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Covariances within the multi-epoch least-squares adjustment problem and their impact on estimating reference epoch coordinates in the modernized National Spatial Reference System

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Contents

Executive Summary

The National Geodetic Survey (NGS) plans to modernized the National Spatial Reference System (NSRS) in 2025. One part of that modernization is the estimation of both geometric and orthometric reference epoch coordinates (RECs) from decades of existing GNSS, classical and leveling observations.

To support the estimation of RECs, NGS published a report outlining the mathematics necessary to not only project observations through time to the reference epoch but also to properly account for the uncertainties in the models used to do the projecting (*The Multi-Epoch Least-Squares Adjustment Problem: Models Relating Estimable Parameters at a Single Epoch to Geodetic Observations, Stochastic Constraints and Fixed Constraints at Multiple Epochs, Volume I: The Projection Method*, Smith, Gillins, Heck and Roman, July 2023, NOAA Technical Report NOS NGS 79). That report outlined how observations (and constraints) across various epochs could be projected through time using a model of changes to parameter values (MCPV). However, the report only provided generalized equations and left unsolved certain practical aspects of the application of the equations.

One of those practical aspects centered on questions of covariances: Will they exist in the models and even if they do not, will they arise out of the method used to create projected observations? That is the focus of this report.

Herein the author explored these questions, and more. The first subject addressed was the relationship between the abstract mathematical construct of the MCPV and the available geodetic value change models (GVCMs), such as the intra-frame deformation model (IFDM2022) and the dynamic geoid model (DGEOID2022). Next, equations are derived which would properly use covariances which might exist in the GVCMs. However, the conclusion reached is that if such covariances exist, then the cofactor matrix of the projected observations becomes full and for large least-squares adjustment problems, this becomes a practical impossibility to invert.

The author then assumes such covariances will be unknown (or ignored) and shows how additional covariances arise simply through the process of projecting, when observations share a common point. The author then addresses the practical impacts of these additional covariances, and finally concludes that the only practical approach is to ignore certain covariances entirely and forcibly construct the cofactor matrix of the projected observations to exactly match the structure (diagonal and/or block-diagonal) of the cofactor matrix of the original observations.

The author closes the paper with a summary, and an appendix which discusses certain special circumstances when the cofactor matrix of the projected observations might be singular.

1 Introduction

In one method of modeling of the multi-epoch least-squares adjustment (ME-LSA) problem (Smith et al., 2023), *observations* ($\boldsymbol{Y}$) taken across a variety of observation epochs are projected through time into *projected observations* ($\overline{\boldsymbol{Y}}$) at the adjustment epoch. The differences between observations and projected observations are a function changes to the parameter values from the observation epoch to the adjustment epoch, which come from a *model of changes to parameter values* (MCPV). Similar statements can be made about *constraints* and *projected constraints*. However, in the modernized national spatial reference system (NSRS; NGS 2021a, NGS 2021b, NGS 2021c), the estimation of reference epoch coordinates (RECs) by the National Geodetic Survey (NGS) will be performed under a strict set of rules, one of which will be that fixed and stochastic constraints will *only* be imposed at the adjustment epoch. As such, in REC adjustment projects the MCPV will only impact the *observation* equations, not the *constraint* equations, and so there will be no difference between *constraints* and *projected constraints*. Therefore, we will focus solely on the impact of the MCPV upon projected *observations*.

Complicating this situation, NGS does not have any models which fulfill the definition of the MCPV as per (ibid)[1]. Rather, NGS has *related* models, such as the intra-frame deformation model of 2022 (IFDM2022), and DGEOID2022. Such models fall into a class called geodetic value change models (GVCM)[2]. Thankfully, the MCPV can be derived from one or more GVCMs. This was discussed in (ibid) and will be expanded in this report.

If the MCPV is known without variance (i.e., *fixed*), then the dispersion matrix of the projected observations, $\Sigma_{\bar{y}}$, will be identical to the dispersion matrix of the observations, Σ_y. This is essentially what was done in the 2011 National Adjustment (Dennis, 2020). See section 3.1.

However, if the MCPV is known *with* variance (i.e., *stochastic*), then the dispersion matrix of the projected observations, $\Sigma_{\bar{y}}$, will be the sum of the dispersion matrix of the observations, Σ_y, *plus* the dispersion matrix of the MCPV contribution to the projected observations, $\Sigma_{\bar{y},m}$.

If all of the GVCMs are fixed, so too is the MCPV. If at least one GVCM is stochastic, the MCPV is stochastic.

The primary issue examined in this report is how to compute matrix $\Sigma_{\bar{y},m}$ when given one or more GVCMs, with special attention placed on which, if any, *covariances* are provided in the GVCM. A related issue is that the matrix $\Sigma_{\bar{y},m}$ might be *significantly* less sparse than Σ_y, making $\Sigma_{\bar{y}}$ less sparse by extension. This report discusses what causes the loss of sparsity in $\Sigma_{\bar{y},m}$, the practical

[1] That is, no model at NGS is constructed as changes over time to Earth-centered Earth-fixed Cartesian coordinates X, Y, Z, as required for geometric adjustments. Nor does a model exist at NGS constructed as changes over time to orthometric heights H, as required for orthometric adjustments.

[2] This term was carefully chosen. First, its acronym does not begin with "M", helping the reader to distinguish between MCPV and GVCM at a glance. Second, it is generic, where *geodetic value* can refer to just about anything. In IFDM2022 it refers to local East-North-Up changes (velocities and displacements). In DGEOID2022 it refers to local changes (velocities only) to the geoid undulation. In theory, the Euler Pole Parameter model of 2022 (EPP2022), will be a GVCM since it will contain rotation rates about the ITRF2020 axes which represent changes to geodetic values, from which changes to X, Y and Z may be derived.

difficulties arising from that loss of sparsity and makes recommendations as to how to alleviate those difficulties.

The bulk of this report is in sections 5 and 6, covering geometric adjustments and orthometric adjustments. As these two different types of adjustments have their own GVCMs and their own particular issues, they must be discussed separately. In each of those two sections, we will begin with a general discussion of projected observations, and show how to compute the diagonal or block-diagonal elements of $\Sigma_{\bar{y},m}$ for each observation type. We will then turn to the off-diagonal elements of $\Sigma_{\bar{y},m}$ by examining certain special cases. We will show in section 7 how knowledge of covariances within the GVCM (and the MCPV, by extension) is one of the two main causes of loss of sparsity in $\Sigma_{\bar{y},m}$, and that the other primary cause is *shared points* between two observations. The shared points issue can cause loss of sparsity if the GVCMs contain grids of linear velocities or grids of discontinuities that are used in common between two projected observations (even if no covariances are available in the GVCM), which happens to be exactly the situation with IFDM2022 and DGEOID2022.

We will close the paper with a discussion of practical issues, specifically seeking a compromise between *knowledge* of off-diagonal elements in $\Sigma_{\bar{y},m}$ and the practical *implications* of those elements being non-zero.

1.1 Terminology and notation

A few words about *dispersion*, *cofactor* and *weight* matrices are warranted. As pointed out in Appendix A of Smith et al. (2023):

> "…*dispersion* matrices (represented by Σ) are not generally available for either observations, stochastic constraints or a GVCM. Rather, one or more *cofactor* matrices (represented by Q) *are* available, with the relationship between dispersion matrices and cofactor matrices being through one or more variance components…".

If a single variance component exists, relating each dispersion matrix to *one* cofactor matrix, this is the Gauss-Markov Model, or GMM (Snow, 2021). If each dispersion matrix is related to multiple cofactor matrices through multiple variance components, this is the Variance Component Model, or VCM (ibid).

Within the GMM, the inverse of a cofactor matrix is called a *weight* matrix, represented by a P. Weight matrices, in general, do not exist in the VCM, as cofactor matrices in that model tend to be singular.

Because the equations for the VCM are different from the GMM, this report will restrict its discussion to *dispersion* matrices in *general*, despite their lack of availability, and will only introduce *cofactor* matrices when necessary.

Additionally, we adopt similar notation as was used in Smith et al. (2023). Specifically, when discussing a particular *observation* epoch, we will use the variable i. The adjustment epoch will not have an index variable. Related to this are the actual *times* of the epochs themselves. The time of observation epoch i will be designated by variable t_i, and the time of the adjustment epoch will be designated by variable t_a.

Finally, because some of the vectors and matrices will get a bit large for the printed page, it will sometimes be difficult to show every element of them. Therefore, we will often look at block-rows, and block-columns. To do so, we adopt the notation that a **bold** variable means *vector*. Thus, variable X (not bold) means the X coordinate (of a Cartesian triad) but $\boldsymbol{X}$ (bold), means a vector containing all three Cartesian coordinates as follows:

$$\boldsymbol{X} = [X \quad Y \quad Z]^T, \tag{1}$$

$$\boldsymbol{\Delta X} = [\Delta X \quad \Delta Y \quad \Delta Z]^T, \tag{2}$$

$$\boldsymbol{E} = [E \quad N \quad h]^T, \tag{3}$$

$$\boldsymbol{\Delta E} = [\Delta E \quad \Delta N \quad \Delta h]^T, \tag{4}$$

$$\dot{\boldsymbol{E}} = [\dot{E} \quad \dot{N} \quad \dot{h}]^T. \tag{5}$$

2 Special case of the multi-epoch least-squares adjustment problem as applied to reference epoch coordinates

In the *projection method* of the ME-LSA problem (Smith et al., 2023), the incremental projected observation equations and incremental projected constraint equations, using a *stochastic* MCPV, take this form:

$$\begin{aligned} \bar{\boldsymbol{y}} &= \bar{A}\boldsymbol{\xi} + \boldsymbol{e}_{\bar{\boldsymbol{y}}}, \\ \bar{\boldsymbol{w}} &= \bar{S}\boldsymbol{\xi} + \boldsymbol{e}_{\bar{\boldsymbol{w}}}, \\ \bar{\boldsymbol{g}} &= \bar{F}_g\boldsymbol{\xi} + \boldsymbol{e}_{\bar{\boldsymbol{g}}}, \\ \bar{\boldsymbol{\gamma}}_{\boldsymbol{a}} &= \bar{F}_{\gamma}\boldsymbol{\xi}. \end{aligned} \tag{6}$$

The random errors in (6) are distributed as:

$$\begin{bmatrix} \boldsymbol{e}_{\bar{\boldsymbol{y}}} \\ \boldsymbol{e}_{\bar{\boldsymbol{w}}} \\ \boldsymbol{e}_{\bar{\boldsymbol{g}}} \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{\bar{y}} & 0 & 0 \\ 0 & \Sigma_{\bar{w}} & 0 \\ 0 & 0 & \Sigma_{\bar{g}} \end{bmatrix} = \begin{bmatrix} \Sigma_y + \Sigma_{\bar{y},m} & 0 & 0 \\ 0 & \Sigma_w + \Sigma_{\bar{w},m} & 0 \\ 0 & 0 & \Sigma_{\bar{g},m} \end{bmatrix} \right). \tag{7}$$

Equations 6 and 7 are the *general* form of the projection method of modeling the ME-LSA problem. However, as mentioned in the introduction, this paper will focus on one *specific* scenario of the ME-LSA problem, with the following criteria:

a) The MCPV is stochastic, and
b) All constraints (fixed or stochastic) will only be provided *at the adjustment epoch.*

This is done for a few reasons. First, using a stochastic MCPV provides a rigorous, consistent way for older observations to be down-weighted relative to more recent observations.

Second, having all constraints at the adjustment epoch allows NGS to restrict them solely to those values that are well-known, without the need to project them through time. For instance, in a geometric adjustment (ibid), defining constraints solely at the adjustment epoch means NGS can constrain the coordinates at CORSs which were actually operational and had a well-defined coordinate function on the date of the adjustment epoch[3]. This restriction on constraints will simplify (6) and (7). With all constraints at the adjustment epoch, the MCPV has no impact on the *constraint* equations, and the equations reduce down to

$$\begin{aligned} \bar{\boldsymbol{y}} &= \bar{A}\boldsymbol{\xi} + \boldsymbol{e}_{\bar{y}}, \\ \bar{\boldsymbol{w}} &= \bar{S}\boldsymbol{\xi} + \boldsymbol{e}_{\bar{w}}, \\ \bar{\boldsymbol{\gamma}}_{\boldsymbol{a}} &= \bar{F}_{\gamma}\boldsymbol{\xi}, \end{aligned} \tag{8}$$

and

$$\begin{bmatrix} \boldsymbol{e}_{\bar{y}} \\ \boldsymbol{e}_{\bar{w}} \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{\bar{y}} & 0 \\ 0 & \Sigma_{\bar{w}} \end{bmatrix} = \begin{bmatrix} \Sigma_y + \Sigma_{\bar{y},m} & 0 \\ 0 & \Sigma_w \end{bmatrix} \right). \tag{9}$$

Note in (9) that $\Sigma_{\bar{w},m}$ has been eliminated, as have all matrices and vectors associated with projected fixed constraints (not at the adjustment epoch), $\bar{\boldsymbol{g}}$.

This scenario will be expected NGS policy with regard to upcoming reference epoch coordinate (REC) and survey epoch coordinate (SEC) adjustments (NGS, 2021c). Therefore, under these two criteria, most of the remaining discussion will focus on observations, and not constraints.

2.1 Two types of adjustments

In the modernized NSRS, NGS will compute RECs in two types of adjustments (NGS, 2021c): *geometric* and *orthometric*. Each type of adjustment has a finite list of allowable observation, constraint and unknown parameter types. In geometric adjustments, the unknown parameters will be Earth-centered Earth-fixed (ECEF) Cartesian coordinates, XYZ. In orthometric adjustments, they will be orthometric heights, H.

These two types of adjustments will be examined in detail, in particular in sections 5 and 6.

3 The relationship between a model of changes to parameter values (MCPV) and a geodetic value change model (GVCM)

Smith et al. (2023) required that the model of changes to parameter values (MCPV) must always refer to the same types of parameters being estimated (XYZ for geometric, H for orthometric). Also, all values coming from the MCPV must be at observation points[4] and each value must be a single value of change between some observation epoch i and the adjustment epoch. For example,

[3] Meaning that older GNSS vectors, for example, which may have a CORS as an endpoint at the observation epoch, but where that CORS no longer exists at the adjustment epoch, will treat neither endpoint as constrained once the observation becomes a projected observation.

[4] The physical points where observations were collected. The MCPV must also, according to Smith et al. (2023) provide values at constraint points, but as mentioned earlier, no constraints will be projected through time, so this point is moot.

in a geometric adjustment the MCPV must, for any given observation provide the change to X, Y and Z between t_i and t_a, at the point where observation i took place. Similar changes to H must come from the MCPV for an orthometric adjustment.

