# NOAA Technical Memorandum NOS NGS 82

## On the Use of *Linear* Units as a Companion to Horizontal Datum Transformations Performed on *Curvilinear* Coordinates

### (or "What does NGS mean when they provide NADCON transformations and error estimates for latitude and longitude in meters?")

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# <u>Versions</u>

| Date | Changes |
|---|---|
| August 30, 2019 | Original Release |
| February 26, 202 | Changed latitude to mean latitude in equations 9 and 11 plus associated text edits, including changes to examples in section 4.3. |

# 1 Introduction

A datum transformation is some formula or tool that takes as input some coordinate(s) in a starting geodetic datum and attempts (often imperfectly) to give as output some new coordinate(s) in a different geodetic datum. The history of datum transformations is long and distinguished, but will not be covered in this paper. The authors also note that this paper covers theory applicable only to horizontal transformations.

Although NGS began in the 1980s writing computer code to do complex datum transformations (such as LEFTI; Vincenty 1980), it wasn't until the release of NADCON 1.0 (Dewhurst 1990) that an NGS transformation tool was fast, accurate, and sufficiently simple to see widespread use across the nation.

When NADCON 1.0 was released, both the software and documentation addressed coordinate differences in both curvilinear and linear units. Yet the two units are not interchangeable, within the context of datum transformations. The actual NADCON transformation (change in latitude or longitude) has always been a *curvilinear* process (i.e., as changes in the angular units of latitude and longitude, each done separately). <u>There is no actual change in latitude or longitude that happens "in meters."</u> Yet NGS has always provided linear (meter) values as "companion" values to the curvilinear transformation results. This subtle, but important, point was generally overlooked in all versions of NADCON prior to NADCON 5.0 (Smith and Bilich, 2017). With the transformation grids in ***NADCON 5.0 release 20160901***, as implemented in the NGS Coordinate Conversion and Transformation Tool (NCAT) came a decision to rely purely on the definable curvilinear coordinate changes. It was felt that providing horizontal datum transformation values "in meters" was both scientifically incorrect and confusing.

While linear values were computed as companion quantities (with all the attendant problems, which this report will cover), they were not released as official transformations in ***NADCON 5.0 release 20160901*** and were therefore not codified into NGS's two primary tools, NCAT and VDatum.

However, it was soon clear that this situation was not entirely satisfactory to the user community. One reason for wanting linear units is to provide a simple means to assess the magnitude of change, in familiar units. Knowing the magnitude helps answer questions about what to expect and whether a transformation is appropriate, for example, if the transformed coordinate differences are smaller than the spatial accuracy of the data. Since spatial accuracy is frequently given in linear units, it is useful to also have an estimate of the transformation in linear units—even if it lacks technical rigor.

That being said, the authors recognize that, in part, users now expect linear units because NGS provided them in the first place. The decades preceding NADCON 5.0, with the general "hand waving" by NGS about linear versus curvilinear coordinate changes, created a desire in the user community to see "coordinate shifts in meters." This paper discusses both what is valid and invalid with reporting linear units in NADCON, and describes how NGS will continue to support this user request, while maintaining scientific rigor.

## 2 Datum transformation basics

Dewhurst (1990), in discussing NADCON 1.0, defined a datum transformation simply enough. Here we use NAD 27 and NAD 83, though this could just as easily be generalized into "new datum" and "old datum", as:

$$\begin{aligned} \phi_{NAD\ 83} &= \phi_{NAD\ 27} + \delta\phi \\ \lambda_{NAD\ 83} &= \lambda_{NAD\ 27} + \delta\lambda \end{aligned} \tag{1}$$

where $\phi$ denotes latitude and $\lambda$ is longitude. Note that the units in this equation are all curvilinear (e.g., decimal degrees or degrees/minutes/seconds, etc.)

