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ABSTRACT. A method is proposed for adjustment of directions and distances in space with heights held fixed and without reductions of observations to the ellipsoid. Programming hints are included to reduce the time needed for forming observation equations. The adjustments can be performed in the rectangular or in the geographic coordinate system. This method is simpler and faster than customary methods.

INTRODUCTION

Adjustments of horizontal networks are traditionally performed on the surface of an ellipsoid of revolution. Directions and distances are reduced to this surface before they can be used in an adjustment as observations.

The problem can be solved in a more straightforward manner without reducing observations to any computational surface by using the formulas from three-dimensional geodesy (Wolf 1963, Mitchell 1963, Heiskanen and Moritz 1967, Bomford 1971, Rapp 1975, and others). In this system the horizontal adjustment is performed in space with all heights held fixed. Astronomic latitudes and longitudes, interpolated if necessary, are needed at all stations at which directions are observed (which is also true of the classical method) and are held fixed. The observations are horizontal unoriented directions, astronomic azimuths, and spatial distances. There are two coordinate unknowns per station.

In any method that divides geodetic computations into horizontal and vertical parts, the accuracy of the results of a horizontal adjustment depends on the accuracy of geodetic heights which in turn depend to some extent on the knowledge of horizontal positions. The adjustment process is therefore iterative. The proposed method of handling horizontal adjustment does not purport to remove this requirement.

NOTATION

a	equatorial radius of the ellipsoid
e	first eccentricity of the ellipsoid
M	radius of curvature in the meridian
N	radius of curvature in the prime vertical
ϕ	geodetic latitude, positive north
λ	geodetic longitude, positive east
h	height above ellipsoid
ϕ'	astronomic latitude
λ'	astronomic longitude
A	astronomic azimuth, clockwise from north
V	vertical angle, positive upwards from astronomic horizon
S	spatial distance

Subscripts 1 and 2 denote the standpoint and the forepoint, respectively.

GENERAL EQUATIONS

Transformation of ϕ, λ, h to x, y, z:

$$x = (N+h) \cos \phi \cos \lambda \tag{1a}$$
$$y = (N+h) \cos \phi \sin \lambda \tag{1b}$$
$$z = [N(1-e^2)+h] \sin \phi \tag{1c}$$

Transformation of x, y, z to ϕ, λ (Bowring 1976). ε is the square of second eccentricity and b is minor semiaxis:

$$p = \sqrt{(x^2+y^2)} \tag{2a}$$

$$\tan u = (z/p)(a/b) \tag{2b}$$

$$\tan \phi = \frac{z + \varepsilon\, b \sin^3 u}{p - e^2 a \cos^3 u} \tag{2c}$$

$$\tan \lambda = y/x \tag{2d}$$

Inverse solution in space:

$$Q_1 = -\Delta x \sin \lambda'_1 + \Delta y \cos \lambda'_1 \tag{3a}$$

$$P_1 = -\sin \phi'(\Delta x \cos \lambda'_1 + \Delta y \sin \lambda'_1) + \Delta z \cos \phi'_1 \tag{3b}$$

$$\tan A_1 = Q_1/P_1 \tag{3c}$$

$$S \sin V_1 = \cos \phi'_1(\Delta x \cos \lambda'_1 + \Delta y \sin \lambda'_1) + \Delta z \sin \phi'_1 \tag{3d}$$

$$S^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{3e}$$

GEOGRAPHIC COORDINATE ADJUSTMENT

The observation equations have the form

$$c_1 d\phi_1 + c_2 d\lambda_1 + c_3 d\phi_2 + c_4 d\lambda_2 + L = v \tag{4}$$

in which L is computed minus observed value. An orientation unknown is added to a set of equations for unoriented directions, as usual. Scale unknowns can also be included in distance observation equations.