As pointed out in (ibid) NGS does not have any models which *directly* provide such values of change. That is, NGS has no MCPVs. What *is* available are various gridded models of change to certain geodetic coordinate values. In particular the intra-frame deformation model of 2022 (IFDM2022) and the dynamic portion of GEOID2022 (i.e., DGEOID2022). The first is made up of linear velocities (as well as their standard deviations) and episodic displacements (with no standard deviations) in a local East-North-Up (ENh, where h represents ellipsoid height as the up direction) frame. The second is made up entirely of linear velocities (as well as their standard deviations) in geoid undulations, L[5]. IFDM2022, and its predecessor, the horizontal time-dependent positioning program, or HTDP (Snay, 1999), belong to a class of models sometimes called *crustal deformation models*. However, DGEOID2022 does not. For this reason, Smith et al. (2023) adopted the more general term *geodetic value change model*, or GVCM to group both IFDM2022, DGEOID2022 and any other such models in the future.

A quick summary of the differences between the MCPV (needed) and the GVCMs (available) is found in Table 1.

Table 1: A comparison between the MCPV and the various GVCMs from which it may be derived.

Model of Changes to Parameters (MCPV)	**Geodetic Value Change Model (GVCM)**
Has no name	IFDM2022 or DGEOID2022
Exists only as a mathematical construct for one least-squares adjustment	Exist as stand-alone models outside of an adjustment
Needed at observation points	Tend to be provided in grid form
Must reflect changes to parameters of the adjustment: either XYZ for geometric or H for orthometric	Tend to be changes to ENh (IFDM2022) or L (DGEOID2022)
Provides a single value, in the units of the appropriate parameter, for any given parameter, point and epoch.	May contain multiple types of values including *velocities* (*length/time*) and episodic displacements (*lengths*).

3.1 An historic example: The National Adjustment of 2011

The 2011 national adjustment of GPS *measured baselines*[6] estimated coordinates at passive control in the NAD 83(2011) frame at epoch 2010.00 (Dennis, 2020). It was a geometric adjustment, whose unknown parameters were Earth-centered, Earth-fixed Cartesian coordinates XYZ. The MCPV (though this term was not yet used) was derived from HTDP, a type of GVCM.

[5] Although N is the traditional variable for geoid undulation, it is already being used to mean "North" in this report.
[6] More often called *GPS vectors*, the term *measured baseline* will be used in this paper to avoid confusion with the term *vector*, which will be used exclusively in its linear algebra context, meaning a one-dimensional column array.

HTDP contains *grids* of linear velocities in the north and east direction, very few linear velocities in the up direction, and *models* of displacing *events*[7] (which can be converted into instantaneous displacement vectors), but no variances on any of these quantities. The HTDP software, which reads and uses the HTDP grids and models, is capable of applying ENh velocities over time and displacement vectors at specific epochs to form total ENh change vectors between two epochs, then rotating the total change vectors into an XYZ frame to serve as the MCPV. However, with no variances, this meant that the MCPV was treated as fixed, and thus older GPS measured baselines were not rigorously and consistently down-weighted relative to more recent ones.[8]

3.2 Changes that will come with the modernized NSRS

For the modernized NSRS, a few things will be different than in the past.

First, the effectively 2-D (East and North) HTDP will be replaced with a 3-D IFDM2022.

Second, the GVCM will not be treated as fixed, but rather will have *some* statistical information. The most obvious and available information will be in the form of grids of standard deviations of velocities, which can be squared to yield *variances*. What is less clear is the availability of *covariances* in the GVCM. Therefore, in this report we will address both the situation when covariances are known and when they are unknown in the GVCM and what that will mean for the sparsity of the dispersion matrix of the projected observations $\Sigma_{\bar{y}}$.

3.3 Covariances and their effect on sparsity

This question of sparsity is very important. As mentioned earlier, the dispersion matrix of the projected observations $\Sigma_{\bar{y}}$ is a combination of the dispersion matrix of the observations Σ_y and the dispersion matrix of the MCPV contribution to projected observations $\Sigma_{\bar{y},m}$. Since nearly every *observation* in the geometric and orthometric adjustments will be a function of the *coordinates* (parameters) at *two or three* points, that means the dispersion matrix $\Sigma_{\bar{y},m}$ should, if possible, account for the covariances between those two or three points to compute the correct dispersion for a single projected observation.

In fact, it gets much more complicated than that. In the discussion of random errors in Smith et al. (2023, Appendix A), some choices were made to ignore certain covariances. The justification given in (ibid) was to keep the dispersion (or cofactor) matrices sparse, rather than full. Herein we will go into greater detail on that exact topic.

One of the choices made in (ibid) was to ignore covariances between epochs in the MCPV. This choice is not made lightly, because, as stated in (ibid):

> "When the GVCM is mostly grids of linear velocities, error covariances *between epochs*…at the same *point* will *always* exist."

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[7] Although the only displacing events in HTDP are earthquakes, it is conceivable that other events, such as landslides or volcanic eruptions could be added to future GVCMs, and so we adopt the generic term "event" for this paper.

[8] Actually, some modifications were made to the projected observation weight matrix to account for some observations in subsidence areas, but no systematic attempt was made to propagate MCPV errors through time.

However, in (ibid) this choice was nonetheless made, so as to simplify the dispersion matrices in the ME-LSA problem. This paper revisits that choice, and provides more details on how to treat the available covariances between epochs in a practical way. We will discuss this in section 7, but as a brief glimpse of what is coming, we will conclude, in sections 5.4 and 6.4, that two uncorrelated *observations* will become two correlated *projected observations* if we rely upon a common grid of velocities from the GVCM *and* either have (a) at least one common point between the two observations or (b) have covariances between two velocities at a distance from one another.

Both of these situations have substantial implications on the sparsity of $\Sigma_{\bar{y},m}$, and thus $\Sigma_{\bar{y}}$, which is why the choice to ignore the GVCM covariances was proposed in (ibid). Some of the missing details from (ibid) have been provided herein, and a practical solution to the lack of sparsity will be discussed in section 7.

Before going any further, we briefly address the complication that GVCMs are currently available on grids, while the MCPVs are needed at observation points.

4 Interpolating the geodetic value change model (GVCM) from grid nodes to observation points

In Table 1, we pointed out the main differences between the GVCMs and the MCPV. One of those differences was *requiring* values from the MCPV at observation points while *having* values from the GVCMs on grids. This is a small complication, and one that can be easily addressed.

The two GVCMs from NGS (IFDM2022 and GEOID2022) will, for the immediate future, consist of gridded data that always come in pairs: one grid for data values and one grid for standard deviations of those data values. Most NGS products and services which use these grids will use interpolation to derive a data value at some point of interest, based on grid nodes that surround that point. As pointed out in Smith (2023), the question of properly computing the standard deviation of a value *interpolated* from grid nodes is complicated by the need for, and general lack of, knowledge about covariances between grid nodes. However, as Smith (ibid) concluded, a realistic, though always slightly pessimistic, estimation of the standard deviation of an interpolated value can be computed by applying the same interpolation scheme to the grid of standard deviations as was applied to the data grid. This can be done without any knowledge of covariances between grid nodes.

As such, though the GVCM is given on grid nodes, and the MCPV is needed at observation points, the use of interpolation from the gridded GVCM (data and standard deviation grids both) will be assumed to yield correct data values and acceptable (if not entirely correct) standard deviations of those data values at the points where the MCPV is needed.

To put it more simply, *further discussion of the GVCMs will assume that their values (and the standard deviations of those values) are available at any given point, and not just at grid nodes.*

The next two sections contain similar discussions about the relation between GVCMs and the MCPV, and the role of covariances. Three types of covariances will be discussed: those that are known in the GVCM, those that arise from frame rotations between the GVCM and MCPV (geometric adjustments only), and those that arise when two observations share common points. Each section is laid out the same, first discussing a single *point*, then a single *observation*, and finally discussing the relation between any *two observations* with zero, one or two points in common. Section 5 covers these topics for geometric adjustments, while section 6 covers them for orthometric adjustments.

5 Geometric Adjustments

Parameters estimated in geometric REC adjustments will be global Cartesian coordinates at a reference epoch, with some nuisance parameters estimated as well. *Observations* will include GNSS vectors and classical (angle/distance) observations, though the possibility of PPP coordinates as an observation type are being considered. *Constraints* will be global Cartesian coordinates at active control stations at the reference epoch.

5.1 Converting the GVCM to the MCPV at a single point

In the case of *geometric adjustments*, the equations which will convert IFDM2022 (as the GVCM) into the MCPV, for any given point (A), at epoch i, are derived in this section.

We begin by stating that IFDM2022 consists of:

- *One* set of *six* grids being three grids of linear velocities, in ENh, and three grids of their standard deviations
- *Multiple* sets of *six* grids of displacements, in ENh, (one set per *event*) each set being three grids of displacements and three grids of their standard deviations.[9]

While the grids of velocities can be easily visualized as having some non-zero value everywhere, that situation is less clear with displacements, since displacing events tend to have a finite radius within which they move the crust. Therefore, we will explicitly make what we will call, our *displacement assumption:*

> For every event, k, within the GVCM, there will be a $[\Delta E, \Delta N, \Delta h]^T$ vector, and its related vector of standard deviations[10], at *every* grid point.

Next, we define rotation matrix R_A.

$$R_A = \begin{bmatrix} -\sin \lambda_A & -\cos \lambda_A \sin \phi_A & \cos \lambda_A \cos \phi_A \\ \cos \lambda_A & -\sin \lambda_A \sin \phi_A & \sin \lambda_A \cos \phi_A \\ 0 & \cos \phi_A & \sin \phi_A \end{bmatrix} \tag{10}$$

In (10), ϕ_A and λ_A are the geodetic latitude and longitude of point A, respectively. Matrix R_A is the rotation matrix used to convert changes in a local ENh horizon system into changes in a

[9] At the time of this writing, IFDM2022 contains no standard deviations of displacements. However, because NGS plans to estimate such values in future versions, the equations which rely on them are provided.

[10] Such vectors of displacement and their standard deviations might have magnitudes of zero (especially if a point is a far distance from the epicenter of a displacement event), but in general should still exist.

global ECEF Cartesian XYZ system. It will be critical for converting any geometric GVCMs that have their various values stored in ENh into the MCPV.

Let the set of all events, k, which falls between epoch i and the adjustment epoch be called $K(i)$. Then, for each $k \in K(i)$ we can convert the ENh displacements *provided by* the GVCM into XYZ displacements, *required* to populate the MCPV in a geometric adjustment as follows:

$$\begin{bmatrix} \Delta X_{A,i} \\ \Delta Y_{A,i} \\ \Delta Z_{A,i} \end{bmatrix}_k = R_A \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta h_{A,k} \end{bmatrix} \quad \forall \; k \in K(i). \tag{11}$$

Subscript k on the left side means "contribution to the MCPV from the k^{th} event."

Unlike displacements from events, linear *velocities* must first be *converted* to displacements, by multiplying by the time span from epoch i to the adjustment epoch, and *then* rotated from ENh to XYZ, as follows:

$$\begin{bmatrix} \Delta X_{A,i} \\ \Delta Y_{A,i} \\ \Delta Z_{A,i} \end{bmatrix}_v = R_A (t_i - t_a) \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix} = \Delta t_i R_A \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix} \tag{12}$$

Subscript v means "contribution to the MCPV from velocities".

Note the replacement of $(t_i - t_a)$ with Δt_i, a shorthand for this scalar quantity which will be used throughout the rest of the paper. Note also the lack of subscript i on $\dot{E}_A$, $\dot{N}_A$ and $\dot{h}_A$. This is because these velocities are constant. We will see later that this is an important point, causing covariances between projected observations that share points. This will be discussed in sections 5.4 and 6.4.

The *total* change to the parameter values (the MCPV), for geometric adjustments, as derived from an ENh-based GVCM (IFDM2022) at grid node A, between observation epoch i and the adjustment epoch is found by combining (12) with the sum of all displacements in (11), to yield:

$$\begin{aligned} \begin{bmatrix} \Delta X_{A,i} \\ \Delta Y_{A,i} \\ \Delta Z_{A,i} \end{bmatrix} &= \begin{bmatrix} \Delta X_{A,i} \\ \Delta Y_{A,i} \\ \Delta Z_{A,i} \end{bmatrix}_v + sgn(\Delta t_i) \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta X_{A,i} \\ \Delta Y_{A,i} \\ \Delta Z_{A,i} \end{bmatrix}_k = R_A \left(\Delta t_i \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix} + sgn(\Delta t_i) \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta h_{A,k} \end{bmatrix} \right) \\ &= R_A \left(\Delta t_i \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta h_{A,k} \end{bmatrix} \right), \end{aligned} \tag{13a}$$

where

$$q_i = sgn(\Delta t_i) = \begin{cases} +1 & \text{if } (\Delta t_i) > 0 \\ -1 & \text{if } (\Delta t_i) < 0 \\ 0 & \text{if } (\Delta t_i) = 0. \end{cases} \tag{13b}$$

Because velocities and displacements are both modeled in the temporal direction of earlier-to-later[11], they must be applied negatively when the observation epoch precedes the adjustment epoch ($\Delta t_i < 0$) and positively when the adjustment epoch precedes the observation epoch ($\Delta t_i > 0$). This need to change the sign is applied automatically to the velocities, since they are multiplied by Δt_i and thus adopt the sign of Δt_i. However, no such situation happens with the displacements, which is why the $sgn(\Delta t_i)$ operator was inserted in (13). However, since inserting that operator into every equation will grow cumbersome, we set it equal to q for the rest of the paper.

