2.0	3.2

Table 11. -- Relative errors of quadrature (in units of  $10^{-3}$ ) for the combination of methods A and B,  $s^\circ = 15^\circ$ ,  $n = 4$ , and  $H_F = 1000$  km

$B_F$	$L_F$				
	$0^\circ$	$30^\circ$	$100^\circ$	$130^\circ$	$180^\circ$
$90^\circ$	108.0				
$80^\circ$	44.8	26.2	31.0	40.4	44.8
$70^\circ$	6.1	-4.8	-0.9	2.4	6.1
$50^\circ$	-5.5	-8.3	-7.6	-6.2	-5.5
$40^\circ$	-4.9	-5.3	-6.6	-5.9	-4.9
$10^\circ$	-0.6	-0.3	-1.7	-1.8	-0.6