Expanding the above equations to include some descriptive text within the context of their actual use in a NADCON example might be instructive. For example, assume the NAD 27 latitude and longitude of a point (say "AB1234") is known, and the intent is to use NADCON to estimate the NAD 83 latitude and longitude of the same point. As such, the latitude part of equation 1 reads like this:

> *(The estimated latitude of AB1234 in NAD 83 in degrees/minutes/seconds)* is equal to *(the published latitude of AB1234 in NAD 27 in degrees/minutes/seconds)* plus *(some value in degrees/minutes/seconds, interpolated from the NADCON latitude grid, at the NAD 27 latitude and longitude[1])[2]*

Similarly for longitude. Note that there is a very subtle point that is easily overlooked: the input and output values refer to the *same point*, but in *different datums*. Put more directly: ***the point did not move***. Rather, NGS has replaced one datum with another. That replacement of one datum with another comes with large changes (such as the location of the geocenter and/or the size and shape of the reference ellipsoid being used) and small changes (such as the impact of local and regional surveys which have improved relational understandings of local geodetic control points from one datum to another). *But the point did not actually move any linear distance. Thus there does not exist any displacement "vector" which connects the point in the old datum with the same point in the new datum.*

NGS has been guilty of using the term "shifts" to describe the coordinate change that comes out of a datum transformation, and that terminology has been embraced by the user community as well. But a "shift" implies that something moved. It is around this word "shift" that a lot of the linear versus curvilinear problems arise. For example, the question might be asked: "How many

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[1] Note that the question of "which latitude and longitude should I use when asking the software to interpolate off of the grid" is not a trivial one. If the coordinate differences between two datums are large enough, and the grid spacing small enough, then one might actually see significant differences when using the "old datum" or the "new datum" when determining which latitude and longitude to give the grid interpolator. *Thankfully this situation does not exist for any of the transformations in NADCON*. Therefore, with equal accuracy, one can give the grid interpolator the latitude/longitude in "old datum" or in "new datum" and receive the same answer for what the datum transformation is between the old datum and the new datum.

[2] The values in the NADCON grids are actually provided in units of arcseconds. This is because even the largest coordinate changes due to a horizontal datum transformation within the datums of the National Spatial Reference System (NSRS) have never exceeded a few arcseconds.

centimeters did my point shift when NAD 83(NSRS2007) was replaced with NAD 83(2011)?" The correct answer, hopefully obvious by now, is "that question has no answer." The next section describes why this is so, and subsequent sections will attempt to avoid the nonsensical while still providing a linear value that is a useful *companion value* to official *coordinate changes* coming out of NADCON.

As a related aside, NADCON 5.0 also provides transformation error estimates in curvilinear units (arcseconds). This is done for the same reason that coordinate differences are in curvilinear units. And, as with the transformation values themselves, there is an expectation in the user community that transformation errors should also be provided in linear units (e.g., meters). As such, although this report is intended to address coordinate differences, the same concepts also apply to NADCON error estimates.

## 3 Units, and a little philosophy

There is essentially one reason that equation 1 works at all, and that is that the values $\phi_{NAD\ 83}$ and $\phi_{NAD\ 27}$ are both in the same units, specifically curvilinear, and that these two coordinates can completely explain the horizontal transformation. From a purely philosophical standpoint, it might be argued that these two different latitudes are not even the same *kind* of things, being determined in different ways at different times, but that does not prevent equation 1 from being useful. Being different kinds of things, but both expressed in curvilinear coordinates, still allows for equation 1 to "work," and as a critical corollary it obviously requires that $\delta\phi$ must also be curvilinear.

Consider now the question "how big is $\delta\phi$ in meters?" Answering this question is not simple. The easiest (and, in fact, long-encoded answer in NADCON, since version 1.0), yet incorrect, way would be to treat $\delta\phi$ as the *separation between two points* on an ellipsoid, and compute the distance between the points. There are two inherent scientific inaccuracies with this approach. First, $\delta\phi$ is the difference in coordinates at *one* point (not *two*). Second, if the two datums involved in $\delta\phi$ have different ellipsoids (such as is the case with NAD 27 and NAD 83), there is no obvious choice of which ellipsoid to use.