The changes to parameter values seen in (13) are only half of what we need. Since the MCPV is stochastic, we also need the *dispersions* of these changes. We therefore apply the law of error propagation to (13), to arrive at the dispersion matrix Σ_{M_i} of the MCPV applied to all parameters for epoch i (see Smith et al., 2023), as follows:

$$\begin{aligned} \Sigma_{M_i} = D\left\{\begin{bmatrix} \Delta X_{A,i} \\ \Delta Y_{A,i} \\ \Delta Z_{A,i} \end{bmatrix}\right\} &= D\left\{ R_A \left(\Delta t_i \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta h_{A,k} \end{bmatrix} \right) \right\} \\ &= R_A D \left\{ \Delta t_i \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta h_{A,k} \end{bmatrix} \right\} R_A^T \\ &= R_A \left(\Delta {t_i}^2 D \left\{\begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix}\right\} + {q_i}^2 \sum_{\substack{k \\ k \in K(i)}} D \left\{\begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta h_{A,k} \end{bmatrix}\right\} \right) R_A^T \\ &= R_A \left(\Delta {t_i}^2 D \left\{\begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{h}_A \end{bmatrix}\right\} + \sum_{\substack{k \\ k \in K(i)}} D \left\{\begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta h_{A,k} \end{bmatrix}\right\} \right) R_A^T. \end{aligned} \tag{14}$$

Note that the sgn function, when squared, is always $+1$ unless applied to zero. In (14) we have assumed that covariances between *velocities* and *displacements*, and covariances between *any two displacements*, will be unknown, and therefore set to zero. Since, in this example, there are only

[11] That is, a positive velocity means that a parameter will grow larger as time goes in the forward direction. Similarly, a positive displacement will mean a parameter grows larger as time goes in the forward direction.

three unknown parameters (X_A, Y_A, Z_A), “all parameters”[12] encompasses only those three, and means that Σ_{M_i} will be a 3×3 matrix.

Equation 14 is useful because it translates the dispersions of our stochastic GVCM into dispersions of the MCPV itself, though only at a single point, A. That doesn’t get us yet to the dispersion of the contribution of the MCPV to the projected *observations*, $\Sigma_{\bar{y},m}$. To derive that, we must look at actual observations, discussed next.

5.2 Geometric Example: One observation

Unlike orthometric adjustments where only one type of observation is allowed (differential orthometric heights; see section 6), geometric adjustments have a variety of allowable observations. Some observations relate to a single point only (e.g., PPP coordinates), some relate two points (e.g., GNSS vectors, slant distances, zenith angles, geodetic azimuths, unoriented directions), and some relate three points (e.g., horizontal angles). Some come as triads of values (e.g., PPP coordinates, GNSS vectors), while the rest are single values. Some *may* be related to an unknown nuisance parameter (e.g., slant distances, zenith angles), while some *must* be related to an unknown nuisance parameter (e.g., unoriented directions).

For simplicity, we group these observations into four types:

- *1* = Related to one point, no nuisance parameters (PPP)
- *2* = Related to two points, no nuisance parameters (slant distances, zenith angles, azimuths, GNSS vectors)
- *2n* = Related to two points and one nuisance parameter (slant distances, zenith angles, unoriented directions)
- *3* = Related to three points (horizontal angles)

Each of these combinations yields a slightly different situation, with slightly different equations. As many of the details are similar between observation types, we will derive the contribution of the MCPV to the projected observations, $\Sigma_{\bar{y},m}$ for a single type of observation, specifically *2n*, (slant distances, zenith angles, unoriented directions) and extrapolate conclusions to the other observation types afterwards.

Consider, in general, one observation (of any type) made at epoch i:

$$\boldsymbol{Y_i} = a_i(\boldsymbol{\Xi_i}) + \boldsymbol{e}_{\boldsymbol{y_i}} \ \ , \ \ \boldsymbol{e}_{\boldsymbol{y_i}} \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{y}_i}\right). \tag{15}$$

As per the previous paragraphs, note that observation vector $\boldsymbol{Y_i}$ can be 1×1 or 3×1, though we may remain general about its size through most of the following derivation.

---

[12] In Smith et al. (2023), matrix Σ_{M_i} is defined as the dispersion of the changes to values of *all parameters* between epoch i and the adjustment epoch, not just the parameters at our point of interest. However, in this case, they are one and the same.

We begin by mapping observations into projected observations, as per Smith et al. (2023). In the derivation below, **bold** indicates a *vector* of values. Subscripts A, B and n refer to points A, B and the nuisance parameter, respectively. Because we are dealing with a single observation, our single observation vector, $\boldsymbol{Y_i}$, is the same as the entire observation vector $\boldsymbol{Y}$. Coefficient (or "design") matrices $\bar{A}_{i,A}$ and $\bar{A}_{i,B}$ are either 1×3 or 3×3 , while design matrix $\bar{A}_{i,n}$ is either 1×1 or 3×1.[13,14]

$$
\begin{aligned}
\overline{\boldsymbol{Y}} = \boldsymbol{Y} - \bar{A}_i \boldsymbol{\Delta P_{i,M}} &= \boldsymbol{Y} - \bar{A}_i \left\{ \begin{bmatrix} \boldsymbol{\Delta X_{A,i}} \\ \boldsymbol{\Delta X_{B,i}} \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{e_{\Delta X_{A,i}}} \\ \boldsymbol{e_{\Delta X_{B,i}}} \\ 0 \end{bmatrix} \right\} \\
&= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_{i,A} & \bar{A}_{i,B} & \bar{A}_{i,n} \end{bmatrix} \left\{ \begin{bmatrix} \boldsymbol{\Delta X_{A,i}} \\ \boldsymbol{\Delta X_{B,i}} \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{e_{\Delta X_{A,i}}} \\ \boldsymbol{e_{\Delta X_{B,i}}} \\ 0 \end{bmatrix} \right\} \\
&= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_{i,A} & \bar{A}_{i,B} & \bar{A}_{i,n} \end{bmatrix} \begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} \boldsymbol{\Delta E_{A,i}} \\ \boldsymbol{\Delta E_{B,i}} \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,i}}} \\ \boldsymbol{e_{\Delta E_{B,i}}} \\ 0 \end{bmatrix} \right\} \\
&= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_{i,A} & \bar{A}_{i,B} & \bar{A}_{i,n} \end{bmatrix} \begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \left(\Delta t_i \begin{bmatrix} \boldsymbol{\dot{E}_A} \\ \boldsymbol{\dot{E}_B} \\ 0 \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \boldsymbol{\Delta E_{A,k}} \\ \boldsymbol{\Delta E_{B,k}} \\ 0 \end{bmatrix} \right) \right. \\
&\qquad \left. + \left(\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix} \right) \right\} \\
&= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_{i,A} & \bar{A}_{i,B} & \bar{A}_{i,n} \end{bmatrix} \begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\Delta t_i \begin{bmatrix} \boldsymbol{\dot{E}_A} \\ \boldsymbol{\dot{E}_B} \\ 0 \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \boldsymbol{\Delta E_{A,k}} \\ \boldsymbol{\Delta E_{B,k}} \\ 0 \end{bmatrix} \right) \\
&\qquad - \begin{bmatrix} \bar{A}_{i,A} & \bar{A}_{i,B} & \bar{A}_{i,n} \end{bmatrix} \begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix} \right)
\end{aligned}
\tag{16}
$$

Equation 16 shows a few things of interest. First, though there is a nuisance parameter involved, it is not projected through time using the MCPV. This is clear from the zeroes in the last row of the vectors in the top line of the equation such as $\begin{bmatrix} \boldsymbol{\Delta X_{A,i}} \\ \boldsymbol{\Delta X_{B,i}} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{e_{\Delta X_{A,i}}} \\ \boldsymbol{e_{\Delta X_{B,i}}} \\ 0 \end{bmatrix}$. Second, the vector $\boldsymbol{e_{\bar{y},m}}$ of the

[13] We note that the only observations which are type 2n have a $\boldsymbol{Y_i}$ whose size is 1×1. Nonetheless, the $\boldsymbol{Y_i}$ vector in (16) will be treated as being either 1×1 or 3×1 , so that it can be extrapolated to observation type 2.

[14] In Smith et al. (2023), the variable $\boldsymbol{\Delta X}$ was used to represent changes to parameter values that were stochastic. However, this paper uses $\boldsymbol{\Delta X}$ to mean a GNSS measured baseline. As such, for clarity, in this paper stochastic changes to parameter values will be designated by $\boldsymbol{\Delta P}$ instead of $\boldsymbol{\Delta X}$. We assume that this will not cause confusion with weight matrix P.

random errors in the MCPV, as they contribute to the projected observations, can be found in the last line of the equation, and is related to the random errors in the GVCM, as follows:

$$\boldsymbol{e_{\bar{y},m}} = -[\bar{A}_{i,A} \quad \bar{A}_{i,B} \quad \bar{A}_{i,n}]\begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix}\right). \tag{17}$$

Recall that we are so far only dealing with one observation, so that $\boldsymbol{Y_i}$=$\boldsymbol{Y}$. Thus, the dispersion of the random errors in (17) will be the 3×3 dispersion matrix $\Sigma_{\bar{y},m}$. We express the dispersion of $\boldsymbol{e_{\bar{y},m}}$ as:

$$\begin{aligned}
\Sigma_{\bar{y},m} = D\{\boldsymbol{e_{\bar{y},m}}\} &= D\left\{-[\bar{A}_{i,A} \quad \bar{A}_{i,B} \quad \bar{A}_{i,n}]\begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix}\right)\right\} \\
&= [\bar{A}_{i,A} \quad \bar{A}_{i,B} \quad \bar{A}_{i,n}]\begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix} D\left\{\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix}\right\}\begin{bmatrix} R_A^T & 0 & 0 \\ 0 & R_B^T & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \bar{A}_{i,A}^T \\ \bar{A}_{i,B}^T \\ \bar{A}_{i,n}^T \end{bmatrix} \\
&= [\bar{A}_{i,A} \quad \bar{A}_{i,B} \quad \bar{A}_{i,n}]\begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(D\left\{\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix}\right\} + D\left\{\sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix}\right\}\right. \\
&\quad + \sum_{\substack{k \\ k\in K(i)}} 2C\left\{\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix}\right\} \\
&\quad \left. + \sum_{\substack{k1 \\ k1\in K(i) \\ k1\neq k2}} \sum_{\substack{k2 \\ k2\in K(i) \\ k1\neq k2}} 2C\left\{\begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k1}}} \\ \boldsymbol{e_{\Delta E_{B,k1}}} \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k2}}} \\ \boldsymbol{e_{\Delta E_{B,k2}}} \\ 0 \end{bmatrix}\right\}\right)\begin{bmatrix} R_A^T & 0 & 0 \\ 0 & R_B^T & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \bar{A}_{i,A}^T \\ \bar{A}_{i,B}^T \\ \bar{A}_{i,n}^T \end{bmatrix} \\
&= [\bar{A}_{i,A} \quad \bar{A}_{i,B} \quad \bar{A}_{i,n}]\begin{bmatrix} R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(D\left\{\Delta t_i \begin{bmatrix} \boldsymbol{e_{\dot{E}_A}} \\ \boldsymbol{e_{\dot{E}_B}} \\ 0 \end{bmatrix}\right\} + D\left\{\sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,k}}} \\ \boldsymbol{e_{\Delta E_{B,k}}} \\ 0 \end{bmatrix}\right\}\right)\begin{bmatrix} R_A^T & 0 & 0 \\ 0 & R_B^T & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \bar{A}_{i,A}^T \\ \bar{A}_{i,B}^T \\ \bar{A}_{i,n}^T \end{bmatrix}
\end{aligned} \tag{18}$$

$$= \begin{bmatrix}\bar{A}_{i,A} & \bar{A}_{i,B} & \bar{A}_{i,n}\end{bmatrix}\begin{bmatrix}R_A & 0 & 0\\ 0 & R_B & 0\\ 0 & 0 & 0\end{bmatrix}\left(\Delta t_i^{\,2}\begin{bmatrix}D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} & C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} & 0\\ C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} & D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} & 0\\ 0 & 0 & 0\end{bmatrix}\right.$$

$$\left. + \sum_{\substack{k\\ k\in K(i)}}\begin{bmatrix}D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\} & C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\} & 0\\ C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\} & D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\} & 0\\ 0 & 0 & 0\end{bmatrix}\right)\begin{bmatrix}R_A^T & 0 & 0\\ 0 & R_B^T & 0\\ 0 & 0 & 0\end{bmatrix}^T\begin{bmatrix}\bar{A}_{i,A}^T\\ \bar{A}_{i,B}^T\\ \bar{A}_{i,n}^T\end{bmatrix}$$

$$= \begin{bmatrix}\bar{A}_{i,A}R_A & \bar{A}_{i,B}R_B & 0\end{bmatrix}\left(\Delta t_i^{\,2}\begin{bmatrix}D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} & C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} & 0\\ C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} & D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} & 0\\ 0 & 0 & 0\end{bmatrix}\right.$$

$$\left. + \sum_{\substack{k\\ k\in K(i)}}\begin{bmatrix}D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\} & C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\} & 0\\ C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\} & D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\} & 0\\ 0 & 0 & 0\end{bmatrix}\right)\begin{bmatrix}R_A^T\bar{A}_{i,A}^T\\ R_B^T\bar{A}_{i,B}^T\\ 0\end{bmatrix}$$

$$= \left(\Delta t_i^{\,2}\begin{bmatrix}\bar{A}_{i,A}R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} + \bar{A}_{i,B}R_B C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} & \bar{A}_{i,A}R_A C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} + \bar{A}_{i,B}R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} & 0\end{bmatrix}\right.$$

$$\left. + \sum_{\substack{k\\ k\in K(i)}}\begin{bmatrix}\bar{A}_{i,A}R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\} + \bar{A}_{i,B}R_B C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\} & \bar{A}_{i,A}R_A C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\} + \bar{A}_{i,B}R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\} & 0\end{bmatrix}\right)\begin{bmatrix}R_A^T\bar{A}_{i,A}^T\\ R_B^T\bar{A}_{i,B}^T\\ 0\end{bmatrix}$$

$$= \Delta t_i^{\,2}\left(\bar{A}_{i,A}R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\}R_A^T\bar{A}_{i,A}^T + \bar{A}_{i,B}R_B C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\}R_A^T\bar{A}_{i,A}^T + \bar{A}_{i,A}R_A C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\}R_B^T\bar{A}_{i,B}^T + \bar{A}_{i,B}R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\}R_B^T\bar{A}_{i,B}^T\right)$$
$$+ \sum_{\substack{k\\ k\in K(i)}}\bar{A}_{i,A}R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\}R_A^T\bar{A}_{i,A}^T + \bar{A}_{i,B}R_B C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\}R_A^T\bar{A}_{i,A}^T + \bar{A}_{i,A}R_A C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\}R_B^T\bar{A}_{i,B}^T + \bar{A}_{i,B}R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\}R_B^T\bar{A}_{i,B}^T$$

Note in (18) that covariances between velocities and displacements, and between displacements and other displacements, have been set to zero as mentioned earlier. However, covariances between *velocities at different points* and *displacements at different points* remain, such as $C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}},\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\}$ or $C\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}},\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\}$. In the future, NGS may have knowledge of such covariances, but for now they will also be set equal to zero since IFDM2022 (called "IFVM2022" in NGS 2021a) currently does not contain them. This reduces (18) to:

$$\Sigma_{\bar{y},m} = \Delta t_i^{\,2}\left(\bar{A}_{i,A}R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\}R_A^T\bar{A}_{i,A}^T + \bar{A}_{i,B}R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\}R_B^T\bar{A}_{i,B}^T\right) + \sum_{\substack{k\\ k\in K(i)}}\bar{A}_{i,A}R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}}\}R_A^T\bar{A}_{i,A}^T + \bar{A}_{i,B}R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}}\}R_B^T\bar{A}_{i,B}^T. \tag{19}$$

A few things can be noted about (19). First, the fact that there is a nuisance parameter associated with this observation had no impact upon matrix $\Sigma_{\bar{y},m}$. As such, (19) is valid for both observation types *2* and *2n*. Second, (19) holds, whether the size of the a $\boldsymbol{Y_i}$ vector is 3×1 or 1×1, though (19) yields a 3×3 or 1×1 matrix, respectively, for those two cases. Third, the dispersion matrices $D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\}$ and $D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\}$ will be 3×3 *diagonal* matrices, since NGS does not have covariance information between velocities in the East and velocities in the North, etc.