The coefficients for azimuths are

$$c_1 = (M_1+h_1)\sin A_1/(S \cos V_1)$$

$$c_2 = -(N_1+h_1)\cos \phi_1 \cos A_1/(S \cos V_1)$$

$$c_3 = -[(M_2+h_2)(\sin \phi_1 \sin \phi_2 \cos \Delta\lambda \sin A_1 + \sin \phi_2 \sin \Delta\lambda \cos A_1$$
$$+ \cos \phi_1 \cos \phi_2 \sin A_1)]/(S \cos V_1)$$

$$c_4 = [(N_2+h_2)\cos \phi_2 (\cos \Delta\lambda \cos A_1 - \sin \phi_1 \sin \Delta\lambda \sin A_1)]/$$
$$(S \cos V_1) .$$

(Note: In Wolf (1963, p. 229) and (Heiskanen and Moritz 1967, p. 221), the azimuth coefficient corresponding to c_3 here contains an inconsequential approximation.)

The coefficients for distances are

$$c_1 = -(M_1+h_1)\cos A_1 \cos V_1$$

$$c_2 = -(N_1+h_1)\cos \phi_1 \sin A_1 \cos V_1$$

$$c_3 = -(M_2+h_2)\cos A_2 \cos V_2$$

$$c_4 = -(N_2+h_2)\cos \phi_2 \sin A_2 \cos V_2 .$$

It is permissible to use $\cos V_2 = \cos V_1 (a + h_1)/(a + h_2)$.

These coefficients are the same as those used in three-dimensional geodesy. They have been simplified as follows:

$$Q_2 = \Delta x \sin \lambda'_2 - \Delta y \cos \lambda'_2$$

$$P_2 = \sin \phi'_2 (\Delta x \cos \lambda'_2 + \Delta y \sin \lambda'_2) - \Delta z \cos \phi'_2$$

$$R = P_1{}^2 + Q_1{}^2$$

Coefficients for azimuths become

$$c_1 = (M_1+h_1)Q_1/R$$

$$c_2 = -(N_1+h_1)\cos\phi_1 P_1/R$$

$$c_3 = -(M_2+h_2)[(\sin\phi_1 \sin\phi_2 \cos\Delta\lambda + \cos\phi_1 \cos\phi_2)Q_1 + \sin\phi_2 \sin\Delta\lambda\ P_1]/R$$

$$c_4 = (N_2+h_2)\cos\phi_2(\cos\Delta\lambda\ P_1 - \sin\phi_1 \sin\Delta\lambda\ Q_1)/R .$$

Coefficients for distances become

$$c_1 = -(M_1+h_1)P_1/S$$

$$c_2 = -(N_1+h_1)\cos\phi_1 Q_1/S$$

$$c_3 = -(M_2+h_2)P_2/S$$

$$c_4 = -(N_2+h_2)\cos\phi_2 Q_2/S .$$

The sines and cosines of astronomic latitudes and longitudes are computed only once. The functions of geodetic latitudes and longitudes can also be computed only once and updated between iterations by differentials:

$$d(\sin\alpha) = \cos\alpha\ d\alpha - \sin\alpha\ d\alpha^2/2$$

$$d(\cos\alpha) = -\sin\alpha\ d\alpha - \cos\alpha\ d\alpha^2/2$$

The radii of curvature, once computed for a point, should be used for all observation equations involving that point. They need not be recomputed between iterations, since they do not change by any appreciable amounts. For the purpose of computation of coefficients of observation equations, M and N are needed to at most five significant figures. In fact, Heiskanen and Moritz (1967) use a radius of Earth's sphere for this purpose, but this is not recommended. The value of N is needed with full precision before the initial adjustment. N changes by at most 0.1 m per 1" of change in latitude, which has practically no effect on the computed azimuth and distance between adjacent points; therefore, the value computed initially can be used in all iterations. Alternatively, N can be updated between iterations by $dN = e^2 a \sin\phi \cos\phi\ d\phi$.