But an even simpler problem exists when one asks "how big is $\delta\phi$ in meters?". In order to fulfill the definition of "datum transformation" in equation 1, we would need to express $\phi_{NAD\ 83}$ and $\phi_{NAD\ 27}$ in meters as well! And this raises the follow-up question: how can you express inherently curvilinear coordinates (latitudes or longitudes) with linear units? The answer is "you can't".

To summarize: the big question, and where philosophy enters in, is:

> **Question: What is this horizontal distance NGS has been calculating and reporting in NADCON for decades?**

The big answer is:

**Answer: It is <u>not a real horizontal distance at all</u>, and thus nobody should use it as anything but a ballpark 'companion value' to understand the general scale of horizontal deformation of a network of points in an area which have undergone a datum change via NADCON.**

# 4 Actual distance

In order to shed light and provide a way forward, it will be instructive to discuss formulae which compute *actual* distances in both curvilinear and linear units. Specifically, consider *two* unique points, both on the same ellipsoid, separated by some non-zero distance, and both with coordinates provided in one datum (NAD 83(2011), for example). The geodetic coordinates of these two points are ($\phi_1$, $\lambda_1$) and ($\phi_2$, $\lambda_2$).

## 4.1 ENU vs arc distance

To ask "how far north and east is point 2 from point 1?" can raise a confusion with regard to what is meant by "north" and "east." Thus, before proceeding, this issue will be clarified.

It is possible to build a local Cartesian coordinate system, centered at point 1, with "east" and "north" axes defining the horizon plane, tangent to the ellipsoid at point 1 (often called a "local geodetic horizon" system). The "up" axis would point along the ellipsoidal normal at point 1. While such a system certainly lends itself to answering questions about "north" and "east" distances, it does not actually describe distances between points *on the ellipsoid*. That is, to travel "north" or "east" in point 1's local ENU system is to travel *off* of the ellipsoid the moment one leaves point 1. Admittedly, the effect on horizontal distance due to departure from the ellipsoid is extremely small for even the largest NADCON 5.0 shifts (about 20 arcseconds). To illustrate, for a shift of 20 arcseconds in latitude, the ENU plane departs from the ellipsoid by 3 cm, yet the ENU distance north (about 617 m at 45° latitude) is shorter than the ellipsoidal distances by only 0.001 *mm*. The effect is the same for a shift in longitude at the equator, but the difference decreases to zero at the poles. Nonetheless, an ENU approach is not strictly appropriate for the curvilinear geometry of the problem, and, moreover, such a solution is unnecessary.

Therefore, rather than worry about a local Cartesian coordinate system, the issue of "distance in the north" and "distance in the east" will be solved by computing *arc distances on the ellipsoid*, in the meridian (for "north") and along a parallel (for "east").

## 4.2 Arc distances

Most of the equations below come from (Rapp, 1991), and describe the distance in "north" (arc along the meridian) and "east" (arc along the parallel) between two points on the same ellipsoid. Consider the differential arc distance in the meridian first:

$$\partial s = M(\phi)\partial\phi \tag{2}$$

where $M(\phi)$ is the radius of curvature in the meridian:

$$M(\phi) = \frac{a\,(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} \tag{3}$$

In the above equation, $a$ is the ellipsoid semi-major axis and $e^2$ is its first eccentricity squared. Integration of Eq. 2 over a finite latitude difference of $\Delta\phi = \phi_2 - \phi_1$ results in an infinite series without an exact closed-form solution. However, an approximation for the distance $s$ can be made if $\Delta\phi$ is assumed to be "small," by using $M_m$ (the evaluation of Eq. 3 at the mean latitude of the end points):

$$s \approx M_m\,\Delta\phi\,, \tag{4}$$

or

$$s \approx \frac{a\,(1-e^2)\,\Delta\phi}{(1-e^2\sin^2\phi_m)^{3/2}} \tag{5}$$

where $\phi_m$ is the mean latitude of $\phi_1$ and $\phi_2$, in radians.