Similar derivations can be performed for observation types *1* and *3*, but rather than provide full details, we simply provide the final equations, using similar assumptions and approaches.

For observation type *1*:

$$\Sigma_{\bar{y},m} = \Delta t_i^{\,2}\left(\bar{A}_{i,A} R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T \bar{A}_{i,A}^T\right) + \sum_{\substack{k \\ k \in K(i)}} \bar{A}_{i,A} R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T \bar{A}_{i,A}^T \tag{20}$$

For observation type *3*:

$$\Sigma_{\bar{y},m} = \Delta t_i^{\,2}\left(\bar{A}_{i,A} R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T \bar{A}_{i,A}^T + \bar{A}_{i,B} R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} R_B^T \bar{A}_{i,B}^T + \bar{A}_{i,C} R_C D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{C}}}\} R_C^T \bar{A}_{i,C}^T\right) + \sum_{\substack{k \\ k \in K(i)}} \bar{A}_{i,A} R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T \bar{A}_{i,A}^T + \bar{A}_{i,B} R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B},\boldsymbol{k}}}\} R_B^T \bar{A}_{i,B}^T + \bar{A}_{i,C} R_C D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{C},\boldsymbol{k}}}\} R_C^T \bar{A}_{i,C}^T \tag{21}$$

Equations 19-21 are critical. They show the relationship between the variance-covariance matrix of the GVCMs (such as IFDM2022) and the contribution of the MCPV to the variance-covariance matrix of projected observations[15].

One thing which might not be obvious in the above equations is that those geometric observations that are triplicates (PPP coordinates and GNSS vectors) that (20), for PPP, and (19), for GNSS vectors, will *always* yield a *full* 3×3 matrix $\Sigma_{\bar{y},m}$. This happens because of the presence of the rotation matrices, even if the dispersion matrices of the GVCM, such as $D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\}$, are diagonal.

Therefore, even if by some strange chance the 3×3 dispersion matrix Σ_y of the *observations* were *diagonal*[16], the addition of a full 3×3 matrix $\Sigma_{\bar{y},m}$ from (19) or (20) to it will guarantee that the 3×3 dispersion matrix $\Sigma_{\bar{y}}$ of the *projected* observations will be *full*. (Recall: we are so far only talking about a *single* observation and haven't begun discussing larger $\Sigma_{\bar{y},m}$ matrices associated with multiple observations. The full 3×3 matrix $\Sigma_{\bar{y},m}$ for this one observation would only be a single on-diagonal block of a much larger $\Sigma_{\bar{y},m}$ in an adjustment with multiple observations.)

[15] This is a good time to remember that we don't actually have variance-covariance matrices, but rather have cofactor matrices, but the conclusion is the same if one simply replaces *variance-covariance* with *cofactor*.

[16] A very unlikely situation, since most modern GNSS processing software provides, at a minimum, a 3 × 3 block-diagonal dispersion matrix for GNSS vector observations. However, some older software did yield such diagonal matrices.

We may therefore conclude that, for a vector of multiple observations ($\boldsymbol{Y}$) which contains only GNSS measured baselines or PPP coordinates, and a GVCM parameterized in an ENh system, it is *guaranteed* that the dispersion matrix $\Sigma_{\bar{y}}$ of the projected observations will be *at least* block-diagonal, with each block being 3×3.

What is more interesting, and has greater implications to the ME-LSA problem, are the *off-diagonal* blocks of $\Sigma_{\bar{y}}$ coming from the *off-diagonal* blocks of $\Sigma_{\bar{y},m}$. To explore those, we will examine a variety of cases between two observations.

For now, we will restrict ourselves solely to the case when the two observations are GNSS vectors. The reason for this will not be immediately clear, but in order to stave off any nagging questions arising at this time, we summarize why: when two otherwise *uncorrelated* observations share a point, they *always* map into *correlated projected* observations. This has the effect of putting a non-zero value into $\Sigma_{\bar{y},m}$, and thus $\Sigma_{\bar{y}}$, that otherwise was a zero value in Σ_y. Since matrix Σ_y is generally always diagonal[17] or block-diagonal with block sizes being multiples of 3×3 (for simultaneously processed GNSS vectors), the addition of new non-zero off-diagonal elements in $\Sigma_{\bar{y},m}$ and thus $\Sigma_{\bar{y}}$ changes the structure from one that is sparse (and also easy to invert) to one that is more complex (and possibly more difficult to invert). It is not at all clear that the additional information is worth the cost of significantly changing the mechanics of how the LSA is executed. Thus, we will conclude, in section 7, that the only time we need consider two observations in computing $\Sigma_{\bar{y},m}$, is when dealing with two GNSS vectors that were processed simultaneously in the same session.

Having gotten ahead of ourselves, we now return to the original discussion of the covariances between two observations, but restrict it only to the case of two GNSS measured baselines. We will look at the situations that two GNSS measured baselines share zero, one or two points.

5.3 Geometric example: Two GNSS measured baselines, no shared points

Consider one observation vector, containing two observations: one being a measured baseline from point A to point B at epoch i and the other from point C to point D at epoch j. The observation equation takes this form:

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y_i} \\ \boldsymbol{Y_j} \end{bmatrix} + \boldsymbol{e_y} \;,\;\; \boldsymbol{e_y} \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma_y}\right). \tag{22}$$

Each observation, $\boldsymbol{Y_i}$ and $\boldsymbol{Y_j}$, is a 3×1 vector. Because the observation equation for GNSS measured baselines is linear, we may expand (22) as such:

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{\Delta X_{AB,i}} \\ \boldsymbol{\Delta X_{CD,j}} \end{bmatrix} + \boldsymbol{e_y} = \bar{A} \begin{bmatrix} \boldsymbol{\Xi_i} \\ \boldsymbol{\Xi_j} \end{bmatrix} + \boldsymbol{e_y} = \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \begin{bmatrix} \boldsymbol{\Xi_i} \\ \boldsymbol{\Xi_j} \end{bmatrix} + \boldsymbol{e_y} \tag{23}$$

[17] A case might be made that *any two* observations, without regard for observation type, made by common observers, at common points, with common instruments on common days are *correlated*, and thus should have a non-zero off-diagonal element (covariance) in Σ_y. However, the only software packages which regularly provide covariances between two observations are simultaneous GNSS vector processors.

$$= \begin{bmatrix} \bar{A}_{i,A} & \bar{A}_{i,B} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_{j,A} & \bar{A}_{j,B} \end{bmatrix} \begin{bmatrix} \boldsymbol{X_{A,i}} \\ \boldsymbol{X_{B,i}} \\ \boldsymbol{X_{C,i}} \\ \boldsymbol{X_{D,i}} \\ \boldsymbol{X_{A,j}} \\ \boldsymbol{X_{B,j}} \\ \boldsymbol{X_{C,j}} \\ \boldsymbol{X_{D,j}} \end{bmatrix} + \boldsymbol{e_y}$$

$$= \begin{bmatrix} -I_3 & +I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_3 & +I_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{X_{A,i}} \\ \boldsymbol{X_{B,i}} \\ \boldsymbol{X_{C,i}} \\ \boldsymbol{X_{D,i}} \\ \boldsymbol{X_{A,j}} \\ \boldsymbol{X_{B,j}} \\ \boldsymbol{X_{C,j}} \\ \boldsymbol{X_{D,j}} \end{bmatrix} + \boldsymbol{e_y}.$$

Notice that the observations between two points at one epoch are shown as functions of the coordinates of *every* point (A, B, C, D) at *every* epoch (i, j), which is how Smith et al. (2023) developed the multi-epoch least-squares adjustment (ME-LSA) problem.

Now, according to (ibid, equation 23) the observations in (23) need to be mapped into *projected* observations, using the stochastic MCPV (based on the GVCM). This mapping yields the projected-observation equation:

$$\boldsymbol{\bar{Y}} = \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta P_{i,M}} \\ \boldsymbol{\Delta P_{j,M}} \end{bmatrix} = \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta \Xi_i + e_{M_i}} \\ \boldsymbol{\Delta \Xi_j + e_{M_j}} \end{bmatrix} = \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \boldsymbol{\Delta X_{A,i}} \\ \boldsymbol{\Delta X_{B,i}} \\ \boldsymbol{\Delta X_{C,i}} \\ \boldsymbol{\Delta X_{D,i}} \\ \boldsymbol{\Delta X_{A,j}} \\ \boldsymbol{\Delta X_{B,j}} \\ \boldsymbol{\Delta X_{C,j}} \\ \boldsymbol{\Delta X_{D,j}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{e_{\Delta X_{A,i}}} \\ \boldsymbol{e_{\Delta X_{B,i}}} \\ \boldsymbol{e_{\Delta X_{C,i}}} \\ \boldsymbol{e_{\Delta X_{D,i}}} \\ \boldsymbol{e_{\Delta X_{A,j}}} \\ \boldsymbol{e_{\Delta X_{B,j}}} \\ \boldsymbol{e_{\Delta X_{C,j}}} \\ \boldsymbol{e_{\Delta X_{D,j}}} \end{bmatrix} \right\}$$

$$= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \begin{bmatrix} R_A & & & & & \text{zeroes} & & \\ & R_B & & & & & & \\ & & R_C & & & & & \\ & & & R_D & & & & \\ & & & & R_A & & & \\ & & & & & R_B & & \\ & \text{zeroes} & & & & & R_C & \\ & & & & & & & R_D \end{bmatrix} \left\{ \begin{bmatrix} \boldsymbol{\Delta E_{A,i}} \\ \boldsymbol{\Delta E_{B,i}} \\ \boldsymbol{\Delta E_{C,i}} \\ \boldsymbol{\Delta E_{D,i}} \\ \boldsymbol{\Delta E_{A,j}} \\ \boldsymbol{\Delta E_{B,j}} \\ \boldsymbol{\Delta E_{C,j}} \\ \boldsymbol{\Delta E_{D,j}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{e_{\Delta E_{A,i}}} \\ \boldsymbol{e_{\Delta E_{B,i}}} \\ \boldsymbol{e_{\Delta E_{C,i}}} \\ \boldsymbol{e_{\Delta E_{D,i}}} \\ \boldsymbol{e_{\Delta E_{A,j}}} \\ \boldsymbol{e_{\Delta E_{B,j}}} \\ \boldsymbol{e_{\Delta E_{C,j}}} \\ \boldsymbol{e_{\Delta E_{D,j}}} \end{bmatrix} \right\}$$

$$= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} R_A \boldsymbol{\Delta E_{A,i}} \\ R_B \boldsymbol{\Delta E_{B,i}} \\ R_C \boldsymbol{\Delta E_{C,i}} \\ R_D \boldsymbol{\Delta E_{D,i}} \\ R_A \boldsymbol{\Delta E_{A,j}} \\ R_B \boldsymbol{\Delta E_{B,j}} \\ R_C \boldsymbol{\Delta E_{C,j}} \\ R_D \boldsymbol{\Delta E_{D,j}} \end{bmatrix} + \begin{bmatrix} R_A \boldsymbol{e_{\Delta E_{A,i}}} \\ R_B \boldsymbol{e_{\Delta E_{B,i}}} \\ R_C \boldsymbol{e_{\Delta E_{C,i}}} \\ R_D \boldsymbol{e_{\Delta E_{D,i}}} \\ R_A \boldsymbol{e_{\Delta E_{A,j}}} \\ R_B \boldsymbol{e_{\Delta E_{B,j}}} \\ R_C \boldsymbol{e_{\Delta E_{C,j}}} \\ R_D \boldsymbol{e_{\Delta E_{D,j}}} \end{bmatrix} \right\} \tag{24}$$

$$= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i R_A \dot{\boldsymbol{E}}_{\boldsymbol{A}} \\ \Delta t_i R_B \dot{\boldsymbol{E}}_{\boldsymbol{B}} \\ \Delta t_i R_C \dot{\boldsymbol{E}}_{\boldsymbol{C}} \\ \Delta t_i R_D \dot{\boldsymbol{E}}_{\boldsymbol{D}} \\ \Delta t_j R_A \dot{\boldsymbol{E}}_{\boldsymbol{A}} \\ \Delta t_j R_B \dot{\boldsymbol{E}}_{\boldsymbol{B}} \\ \Delta t_j R_C \dot{\boldsymbol{E}}_{\boldsymbol{C}} \\ \Delta t_j R_D \dot{\boldsymbol{E}}_{\boldsymbol{D}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \\ R_D \boldsymbol{\Delta E_{D,k}} \\ R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \\ R_D \boldsymbol{\Delta E_{D,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \\ R_D \boldsymbol{\Delta E_{D,k}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + q_j \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \\ R_D \boldsymbol{\Delta E_{D,k}} \end{bmatrix} \right\}$$

$$- \begin{bmatrix} \bar{A}_i & 0_{3,12} \\ 0_{3,12} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i R_A \boldsymbol{e_{\dot{E}_A}} \\ \Delta t_i R_B \boldsymbol{e_{\dot{E}_B}} \\ \Delta t_i R_C \boldsymbol{e_{\dot{E}_C}} \\ \Delta t_i R_D \boldsymbol{e_{\dot{E}_D}} \\ \Delta t_j R_A \boldsymbol{e_{\dot{E}_A}} \\ \Delta t_j R_B \boldsymbol{e_{\dot{E}_B}} \\ \Delta t_j R_C \boldsymbol{e_{\dot{E}_C}} \\ \Delta t_j R_D \boldsymbol{e_{\dot{E}_D}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} R_A \boldsymbol{e_{\Delta E_{A,k}}} \\ R_B \boldsymbol{e_{\Delta E_{B,k}}} \\ R_C \boldsymbol{e_{\Delta E_{C,k}}} \\ R_D \boldsymbol{e_{\Delta E_{D,k}}} \\ R_A \boldsymbol{e_{\Delta E_{A,k}}} \\ R_B \boldsymbol{e_{\Delta E_{B,k}}} \\ R_C \boldsymbol{e_{\Delta E_{C,k}}} \\ R_D \boldsymbol{e_{\Delta E_{D,k}}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} R_A \boldsymbol{e_{\Delta E_{A,k}}} \\ R_B \boldsymbol{e_{\Delta E_{B,k}}} \\ R_C \boldsymbol{e_{\Delta E_{C,k}}} \\ R_D \boldsymbol{e_{\Delta E_{D,k}}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + q_j \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ R_A \boldsymbol{e_{\Delta E_{A,k}}} \\ R_B \boldsymbol{e_{\Delta E_{B,k}}} \\ R_C \boldsymbol{e_{\Delta E_{C,k}}} \\ R_D \boldsymbol{e_{\Delta E_{D,k}}} \end{bmatrix} \right\}.$$

Note in (24) that $q_j = sgn(\Delta t_j)$ and also that for the first of the three displacing event summations, that $q_i = q_j$, since events which impact both observations must both be on the same side of the adjustment epoch, and so we choose q_i, though could just as easily have used q_j. This holds only for the first of the three displacing event summations.