RECTANGULAR COORDINATE ADJUSTMENT

The observation equations with three unknowns have the form

$$a_1(dx_2-dx_1) + a_2(dy_2-dy_1) + a_3(dz_2-dz_1) + L = v. \tag{5}$$

The coefficients for azimuths are

$$a_1 = (\sin \phi'_1 \cos \lambda'_1 Q_1 - \sin \lambda'_1 P_1)/R$$

$$a_2 = (\sin \phi'_1 \sin \lambda'_1 Q_1 + \cos \lambda'_1 P_1)/R$$

$$a_3 = -\cos \phi'_1 Q_1/R.$$

The coefficients for distances are

$$a_1 = \Delta x/S \qquad a_2 = \Delta y/S \qquad a_3 = \Delta z/S.$$

In order to hold the heights fixed, one unknown is eliminated and expressed in terms of the other two. The three unknowns must satisfy the condition

$$x\,dx + y\,dy + \frac{z}{1 - e_o^2}\,dz = 0 \tag{6}$$

where $e_o^2 = e^2/(1 + h/N)$. This states that each point in space lies on the surface of an auxiliary ellipsoid having a different eccentricity and equatorial radius than the adopted ellipsoid. For practical purposes the approximation $e_o^2 = e^2/(1+h/a)$ or even $e_o^2 = e^2$ is harmless. There are three cases.

Case 1. The unknown dx is eliminated.

$$b_1\,dy_1 + b_2\,dz_1 + b_3\,dy_2 + b_4\,dz_2 + L = v, \tag{7a}$$

where

$$b_1 = -a_2 + a_1 y_1/x_1$$

$$b_2 = -a_3 + a_1 \frac{z_1}{(1-e^2)x_1}$$

$$b_3 = a_2 - a_1 y_2/x_2$$

$$b_4 = a_3 - a_1 \frac{z_2}{(1-e^2)x_2}.$$

Case 2. The unknown dy is eliminated.

$$b_1\,dx_1 + b_2\,dz_1 + b_3\,dx_2 + b_4\,dz_2 + L = v, \tag{7b}$$

where

$$b_1 = -a_3 + a_2 x_1/y_1$$

$$b_2 = -a_1 + a_2 \frac{z_1}{(1-e^2)y_1}$$

$$b_3 = a_3 - a_2 x_2/y_2$$

$$b_4 = a_1 - a_2 \frac{z_2}{(1-e^2)y_2} .$$

Case 3. The unknown dz is eliminated.

$$b_1\, dx_1 + b_2\, dy_1 + b_3\, dx_2 + b_4\, dy_2 + L = v, \tag{7c}$$

where

$$b_1 = -a_1 + a_3(1-e^2)x_1/z_1$$

$$b_2 = -a_2 + a_3(1-e^2)y_1/z_1$$

$$b_3 = a_1 - a_3(1-e^2)x_2/z_2$$

$$b_4 = a_2 - a_3(1-e^2)y_2/z_2 .$$

In practice, case 3 would be used with points for which $\cos^2\phi < 0.5$. In the remaining equatorial belt, case 1 is chosen if $\cos^2\lambda < 0.5$, otherwise case 2 applies. It is not a requirement that the same unknown be eliminated for all points in the same solution.

The eliminated unknown is computed after the adjustment by (6). Geographic coordinates are obtained by equations (2).

CONCLUDING REMARKS

The classical method reduces observations to the surface of the ellipsoid. The reduction of azimuths is effected on the basis of provisional geodetic positions. Therefore, Laplace azimuths are slightly wrong, and a term is included in the observation equation to make allowance for the change in geodetic longitude. Strictly speaking, correction terms should also be included in observation equations for azimuths and unoriented directions to compensate for the changes in latitude and longitude which were used in the computation of deflection corrections.

The proposed method handles observations without reducing them to the surface of the ellipsoid. It avoids time consuming computations of geodetic azimuths and distances before iterations of an adjustment and uses relatively simple closed equations from three-dimensional geodesy. It is also more accurate, at least theoretically, because it does not impose any restrictions on the lengths of the lines or on the extent of the network.

This method is not to be confused with the general three-dimensional method in which there are three coordinate unknowns per station and astronomic latitudes and longitudes may be permitted to acquire corrections.

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