Consider next the differential arc length along the parallel:

$$\partial L = N(\phi)\cos\phi\,\partial\lambda \tag{6}$$

where $N(\phi)$ is the prime vertical radius of curvature given by

$$N(\phi) = \frac{a}{\sqrt{1-e^2\sin^2\phi}} \tag{7}$$

The integration of this equation is simpler, with an exact solution:

$$L = N(\phi)\cos\phi\,\Delta\lambda \tag{8}$$

and thus, at the same mean latitude $\phi_m$ used in equation 5, the arc length $L$ becomes

$$L = \frac{a\cos\phi_m\,\Delta\lambda}{\sqrt{1-e^2\sin^2\phi_m}} \tag{9}$$

For consistency with the coordinate differences in NADCON, equations 5 and 9 will be expressed with $\Delta\phi$ and $\Delta\lambda$ in arcseconds:

$$s \approx \frac{a\,(1-e^2)\,\Delta\phi_{sec}}{(1-e^2\sin^2\phi_m)^{3/2}}\cdot\frac{\pi}{180^\circ}\cdot\frac{1^\circ}{3600\,sec} \tag{10}$$

$$L = \frac{a \cos \phi_m \, \Delta\lambda_{sec}}{\sqrt{1 - e^2 \sin^2 \phi_m}} \cdot \frac{\pi}{180^{\circ}} \cdot \frac{1^{\circ}}{3600\, sec} \tag{11}$$

Equations 10 and 11 therefore express the distance in "north" (arc along the meridian) and "east" (arc along the parallel) between two points on the same ellipsoid, where $\Delta\phi$ is assumed "small," and all coordinate differences are provided in arcseconds.

It so happens that equations 10 and 11 are the exact formulae that have been used in NADCON, for the reporting of transformations "in meters," with these GRS 80 ellipsoid parameters:

$a$ = 6,378,137 meters (exact)

$e^2$ = 0.00669438002290… (unitless, not exact)

However, it must be re-emphasized: <u>the only official transformations coming out of NADCON, for latitude and longitude, are *coordinate differences* expressed in curvilinear values (arcseconds)</u>. The encoding of equations 10 and 11 since the first version of NADCON have been done with the best of intentions, but is philosophically incorrect[3].

Nonetheless, because these values have been helpful for visualizing and understanding deformations of survey point networks induced by datum transformations, NGS plans to encode them into NGS's two primary tools: NCAT and VDatum, as appendices to NADCON, for both coordinate differences and error estimates. However, both will be quantities computed by NCAT and VDatum (or other software applications) but will not be official products of the NADCON transformation itself.

Because of this decision, it may be worthwhile to close with some words about the accuracy of equations 10 and 11, without regard to the correctness, or lack thereof, of their application.

## 4.3 Accuracy

While equation 11 is exact at any latitude, use of the *mean* latitude in equation 10 introduces a level of approximation which should be quantified.

Equation 10 is accurate to within ±0.4 mm for $\Delta\phi_{sec}$ < 900" = 0.25° ≈ 28 km, at any location on the GRS 80 ellipsoid. The error magnitude is maximum at the equator and poles and minimum at $\phi_m$ = 45° (−0.002 mm for $\Delta\phi_{sec}$ = 900"). The maximum error (at the poles and equator) decreases to ±0.00013 mm for $\Delta\phi_{sec}$ = 60" (1 arcminute), which is about 1.85 km at $\phi_m$ = 45°. The bottom line is that equation 10 is accurate to better than ±0.00001 mm for any $\Delta\phi_{sec}$ values

[3] In fact, they are also numerically incorrect, in that the Clarke 1866 ellipsoid, used in NAD 27, was completely ignored in computing linear values from the curvilinear coordinate changes in the NAD 27/NAD 83(1986) transformation. For reference, the Clarke 1866 ellipsoid has $a$ = 6,378,206.4 m (exact) and $e^2$ = 0.00676865799729... (not exact).

which might come from NADCON 5.0 (for an assumed maximum $\Delta\phi_{sec} \approx 20''$, for the American Samoa region).