Note in (24) that it was necessary to distinguish between events that affect *both* projected observations ($k \in K(i)$ and $k \in K(j)$), those that affect *only* the projected observations from epoch i ($k \in K(i)$ and $k \notin K(j)$), and those that affect *only* the projected observations from epoch j ($k \notin K(i)$and $k \in K(j)$).

Also note in (24) there are two places where random errors contribute to the projected observations. The first, subtler place, is that the observation vector $\boldsymbol{Y}$ has random errors, $\boldsymbol{e_y}$, embedded in it. That random error vector is not particularly relevant to the following derivations, which is why it was not separated from $\boldsymbol{Y}$. The other place is the last half of the last line of (24), where we see the error vector of the stochastic MCPV, as it impacts the observation vector. We have called that error vector $\boldsymbol{e_{\bar{y},m}}$, and show its relationship to the error vector of the MCPV directly, $\boldsymbol{e_M}$, as follows:

$$\boldsymbol{e}_{\overline{y},\boldsymbol{m}} = -\begin{bmatrix} \overline{A}_i & 0_{3,12} \\ 0_{3,12} & \overline{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i R_A \boldsymbol{e}_{\dot{E}_A} \\ \Delta t_i R_B \boldsymbol{e}_{\dot{E}_B} \\ \Delta t_i R_C \boldsymbol{e}_{\dot{E}_C} \\ \Delta t_i R_D \boldsymbol{e}_{\dot{E}_D} \\ \Delta t_j R_A \boldsymbol{e}_{\dot{E}_A} \\ \Delta t_j R_B \boldsymbol{e}_{\dot{E}_B} \\ \Delta t_j R_C \boldsymbol{e}_{\dot{E}_C} \\ \Delta t_j R_D \boldsymbol{e}_{\dot{E}_D} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} R_A \boldsymbol{e}_{\Delta E_{A,k}} \\ R_B \boldsymbol{e}_{\Delta E_{B,k}} \\ R_C \boldsymbol{e}_{\Delta E_{C,k}} \\ R_D \boldsymbol{e}_{\Delta E_{D,k}} \\ R_A \boldsymbol{e}_{\Delta E_{A,k}} \\ R_B \boldsymbol{e}_{\Delta E_{B,k}} \\ R_C \boldsymbol{e}_{\Delta E_{C,k}} \\ R_D \boldsymbol{e}_{\Delta E_{D,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} R_A \boldsymbol{e}_{\Delta E_{A,k}} \\ R_B \boldsymbol{e}_{\Delta E_{B,k}} \\ R_C \boldsymbol{e}_{\Delta E_{C,k}} \\ R_D \boldsymbol{e}_{\Delta E_{D,k}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + q_j \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ R_A \boldsymbol{e}_{\Delta E_{A,k}} \\ R_B \boldsymbol{e}_{\Delta E_{B,k}} \\ R_C \boldsymbol{e}_{\Delta E_{C,k}} \\ R_D \boldsymbol{e}_{\Delta E_{D,k}} \end{bmatrix} \right\} \tag{25}$$

$$= -\overline{A}\boldsymbol{e}_{\boldsymbol{M}}.$$

The relationship between the dispersion matrices of $\boldsymbol{e}_{\overline{y},\boldsymbol{m}}$ and $\boldsymbol{e}_{\boldsymbol{M}}$ is seen as follows:

$$\Sigma_{\overline{y},m} = D\{\boldsymbol{e}_{\overline{y},\boldsymbol{m}}\} = \overline{A}D\{\boldsymbol{e}_{\boldsymbol{M}}\}\overline{A}^T = \overline{A}\Sigma_M\overline{A}^T. \tag{26}$$

This is a good place to provide a reminder and clarification about the difference between the geodetic value change model (GVCM) and model of changes to parameter values (MCPV). We will do so by examining the last line of (24), replacing the displacements which affect only one projected observation with ellipses, for brevity. See Figure 1. Note we have dropped the q_i for space consideration.

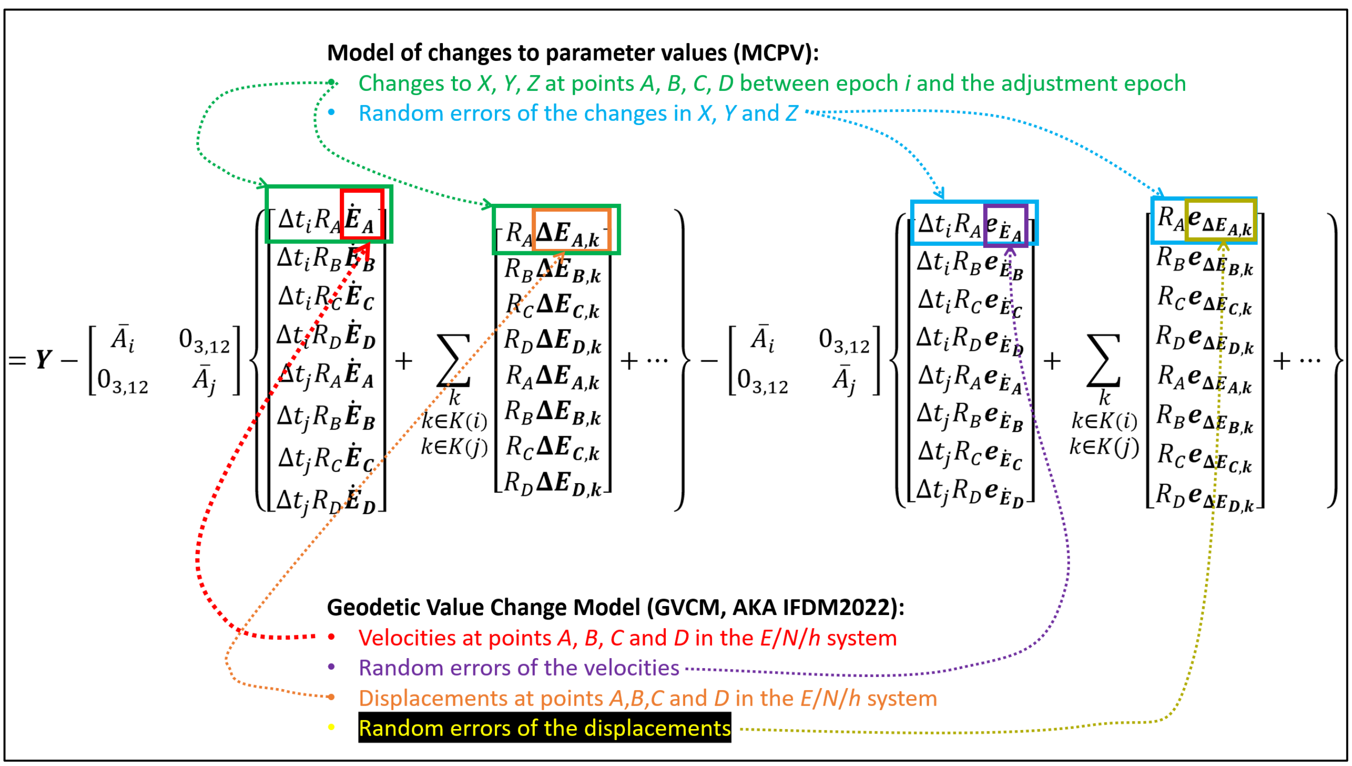

Figure 1: The relationship between the MCPV and GVCM (IFDM2022) in projected observations within a geometric adjustment

We turn our attention back to the 6×6 $\Sigma_{\bar{y},m}$ matrix. Because $\Sigma_{\bar{y},m}$ in (26) is a bit complicated, it will be examined in 3×3 blocks below.

Just examining the velocity-based contribution to $\Sigma_{\bar{y},m}$, we have:

$$\Sigma_{\bar{y},m_{v,11}} = \Delta t_i^{\,2} \Big(\bar{A}_{i,A} R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_A}\} R_A^T \bar{A}_{i,A}^{\;T} - \bar{A}_{i,B} R_B C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_B}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_A}\} R_A^T \bar{A}_{i,A}^{\;T} - \bar{A}_{i,A} R_A C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_A}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_B}\} R_B^T \bar{A}_{i,B}^{\;T} + \bar{A}_{i,B} R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_B}\} R_B^T \bar{A}_{i,B}^{\;T} \Big), \tag{27a}$$

$$\Sigma_{\bar{y},m_{v,21}} = \Delta t_i \Delta t_j \Big(\bar{A}_{j,C} R_C C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_C}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_A}\} R_A^T \bar{A}_{i,A}^{\;T} - \bar{A}_{j,D} R_D C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_D}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_A}\} R_A^T \bar{A}_{i,A}^{\;T} - \bar{A}_{j,C} R_C C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_C}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_B}\} R_B^T \bar{A}_{i,B}^{\;T} + \bar{A}_{j,D} R_D C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_D}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_B}\} R_B^T \bar{A}_{i,B}^{\;T} \Big), \tag{27b}$$

$$\Sigma_{\bar{y},m_{v,12}} = \Delta t_i \Delta t_j \Big(\bar{A}_{i,A} R_A C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_A}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_C}\} R_C^T \bar{A}_{j,C}^{\;T} - \bar{A}_{i,A} R_A C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_A}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_D}\} R_D^T \bar{A}_{j,D}^{\;T} - \bar{A}_{i,B} R_B C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_B}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_C}\} R_C^T \bar{A}_{j,C}^{\;T} + \bar{A}_{i,B} R_B C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_B}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_D}\} R_D^T \bar{A}_{j,D}^{\;T} \Big), \tag{27c}$$

$$\Sigma_{\bar{y},m_{v,22}} = \Delta t_i^{\,2} \Big(\bar{A}_{j,C} R_C D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_C}\} R_C^T \bar{A}_{j,C}^{\;T} - \bar{A}_{j,C} R_C C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_C}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_D}\} R_D^T \bar{A}_{j,D}^{\;T} - \bar{A}_{j,D} R_D C\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_D}, \boldsymbol{e}_{\dot{\boldsymbol{E}}_C}\} R_C^T \bar{A}_{j,C}^{\;T} + \bar{A}_{j,D} R_D D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_D}\} R_D^T \bar{A}_{j,D}^{\;T} \Big). \tag{27d}$$

The next three sets of equations show the blocks for the *displacement* errors.

First, for any $k \in K(i)$ and $k \in K(j)$ we have:

$$\Sigma_{\bar{y},m_{k,11}} = R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}\} R_A^T - R_B C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}\} R_A^T - R_A C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}\} R_B^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}\} R_B^T, \tag{27e}$$

$$\Sigma_{\bar{y},m_{k,21}} = R_C C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{C,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}\} R_A^T - R_D C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{D,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}\} R_A^T - R_C C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{C,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}\} R_B^T + R_D C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{D,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}\} R_B^T, \tag{27f}$$

$$\Sigma_{\bar{y},m_{k,12}} = R_A C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{C,k}}\} R_C^T - R_B C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{C,k}}\} R_C^T - R_A C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{D,k}}\} R_D^T + R_B C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{D,k}}\} R_D^T, \tag{27g}$$

$$\Sigma_{\bar{y},m_{k,22}} = R_C D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{C,k}}\} R_C^T - R_D C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{D,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{C,k}}\} R_C^T - R_C C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{C,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{D,k}}\} R_D^T + R_D D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{D,k}}\} R_D^T. \tag{27h}$$

Second, for any $k \in K(i)$ and $k \notin K(j)$ we have:

$$\Sigma_{\bar{y},m_{k,11}} = R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}\} R_A^T - R_B C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}\} R_A^T - R_A C\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{A,k}}, \boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}\} R_B^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{B,k}}\} R_B^T, \tag{27i}$$

$$\Sigma_{\bar{y},m_{k,21}} = 0, \tag{27j}$$

$$\Sigma_{\bar{y},m_{k,12}} = 0, \tag{27k}$$

$$\Sigma_{\bar{y},m_{k,22}} = 0. \tag{27l}$$

And third, for any $k \notin K(i)$ and $k \in K(j)$ we have:

$$\Sigma_{\bar{y},m_{k,11}} = 0, \tag{27m}$$

$$\Sigma_{\bar{y},m_{k,21}} = 0, \tag{27n}$$

$$\Sigma_{\bar{y},m_{k,12}} = 0, \tag{27o}$$

$$\Sigma_{\bar{y},m_{k,22}} = R_C D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{C,k}}\} R_C^T - R_D C\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{D,k}}, \boldsymbol{e}_{\Delta \boldsymbol{E}_{C,k}}\} R_C^T - R_C C\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{C,k}}, \boldsymbol{e}_{\Delta \boldsymbol{E}_{D,k}}\} R_D^T + R_D D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{D,k}}\} R_D^T. \tag{27p}$$

Equation 27a (and 27d by replacing points C and D with A and B) is identical to the velocity contribution at the end of (18). This is expected, as (18) deals with *one* observation, as does the block matrices found in (27a) and (27d). Similarly do equation 27e or 27i (and 27h or 27p), equal the displacement portion of (18).