Note, linear arc lengths obtained using $\Delta\phi_{sec}$ and $\Delta\lambda_{sec}$ should not in general be combined to compute an accurate geodesic distance (i.e., as the square root of the sum of their squares). However, for all NADCON 5.0 transformations, the $\Delta\phi_{sec}$ and $\Delta\lambda_{sec}$ values are sufficiently small that their conversion into linear units (using Eq. 10 and 11) agree to better than 0.01 mm with actual geodesic distances (e.g., using the Vincenty method; Vincenty 1980). As an example using large delta values of $\Delta\phi_{sec} = \Delta\lambda_{sec} = 60''$ (much larger than any NADCON 5.0 transformation), $s = 1850.57988$ m and $L = 1423.05792$ m at $\phi = 40°$. The "horizontal"[4] distance implied by these deltas can be approximated by $\sqrt{s^2 + L^2}$, = 2334.46776 m, which is only 0.01 mm greater than the actual geodesic distance of 2334.46775, and the error is even less for smaller deltas. Despite this excellent agreement, the error increases rapidly with increasing deltas, to 1.3 mm for $\Delta\phi_{sec} = \Delta\lambda_{sec} = 300''$ (5 arcminutes) and 36 mm for $\Delta\phi_{sec} = \Delta\lambda_{sec} = 900''$ (15 arcminutes). For this reason, caution should be exercised when attempting to estimate "horizontal" distances using delta latitude and longitude values.

As mentioned previously, another source of error (and ambiguity) when computing linear coordinate differences occurs when the transformation involves two different ellipsoids. Consider the same large delta values of $\Delta\phi_{sec} = \Delta\lambda_{sec} = 60''$ at $\phi = 40°$ from the previous example, but with the linear distances based on the Clarke 1866 ellipsoid. The resulting distances are shorter by 33 mm for latitude and longer by 37 mm for longitude than those based on the GRS 80 ellipsoid. The maximum magnitudes of linear differences in latitude for this case are 118 mm shorter than GRS 80 at the equator and 90 mm longer at the poles (whereas the linear difference in longitude is relatively constant at most latitudes and goes to zero at the poles). Although these differences decrease proportionally with decrease in $\Delta\phi_{sec}$ and $\Delta\lambda_{sec}$, it clearly shows the difficulty in determining meaningful, accurate linear values for curvilinear coordinate differences when the input and output ellipsoids are not the same, as occurs when transforming between NAD 27 and NAD 83 (1986).

## 4.4 Summary

Since its inception, NADCON has computed actual curvilinear coordinate changes (arcseconds), while also reporting linear distances as companion values. The ingestion of ***NADCON 5.0 release 20160901*** into NCAT and VDatum did not originally include these linear companion values, due to their philosophical incorrectness. This is true for two reasons: (a) the transformation values within ***NADCON 5.0 release 20160901*** are not separations between two points and (b) two different ellipsoids might be in play, such as the NAD 27 to NAD 83(1986) transformations.

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[4] The term "horizontal" is in quotes because it is an amalgamation of things: on the one hand $s$ and $L$ represent arc distances on an ellipsoid, but combining them as $\sqrt{s^2 + L^2}$ treats them as linear distances in a *Cartesian* frame. For this reason, while NGS will be providing the $s$ and $L$ values as "companion" values to the true transformations, there will not be a corresponding "horizontal" value.

Nonetheless, users have requested that NGS reinstate linear distances as a companion to NADCON official datum transformations. In order to address this situation, NGS has encoded equations 10 and 11 into NCAT and VDatum (based on the GRS 80 ellipsoid) as a companion to ***NADCON 5.0 release 20160901***, for both transformation coordinate differences and error estimates. As part of the decision to add such functionality to NCAT and VDatum, NGS has provided this paper as an explanation about why these linear values should be treated with caution.

## Bibliography


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