However, something new is seen in (27) that hasn't been encountered before (aside from separating out displacements into three categories, depending on what points they effect): the non-zero off-diagonal blocks $\Sigma_{\bar{y},m_{v,21}}$ and $\Sigma_{\bar{y},m_{v,12}}$ seen in (27b) and (27c) and $\Sigma_{\bar{y},m_{k,21}}$ and $\Sigma_{\bar{y},m_{k,12}}$ seen in (27f) and (27g). Blocks $\Sigma_{\bar{y},m_{v,21}}$ and $\Sigma_{\bar{y},m_{v,12}}$ are transposes of one another, as are $\Sigma_{\bar{y},m_{k,21}}$ and $\Sigma_{\bar{y},m_{k,12}}$, an expected result since the matrix $\Sigma_{\bar{y},m}$ should be symmetric. But more importantly they are non-zero because it has been assumed that covariances within the GVCM (between velocities at different points or between displacements at different points) are known. This brings us to our first conclusion about covariances and observations:

> *Conclusion 1: If covariances inside the GVCM are* known*, and even if two observations have* no common points*, then* $\Sigma_{\bar{y},m}$ will have non-zero off-diagonal components.

This is a critical point, and one with devastating computational impacts. Recall that if two observations have no common points, and are themselves otherwise uncorrelated, there will be no off-diagonal components in matrix Σ_y for these two observations. But if we have covariances inside the GVCM, this adds off-diagonal elements to $\Sigma_{\bar{y},m}$ and thus off-diagonal elements in $\Sigma_{\bar{y}}$, where such off-diagonal blocks were zero in Σ_y. This is not just a loss of a loss of sparsity, it means the $\Sigma_{\bar{y},m}$, and by extension $\Sigma_{\bar{y}}$, will be *full*. The implications of that are hopefully obvious, but will be discussed in section 7.

At this time, however, IFDM2022 has no known covariances. As such, if we set them to zero, then the dispersion matrix $\Sigma_{\bar{y},m}$ simplifies substantially to:

$$
\Sigma_{\bar{y},m} = \left(\begin{bmatrix} \Delta t_i{}^2 R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_A}\} R_A^T + \Delta t_i{}^2 R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_B}\} R_B^T & 0 \\ 0 & \Delta t_j{}^2 R_C D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_C}\} R_C^T + \Delta t_j{}^2 R_D D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_D}\} R_D^T \end{bmatrix} \right.
$$

$$
+ \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} R_A D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{A,k}}\} R_A^T + R_B D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{B,k}}\} R_B^T & 0 \\ 0 & R_C D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{C,k}}\} R_C^T + R_D D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{D,k}}\} R_D^T \end{bmatrix}
$$

$$
+ \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} R_A D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{A,k}}\} R_A^T + R_B D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{B,k}}\} R_B^T & 0 \\ 0 & 0 \end{bmatrix} \tag{28}
$$

$$
\left. + \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & R_C D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{C,k}}\} R_C^T + R_D D\{\boldsymbol{e}_{\Delta \boldsymbol{E}_{D,k}}\} R_D^T \end{bmatrix} \right).
$$

Examination of (28) leads to another very important conclusion:

> *Conclusion 2: If covariances inside the GVCM are unknown, and* set to zero*, and two observations have* no common points*, then $\Sigma_{\bar{y},m}$ will have* no off-diagonal components*.*

This conclusion actually holds for all geometric observations, not just GNSS vectors. It also holds whether the two GNSS vectors were processed in different sessions ($i \neq j$, and thus have no known correlation/covariance) or simultaneously processed in the same session ($i = j$, and thus *do* have a known correlation/covariance).

Consider what this means to the sparsity of $\Sigma_{\bar{y}}$: if we assume that two GNSS measured baselines at two different epochs, i and j, are themselves uncorrelated then the 6×6 dispersion matrix Σ_y of the original two observations is *block-diagonal*. With a stochastic GVCM, but in the absence of covariances in that GVCM, and with no common points between the observations, (28) shows that the 6×6 dispersion matrix $\Sigma_{\bar{y},m}$ will also be *block-diagonal*. This means their sum, $\Sigma_{\bar{y}}$, the 6×6 dispersion matrix of the two *projected* observations at the adjustment epoch, will also be *block-diagonal*. That is, the sparsity of $\Sigma_{\bar{y}}$ matches that of Σ_y. The next section will examine the case when the two GNSS measured baselines have one point in common.

5.4 Geometric example: Two GNSS measured baselines, one shared point

Consider now two GNSS measured baselines are from point A to point B at epoch i and another from point A to point C at epoch j, removing point D from consideration entirely. As this situation closely resembles the previous section, the derivations will be much briefer.

The observation equation is:

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{\Delta X_{AB,i}} \\ \boldsymbol{\Delta X_{AC,j}} \end{bmatrix} + \boldsymbol{e_y} \quad , \quad \boldsymbol{e_y} \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma_y}\right). \tag{29}$$

This expands to:

$$\boldsymbol{Y} = \begin{bmatrix} -I_3 & +I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_3 & 0 & +I_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{X_{A,i}} \\ \boldsymbol{X_{B,i}} \\ \boldsymbol{X_{C,i}} \\ \boldsymbol{X_{A,j}} \\ \boldsymbol{X_{B,j}} \\ \boldsymbol{X_{C,j}} \end{bmatrix} + \boldsymbol{e_y}. \tag{30}$$

This is then mapped into a *projected* observation vector, yielding:

$$\begin{aligned} \overline{\boldsymbol{Y}} = \boldsymbol{Y} - \begin{bmatrix} \overline{A}_i & 0_{3,9} \\ 0_{3,9} & \overline{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i R_A \dot{\boldsymbol{E}}_{\boldsymbol{A}} \\ \Delta t_i R_B \dot{\boldsymbol{E}}_{\boldsymbol{B}} \\ \Delta t_i R_C \dot{\boldsymbol{E}}_{\boldsymbol{C}} \\ \Delta t_j R_A \dot{\boldsymbol{E}}_{\boldsymbol{A}} \\ \Delta t_j R_B \dot{\boldsymbol{E}}_{\boldsymbol{B}} \\ \Delta t_j R_C \dot{\boldsymbol{E}}_{\boldsymbol{C}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \\ R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + q_j \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ R_A \boldsymbol{\Delta E_{A,k}} \\ R_B \boldsymbol{\Delta E_{B,k}} \\ R_C \boldsymbol{\Delta E_{C,k}} \end{bmatrix} \right\} \\ - \begin{bmatrix} \overline{A}_i & 0_{3,9} \\ 0_{3,9} & \overline{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i R_A \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}} \\ \Delta t_i R_B \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}} \\ \Delta t_i R_C \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{C}}} \\ \Delta t_j R_A \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}} \\ \Delta t_j R_B \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}} \\ \Delta t_j R_C \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{C}}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} R_A \boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}} \\ R_C \boldsymbol{e}_{\boldsymbol{\Delta E_{C,k}}} \\ R_A \boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}} \\ R_C \boldsymbol{e}_{\boldsymbol{\Delta E_{C,k}}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} R_A \boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}} \\ R_C \boldsymbol{e}_{\boldsymbol{\Delta E_{C,k}}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + q_j \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ R_A \boldsymbol{e}_{\boldsymbol{\Delta E_{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E_{B,k}}} \\ R_C \boldsymbol{e}_{\boldsymbol{\Delta E_{C,k}}} \end{bmatrix} \right\}. \end{aligned} \tag{31}$$

The error vector $\boldsymbol{e}_{\overline{\boldsymbol{y}},\boldsymbol{m}}$ of the MCPV contribution to projected observations is the last line in (31). As before, the dispersion matrix, $\Sigma_{\overline{y},m}$, is a bit complicated if the covariances in the GVCM are known. As this is not the case for IFDM2022, and such complications will only form a distraction going forward, we will assume that the covariances in the GVCM are *not* known, and will be set to zero. In that case, the dispersion matrix $\Sigma_{\overline{y},m}$ simplifies to:

$$
\begin{aligned}
\Sigma_{\bar{y},m} = \Bigg(& \begin{bmatrix} \Delta t_i{}^2 R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T + \Delta t_i{}^2 R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} R_B^T & \Delta t_i \Delta t_j R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T \\ \Delta t_i \Delta t_j R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T & \Delta t_j{}^2 R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T + \Delta t_j{}^2 R_C D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{C}}}\} R_C^T \end{bmatrix} \\
& + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B},\boldsymbol{k}}}\} R_B^T & R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T \\ R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T & R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T + \Delta t_j{}^2 R_C D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{C}}}\} R_C^T \end{bmatrix} \\
& + \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B},\boldsymbol{k}}}\} R_B^T & 0 \\ 0 & 0 \end{bmatrix} \\
& + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A},\boldsymbol{k}}}\} R_A^T + \Delta t_j{}^2 R_C D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{C}}}\} R_C^T \end{bmatrix} \Bigg).
\end{aligned}
\tag{32}
$$

Of special interest are the non-zero off-diagonal blocks in the first two lines of (32). From (32) we draw a new, and critical conclusion.

> *Conclusion 3: If covariances inside the GVCM are unknown, and* set to zero*, and two observations have* one common point*, then $\Sigma_{\bar{y},m}$ will have* non-zero off-diagonal components*.*

The change in dispersion matrix $\Sigma_{\bar{y},m}$ from block-diagonal (28) to full (32) is due entirely to the common point, A, between the two observations, and thus the reliance of both observations upon the same velocities and/or at least one common displacement at point A.

The implications of this are important. When two GNSS measured baselines (*AB* and *AC*) occur at different epochs (i and j), most standard GNSS processing software will treat them as two *independent observations*. But the *projected* observations will *not* be independent, as evidenced by the non-zero off-diagonal blocks in (32). To put it simply, if the *observations* are independent, their dispersion matrix Σ_y is expected to be block diagonal, but the dispersion matrix $\Sigma_{\bar{y},m}$ of the contribution of the MCPV to the projected observations is *full*, and therefore their sum, being the dispersion of projected observations $\Sigma_{\bar{y}}$ must also be *full*.

The non-zero off-diagonal blocks of $\Sigma_{\bar{y}}$ (the covariances between projected observations) will therefore be identical to those same blocks in $\Sigma_{\bar{y},m}$. These non-zero blocks are potentially problematic, and will be discussed in detail in section 7.

Before doing so, however, we ask this question: What if the two vectors had been at the *same* epoch? Would that still present a problem? The answer is "probably not", and is discussed in the next sub-section.

5.4.1 Special case: observations at the same epoch

Consider the above derivation of $\Sigma_{\bar{y},m}$, but with both vectors at observation epoch i. First, we must consider whether the dispersion matrix of the observations Σ_y is full or block-diagonal. The answer depends on the processing strategy. If the GNSS data are processed *simultaneously*, then Σ_y will be a *full* 6×6 matrix, not block-diagonal. However, even if the vectors are processed *sequentially*, standard operating procedure in many GNSS software suites is to process all measured baselines between all points that have a GNSS occupation (in this case AB, AC *and* BC), getting one 3×3 dispersion matrix for each measured baseline (which would form a 9×9 block-diagonal Σ_y). However, since one of the measured baselines is redundant, the *three* separately computed measured baselines and their respective 3×3 dispersion matrices can be mathematically combined into *two* correlated vectors with a full 6×6 dispersion matrix for the pair. Therefore, in general, it can be assumed that the dispersion matrix for the two observations Σ_y, assuming they both occur at epoch i, will be *full*.

Since the Σ_y matrix is full, adding a full $\Sigma_{\bar{y},m}$ matrix does not add any new non-zero blocks to $\Sigma_{\bar{y}}$. Put another way, if your least squares adjustment software is capable of handling some off-diagonal blocks of Σ_y within one session, it can certainly handle them from $\Sigma_{\bar{y}}$.

So, generally speaking, two measured baselines with a common point only presents new non-zero off-diagonal blocks to the LSA if the baselines were measured at *different* epochs. This will be discussed in section 7. We turn now to one final case: when the two measured baselines have *both* points in common.

5.5 Geometric example: Two GNSS measured baselines, two shared points

Consider one observation vector, containing two GNSS observations, one with a measured baseline from point A to point B at epoch i and another from point A to point B at epoch j, removing point C from consideration. The observation equation is:

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{\Delta X_{AB,i}} \\ \boldsymbol{\Delta X_{AB,j}} \end{bmatrix} + \boldsymbol{e_y} \quad , \quad \boldsymbol{e_y} \sim (\boldsymbol{0}, \boldsymbol{\Sigma_y}). \tag{33}$$

This expands to:

$$\boldsymbol{Y} = \begin{bmatrix} -I_3 & +I_3 & 0 & 0 \\ 0 & 0 & -I_3 & +I_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{X_{A,i}} \\ \boldsymbol{X_{B,i}} \\ \boldsymbol{X_{A,j}} \\ \boldsymbol{X_{B,j}} \end{bmatrix} + \boldsymbol{e_y}. \tag{34}$$

This is then mapped into a *projected* observation vector, yielding:

$$\overline{\boldsymbol{Y}} = \boldsymbol{Y} - \begin{bmatrix} \overline{A}_i & 0_{3,6} \\ 0_{3,6} & \overline{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i R_A \dot{\boldsymbol{E}}_{\boldsymbol{A}} \\ \Delta t_i R_B \dot{\boldsymbol{E}}_{\boldsymbol{B}} \\ \Delta t_j R_A \dot{\boldsymbol{E}}_{\boldsymbol{A}} \\ \Delta t_j R_B \dot{\boldsymbol{E}}_{\boldsymbol{B}} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} R_A \boldsymbol{\Delta E}_{\boldsymbol{A,k}} \\ R_B \boldsymbol{\Delta E}_{\boldsymbol{B,k}} \\ R_A \boldsymbol{\Delta E}_{\boldsymbol{A,k}} \\ R_B \boldsymbol{\Delta E}_{\boldsymbol{B,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} R_A \boldsymbol{\Delta E}_{\boldsymbol{A,k}} \\ R_B \boldsymbol{\Delta E}_{\boldsymbol{B,k}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + q_j \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ R_A \boldsymbol{\Delta E}_{\boldsymbol{A,k}} \\ R_B \boldsymbol{\Delta E}_{\boldsymbol{B,k}} \end{bmatrix} \right\}$$
$$- \begin{bmatrix} \overline{A}_i & 0_{3,6} \\ 0_{3,6} & \overline{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i R_A \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}} \\ \Delta t_i R_B \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}} \\ \Delta t_j R_A \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}} \\ \Delta t_j R_B \boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} R_A \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}} \\ R_A \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} R_A \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + q_j \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ R_A \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}} \\ R_B \boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}} \end{bmatrix} \right\}. \tag{35}$$

The error vector $\boldsymbol{e}_{\overline{\boldsymbol{y}},\boldsymbol{m}}$ of the MCPV contribution to projected observations is the last line in (35). We continue as before, assuming unknown covariances in the GVCM, and setting them to zero. In that case, the dispersion matrix $\Sigma_{\overline{y},m}$ simplifies to:

$$\Sigma_{\overline{y},m} = \Bigg(\begin{bmatrix} \Delta t_i{}^2 R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T + \Delta t_i{}^2 R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} R_B^T & \Delta t_i \Delta t_j R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T + \Delta t_i \Delta t_j R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} R_B^T \\ \Delta t_i \Delta t_j R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T + \Delta t_i \Delta t_j R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} R_B^T & \Delta t_j{}^2 R_A D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{A}}}\} R_A^T + \Delta t_j{}^2 R_B D\{\boldsymbol{e}_{\dot{\boldsymbol{E}}_{\boldsymbol{B}}}\} R_B^T \end{bmatrix}$$
$$+ \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}}\} R_B^T & R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}}\} R_B^T \\ R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}}\} R_B^T & R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}}\} R_B^T \end{bmatrix} \tag{36}$$
$$+ \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}}\} R_B^T & 0 \\ 0 & 0 \end{bmatrix} + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & R_A D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{A,k}}}\} R_A^T + R_B D\{\boldsymbol{e}_{\boldsymbol{\Delta E}_{\boldsymbol{B,k}}}\} R_B^T \end{bmatrix} \Bigg).$$

As was the case in (32), we once again find non-zero elements in the off-diagonal blocks of $\Sigma_{\overline{y},m}$. However, we note certain patterns in the matrices of (36) that are known to cause matrix singularities. We point out that (36) can be written in this generalized form:

$$\Sigma_{\overline{y},m} = \left(\begin{bmatrix} a^2 H & abH \\ abH & b^2 H \end{bmatrix} + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} H_k & H_k \\ H_k & H_k \end{bmatrix} + \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} H_k & 0 \\ 0 & 0 \end{bmatrix} + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & H_k \end{bmatrix} \right). \tag{37}$$

We examine the implications of these patterns in Appendix A. Those implications lead to this conclusion:

Conclusion 4: If covariances inside the GVCM are unknown, and set to zero*, and two observations have* two common points*, then $\Sigma_{\bar{y},m}$ will have* non-zero off-diagonal components*, and under certain conditions, may be singular.*

As pointed out in the appendix, the singularity of $\Sigma_{\bar{y},m}$ is not guaranteed. But the fullness of $\Sigma_{\bar{y},m}$ is guaranteed by (36), and thus will have the same impact on the sparseness of $\Sigma_{\bar{y}}$ as in the previous section. The potential difficulties from this loss of sparsity will be discussed in section 7.

6 Orthometric Adjustments

Parameters estimated in orthometric REC adjustments will be orthometric heights at a reference epoch. *Observations* will include differential orthometric heights reduced from leveling and classical surveys. *Constraints* will be orthometric heights at passive geodetic control marks, derived from ellipsoid height RECs (from a geometric REC adjustment) and GEOID2022.

Much of this section parallels that of the geometric adjustments section, so when possible certain details will be skipped.

6.1 Converting the GVCM to the MCPV at a single point

In the case of *orthometric adjustments*, the equations which will convert IFDM2022 and DGEOID2022 (as the *two* GVCMs) into the MCPV, for any given point (A), at epoch i, are derived below.

We use the same IFDM2022 as in geometric adjustments, but restrict ourselves solely to the ellipsoid height information therein, that is:

- *One* set of *two* grids containing one grid of linear velocities, in h, and one grid of their standard deviations
- *Multiple* sets of *two* grids of displacements, in h, (one set per event) each set containing one grid of displacements and one grid of their standard deviations.

The DGEOID2022 model will contain:

- *One* set of *two* grids containing one grid of linear velocities, in L, and one grid of their standard deviations

There are currently no displacing events as part of the DGEOID2022 model, nor plans to introduce them.

Orthometric adjustments will have a few noteworthy differences relative to geometric adjustments. First, two GVCMs (IFDM2022 and DGEOID2022) must be considered rather than just one. Second, the adjustment is one-dimensional, not three-dimensional. And finally, orthometric heights can be assumed parallel with ellipsoid heights and geoid undulations, which means *no rotation matrix is needed* to relate our MCPV to our GVCMs.

We begin with the equation for an orthometric height at some grid node, A:

$$H_A = h_A - L_A. \tag{38}$$

As before, we let the set of all events, k, which falls between epoch i and the adjustment epoch be called $K(i)$. Then, for each $k \in K(i)$, the orthometric height change is identical to the ellipsoid height change, since DGEOID2022 will contain no events. Thus, we have:

$$\left[\Delta H_{A,i}\right]_k = \left[\Delta h_{A,k}\right] - \left[\Delta L_{A,k}\right] = \left[\Delta h_{A,k}\right] \;\; \forall \;\; k \in K(i). \tag{39}$$

As before, subscript k means "contribution from the k^{th} displacement." Note the purposeful use of brackets. Though the values in (39) are 1×1 vectors, they will be related to larger vectors and matrices soon, and so it is worth remembering that these are vectors.

Again, *velocities* must first be converted to *displacements*, by multiplying by the time span from epoch i to the adjustment epoch. In this case, both IFDM2022 and DGEOID2022 will contain velocities. The conversion is:

$$\left[\Delta H_{A,i}\right]_v = \Delta t_i \left[\dot{h}_A - \dot{L}_A\right]. \tag{40}$$

Subscript v means "contribution from velocities", and we note the lack of subscript i on both $\dot{h}_A$ and $\dot{L}_A$. Again, this means there is a common field of velocities that are treated as constant through time.

The *total* change to the parameter values (the MCPV), for orthometric adjustments, as derived from two GVCMs at grid node A, between observation epoch i and the adjustment epoch is found by combining (40) with the sum of all displacements in (39), to yield:

$$\left[\Delta H_{A,i}\right] = \Delta t_i \left[\dot{h}_A - \dot{L}_A\right] + q_i \sum_{\substack{k \\ k \in K(i)}} \left[\Delta h_{A,k}\right]. \tag{41}$$

As in the geometric adjustment, we are also interested in the dispersions of the changes to parameter values. To get them, we apply the law of error propagation to (41) to arrive at the dispersion matrix Σ_{M_i} of the MCPV applied to all parameters for epoch i as follows:

$$\Sigma_{M_i} = D\{[\Delta H_{A,i}]\} = D\left\{\Delta t_i[\dot{h}_A - \dot{L}_A] + q_i \sum_{\substack{k \\ k\in K(i)}} [\Delta h_{A,k}]\right\}$$
$$= \left(\Delta t_i{}^2\left(D\{[\dot{h}_A]\} + D\{[\dot{L}_A]\}\right) \sum_{\substack{k \\ k\in K(i)}} D\{[\Delta h_{A,k}]\}\right). \tag{42}$$

As in geometric adjustments we assume no covariances between velocities and displacements nor between two different displacements. However, we *also* assume no covariances between ellipsoid height velocities and geoid undulation velocities[18]. In (42), there is only one unknown parameter H_A, so "all parameters" encompasses only that one, and means Σ_{M_i} will be a 1×1 matrix. That doesn't get us yet to the dispersion of the contribution of the MCPV to the projected observations, $\Sigma_{\bar{y},m}$. To derive that, as before, we now examine the cases of one observation, and then two observations that share zero, one and two points respectively.

6.2 Orthometric Example: One orthometric height difference

This and the three following sections have close parallels with sections 5.2 through 5.5, which covered GNSS observations. As such, some common details may be skipped.

Consider an observation vector $\boldsymbol{Y}$ consisting of a single orthometric height difference, between two points A and B, at observation epoch i. The observation equation is:

$$\boldsymbol{Y} = [\Delta H_{AB,i}] + \boldsymbol{e_y} \quad , \quad \boldsymbol{e_y} \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma_y}\right). \tag{43}$$

Now we must map observations to projected observations, as:

[18] There can be a correlation, such as in the Hudson Bay where a substantial amount of mass is actually entering the region via the mantle, causing both a land uplift ($\dot{h}_A$) and geoid change ($\dot{L}_A$). However, this is more the exception than the rule. Often, local uplift or subsidence is due to small mass changes or no mass change (such as compaction), which would yield no discernable $\dot{L}_A$ value at all. Regardless, the ability to compute such a correlation is difficult, and for the purposes of this paper, the assumption of non-correlation will be used. Any researchers expanding this work may wish to consider the correlation between $\dot{h}_A$ and $\dot{L}_A$.

$$
\begin{aligned}
\overline{\boldsymbol{Y}} = \boldsymbol{Y} - \bar{A}_i \boldsymbol{\Delta P_{i,M}} &= \boldsymbol{Y} - \bar{A}_i \left\{ \begin{bmatrix} \Delta H_{A,i} \\ \Delta H_{B,i} \end{bmatrix} + \begin{bmatrix} e_{\Delta H_{A,i}} \\ e_{\Delta H_{B,i}} \end{bmatrix} \right\} \\
&= \boldsymbol{Y} - \bar{A}_i \left\{ \left(\begin{bmatrix} \Delta h_{A,i} \\ \Delta h_{B,i} \end{bmatrix} - \begin{bmatrix} \Delta L_{A,i} \\ \Delta L_{B,i} \end{bmatrix} \right) + \left(\begin{bmatrix} e_{\Delta h_{A,i}} \\ e_{\Delta h_{B,i}} \end{bmatrix} - \begin{bmatrix} e_{\Delta L_{A,i}} \\ e_{\Delta L_{B,i}} \end{bmatrix} \right) \right\} \\
&= \boldsymbol{Y} - \bar{A}_i \left\{ \Delta t_i \begin{bmatrix} \dot{h}_A \\ \dot{h}_B \end{bmatrix} - \Delta t_i \begin{bmatrix} \dot{L}_A \\ \dot{L}_B \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta h_{A,k} \\ \Delta h_{B,k} \end{bmatrix} \right\} \\
&\quad - \bar{A}_i \left\{ \Delta t_i \begin{bmatrix} e_{\dot{h}_A} \\ e_{\dot{h}_B} \end{bmatrix} - \Delta t_i \begin{bmatrix} e_{\dot{L}_A} \\ e_{\dot{L}_B} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \end{bmatrix} \right\}
\end{aligned}
\tag{44}
$$

Compare (44) with (16) and note that the one-dimensional nature of the adjustment means we need not rely as heavily upon **bold** to indicate vectors, but rather can write out each element of each vector or matrix as is.

The vector $\boldsymbol{e_{\bar{y},m}}$, can be found in the last line of (44), as follows:

$$
\boldsymbol{e_{\bar{y},m}} = -\bar{A}_i \left\{ \Delta t_i \begin{bmatrix} e_{\dot{h}_A} \\ e_{\dot{h}_B} \end{bmatrix} - \Delta t_i \begin{bmatrix} e_{\dot{L}_A} \\ e_{\dot{L}_B} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \end{bmatrix} \right\}
\tag{45}
$$

We express the dispersion of $\boldsymbol{e_{\bar{y},m}}$ as:

$$
\begin{aligned}
\Sigma_{\bar{y},m} = D\{\boldsymbol{e_{\bar{y},m}}\} &= D\left\{ -\bar{A}_i \left\{ \Delta t_i \begin{bmatrix} e_{\dot{h}_A} \\ e_{\dot{h}_B} \end{bmatrix} - \Delta t_i \begin{bmatrix} e_{\dot{L}_A} \\ e_{\dot{L}_B} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \end{bmatrix} \right\} \right\} \\
&= [-1 \quad +1](\Delta t_i)^2 \begin{bmatrix} D\{e_{\dot{h}_A}\} & C\{e_{\dot{h}_A}, e_{\dot{h}_B}\} \\ C\{e_{\dot{h}_B}, e_{\dot{h}_A}\} & D\{e_{\dot{h}_B}\} \end{bmatrix} [-1 \quad +1]^T \\
&\quad + [-1 \quad +1](\Delta t_i)^2 \begin{bmatrix} \sigma^2_{e_{\dot{L}_A}} & \sigma_{e_{\dot{L}_A}, e_{\dot{L}_B}} \\ \text{sym} & \sigma^2_{e_{\dot{L}_B}} \end{bmatrix} [-1 \quad +1]^T \\
&\quad + [-1 \quad +1] \left(\sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \sigma^2_{e_{\Delta h_{A,k}}} & \sigma_{e_{\Delta h_{A,k}}, e_{\Delta h_{B,k}}} \\ \text{sym} & \sigma^2_{e_{\Delta h_{B,k}}} \end{bmatrix} \right) [-1 \quad +1]^T \\
&= (\Delta t_i)^2 \left[D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} - 2C\{e_{\dot{h}_A}, e_{\dot{h}_B}\} \right] \\
&\quad + (\Delta t_i)^2 \left[D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\} - 2C\{e_{\dot{L}_A}, e_{\dot{L}_B}\} \right] \\
&\quad + \sum_{\substack{k \\ k \in K(i)}} \left[D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} - 2C\{e_{\Delta h_{A,k}}, e_{\Delta h_{B,k}}\} \right].
\end{aligned}
\tag{46}
$$

Note in (46) that covariances between velocities and displacements, and between displacements and other displacements, and between geoid velocities and ellipsoid height velocities have all been set to zero as mentioned earlier. However, covariances between *velocities at different points* and *events at different points* remain. In the future, NGS may have knowledge of such covariances, but for now they will also be set equal to zero since neither IFDM2022 nor DGEOID2022 contain them. This reduces (46) to:

$$\Sigma_{\bar{y},m} = \Delta t_i^{\,2}\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\}\right) + \Delta t_i^{\,2}\left(D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) + \sum_{\substack{k \\ k\in K(i)}} \left(D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\}\right). \tag{47}$$

In (47) we have kept the three contributions (ellipsoid height velocities, geoid height velocities and ellipsoid height displacements) separate for ease of understanding.

We now turn to the same three situations which arose for geometric adjustments, but using orthometric height differences: no points in common, one point in common, and two points in common. However, we will go through it much more quickly, skipping many details common to the geometric examples.

6.3 Orthometric example: Two height differences, no shared points

Consider one observation vector, containing two orthometric height differences, one from point A to point B at epoch i and the other from point C to point D at epoch j. The observation equation takes this form:

$$\boldsymbol{Y} = \begin{bmatrix} \Delta H_{AB,i} \\ \Delta H_{CD,j} \end{bmatrix} + \boldsymbol{e_y} \ , \ \ \boldsymbol{e_y} \sim (\boldsymbol{0}, \boldsymbol{\Sigma_y}). \tag{48}$$

Expanding (48) yields:

$$\boldsymbol{Y} = \begin{bmatrix} \Delta H_{AB,i} \\ \Delta H_{CD,j} \end{bmatrix} + \boldsymbol{e_y} = \bar{A}\begin{bmatrix} \boldsymbol{\Xi_i} \\ \boldsymbol{\Xi_j} \end{bmatrix} + \boldsymbol{e_y} = \begin{bmatrix} \bar{A}_i & 0_{1,4} \\ 0_{1,4} & \bar{A}_j \end{bmatrix}\begin{bmatrix} \boldsymbol{\Xi_i} \\ \boldsymbol{\Xi_j} \end{bmatrix} + \boldsymbol{e_y} = \begin{bmatrix} -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 \end{bmatrix}\begin{bmatrix} H_{A,i} \\ H_{B,i} \\ H_{C,i} \\ H_{D,i} \\ H_{A,j} \\ H_{B,j} \\ H_{C,j} \\ H_{D,j} \end{bmatrix} + \boldsymbol{e_y}. \tag{49}$$

Mapping observations into projected observations yields:

$$\bar{\boldsymbol{Y}} = \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{1,4} \\ 0_{1,4} & \bar{A}_j \end{bmatrix}\begin{bmatrix} \boldsymbol{\Delta X_{i,M}} \\ \boldsymbol{\Delta X_{j,M}} \end{bmatrix} = \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{1,4} \\ 0_{1,4} & \bar{A}_j \end{bmatrix}\left\{\begin{bmatrix} \Delta H_{A,i} \\ \Delta H_{B,i} \\ \Delta H_{C,i} \\ \Delta H_{D,i} \\ \Delta H_{A,j} \\ \Delta H_{B,j} \\ \Delta H_{C,j} \\ \Delta H_{D,j} \end{bmatrix} + \begin{bmatrix} e_{\Delta H_{A,i}} \\ e_{\Delta H_{B,i}} \\ e_{\Delta H_{C,i}} \\ e_{\Delta H_{D,i}} \\ e_{\Delta H_{A,j}} \\ e_{\Delta H_{B,j}} \\ e_{\Delta H_{C,j}} \\ e_{\Delta H_{D,j}} \end{bmatrix}\right\}$$

$$
\begin{aligned}
&= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{1,4} \\ 0_{1,4} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta h_{A,i} \\ \Delta h_{B,i} \\ \Delta h_{C,i} \\ \Delta h_{D,i} \\ \Delta h_{A,j} \\ \Delta h_{B,j} \\ \Delta h_{C,j} \\ \Delta h_{D,j} \end{bmatrix} - \begin{bmatrix} \Delta L_{A,i} \\ \Delta L_{B,i} \\ \Delta L_{C,i} \\ \Delta L_{D,i} \\ \Delta L_{A,j} \\ \Delta L_{B,j} \\ \Delta L_{C,j} \\ \Delta L_{D,j} \end{bmatrix} + \begin{bmatrix} e_{\Delta h_{A,i}} \\ e_{\Delta h_{B,i}} \\ e_{\Delta h_{C,i}} \\ e_{\Delta h_{D,i}} \\ e_{\Delta h_{A,j}} \\ e_{\Delta h_{B,j}} \\ e_{\Delta h_{C,j}} \\ e_{\Delta h_{D,j}} \end{bmatrix} - \begin{bmatrix} e_{\Delta L_{A,i}} \\ e_{\Delta L_{B,i}} \\ e_{\Delta L_{C,i}} \\ e_{\Delta L_{D,i}} \\ e_{\Delta L_{A,j}} \\ e_{\Delta L_{B,j}} \\ e_{\Delta L_{C,j}} \\ e_{\Delta L_{D,j}} \end{bmatrix} \right\} \\
&= \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{1,4} \\ 0_{1,4} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i \dot{h}_A \\ \Delta t_i \dot{h}_B \\ \Delta t_i \dot{h}_C \\ \Delta t_i \dot{h}_D \\ \Delta t_j \dot{h}_A \\ \Delta t_j \dot{h}_B \\ \Delta t_j \dot{h}_C \\ \Delta t_j \dot{h}_D \end{bmatrix} - \begin{bmatrix} \Delta t_i \dot{L}_A \\ \Delta t_i \dot{L}_B \\ \Delta t_i \dot{L}_C \\ \Delta t_i \dot{L}_D \\ \Delta t_j \dot{L}_A \\ \Delta t_j \dot{L}_B \\ \Delta t_j \dot{L}_C \\ \Delta t_j \dot{L}_D \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \\ \Delta h_{D,k} \\ \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \\ \Delta h_{D,k} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \\ \Delta h_{D,k} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q_j \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \\ \Delta h_{D,k} \end{bmatrix} \right\} \\
&- \begin{bmatrix} \bar{A}_i & 0_{1,4} \\ 0_{1,4} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i e_{\Delta \dot{h}_A} \\ \Delta t_i e_{\Delta \dot{h}_B} \\ \Delta t_i e_{\Delta \dot{h}_C} \\ \Delta t_i e_{\Delta \dot{h}_D} \\ \Delta t_j e_{\Delta \dot{h}_A} \\ \Delta t_j e_{\Delta \dot{h}_B} \\ \Delta t_j e_{\Delta \dot{h}_C} \\ \Delta t_j e_{\Delta \dot{h}_D} \end{bmatrix} - \begin{bmatrix} \Delta t_i e_{\Delta \dot{L}_A} \\ \Delta t_i e_{\Delta \dot{L}_B} \\ \Delta t_i e_{\Delta \dot{L}_C} \\ \Delta t_i e_{\Delta \dot{L}_D} \\ \Delta t_j e_{\Delta \dot{L}_A} \\ \Delta t_j e_{\Delta \dot{L}_B} \\ \Delta t_j e_{\Delta \dot{L}_C} \\ \Delta t_j e_{\Delta \dot{L}_D} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q_j \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \end{bmatrix} \right\}.
\end{aligned}
\tag{50}
$$

Comparing (50) to (24) we see similarities, but the largest differences are *no* rotation matrices, and *two* sets of velocities rather than one. The random error vector $\boldsymbol{e}_{\bar{\boldsymbol{y}},\boldsymbol{m}}$ of the contribution of the MCPV to the projected observations is found in the last line of (50), as:

$$
\boldsymbol{e}_{\bar{\boldsymbol{y}},\boldsymbol{m}} = - \begin{bmatrix} \bar{A}_i & 0_{1,4} \\ 0_{1,4} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i e_{\Delta \dot{h}_A} \\ \Delta t_i e_{\Delta \dot{h}_B} \\ \Delta t_i e_{\Delta \dot{h}_C} \\ \Delta t_i e_{\Delta \dot{h}_D} \\ \Delta t_j e_{\Delta \dot{h}_A} \\ \Delta t_j e_{\Delta \dot{h}_B} \\ \Delta t_j e_{\Delta \dot{h}_C} \\ \Delta t_j e_{\Delta \dot{h}_D} \end{bmatrix} - \begin{bmatrix} \Delta t_i e_{\Delta \dot{L}_A} \\ \Delta t_i e_{\Delta \dot{L}_B} \\ \Delta t_i e_{\Delta \dot{L}_C} \\ \Delta t_i e_{\Delta \dot{L}_D} \\ \Delta t_j e_{\Delta \dot{L}_A} \\ \Delta t_j e_{\Delta \dot{L}_B} \\ \Delta t_j e_{\Delta \dot{L}_C} \\ \Delta t_j e_{\Delta \dot{L}_D} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \in K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k \in K(i) \\ k \notin K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q_j \sum_{\substack{k \\ k \notin K(i) \\ k \in K(j)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{D,k}} \end{bmatrix} \right\}. \tag{51}
$$

Because the relationship between MCPV and GVCM is slightly different than in the geometric case, we again use a figure to exemplify it. Once again, we skip the displacements which affect only one projected observation, for brevity. See Figure 2, and note that each sub-matrix $\bar{A}$ is of size 1×4, and we have again dropped q_i for space considerations.

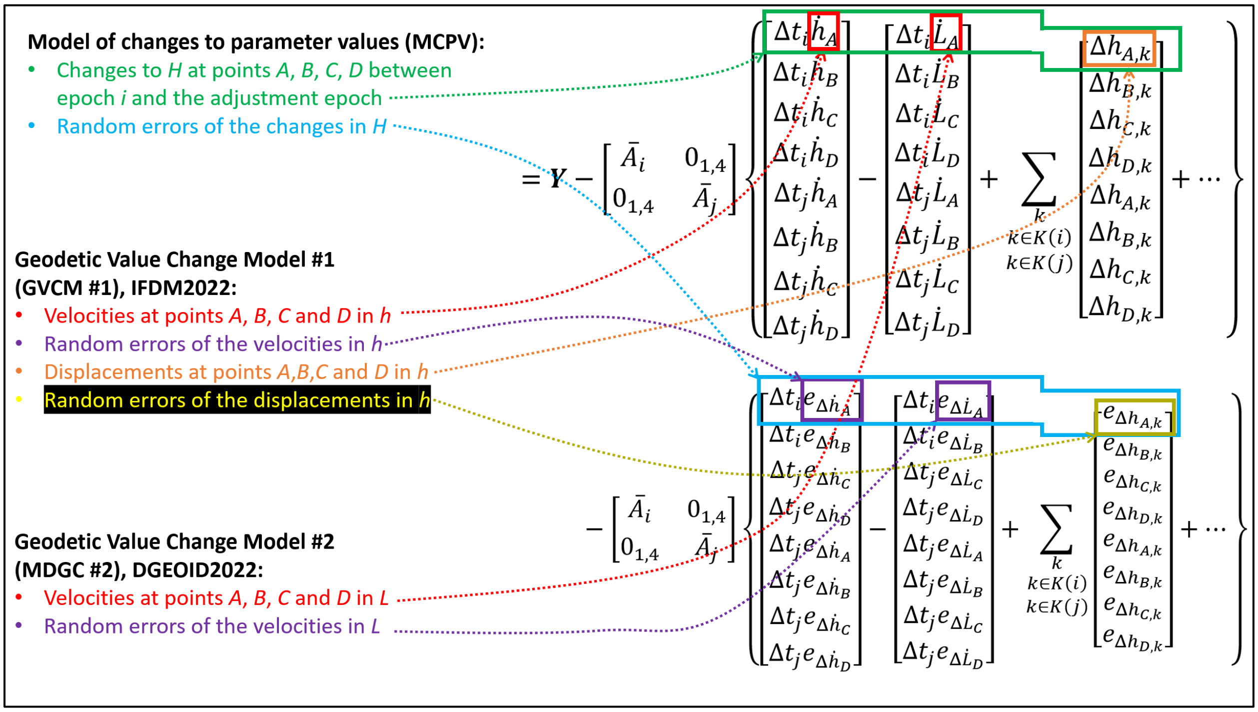

Figure 2: The relationship between the MCPV and GVCM (IFDM2022) in projected observations within an orthometric adjustment.

We again examine matrix $\Sigma_{\bar{y},m}$ in parts, but in this case element-by-element, rather than in 3×3 blocks (as was done for the geometric case), due to the one-dimensional nature of the adjustment. The velocity components are:

$$\Sigma_{\bar{y},m_{v,11}} = \Delta t_i^{\,2}\left[\left(D\{e_{\dot{h}_A}\} - 2C\{e_{\dot{h}_A}, e_{\dot{h}_B}\} + D\{e_{\dot{h}_B}\}\right) + \left(D\{e_{\dot{L}_A}\} - 2C\{e_{\dot{L}_A}, e_{\dot{L}_B}\} + D\{e_{\dot{L}_B}\}\right)\right], \tag{52a}$$

$$\begin{aligned}\Sigma_{\bar{y},m_{v,21}} = \Sigma_{\bar{y},m_{v,12}} \\ &= \Delta t_i \Delta t_j\left[\left(C\{e_{\dot{h}_A}, e_{\dot{h}_C}\} - C\{e_{\dot{h}_A}, e_{\dot{h}_D}\} - C\{e_{\dot{h}_B}, e_{\dot{h}_C}\} + C\{e_{\dot{h}_B}, e_{\dot{h}_D}\}\right)\right. \\ &\quad \left. + \left(C\{e_{\dot{L}_A}, e_{\dot{L}_C}\} - C\{e_{\dot{L}_A}, e_{\dot{L}_D}\} - C\{e_{\dot{L}_B}, e_{\dot{L}_C}\} + C\{e_{\dot{L}_B}, e_{\dot{L}_D}\}\right)\right],\end{aligned} \tag{52b}$$

$$\Sigma_{\bar{y},m_{v,22}} = \Delta t_i^{\,2}\left[\left(D\{e_{\dot{h}_C}\} - 2C\{e_{\dot{h}_C}, e_{\dot{h}_D}\} + D\{e_{\dot{h}_C}\}\right) + \left(D\{e_{\dot{L}_C}\} - 2C\{e_{\dot{L}_C}, e_{\dot{L}_D}\} + D\{e_{\dot{L}_C}\}\right)\right]. \tag{52c}$$

As before, we examine the displacing event components in three sections: those that effect both observations, those that affect only the observation at epoch i and those that affect only the observation at epoch j.

First, for any $k \in K(i)$ and $k \in K(j)$ we have:

$$\Sigma_{\bar{y},m_{k,11}} = D\{e_{\Delta h_{A,k}}\} - 2C\{e_{\Delta h_{A,k}}, e_{\Delta h_{B,k}}\} + D\{e_{\Delta h_{B,k}}\}, \tag{52d}$$

$$\Sigma_{\bar{y},m_{k,21}} = \Sigma_{\bar{y},m_{k,21}} = C\{e_{\Delta h_{A,k}}, e_{\Delta h_{C,k}}\} - C\{e_{\Delta h_{A,k}}, e_{\Delta h_{D,k}}\} - C\{e_{\Delta h_{B,k}}, e_{\Delta h_{C,k}}\} + C\{e_{\Delta h_{B,k}}, e_{\Delta h_{D,k}}\}, \tag{52e}$$

$$\Sigma_{\bar{y},m_{k,22}} = D\{e_{\Delta h_{C,k}}\} - 2C\{e_{\Delta h_{C,k}}, e_{\Delta h_{D,k}}\} + D\{e_{\Delta h_{D,k}}\}. \tag{52f}$$

Second, for any $k \in K(i)$ and $k \notin K(j)$ we have:

$$\Sigma_{\bar{y},m_{k,11}} = D\{e_{\Delta h_{A,k}}\} - 2C\{e_{\Delta h_{A,k}}, e_{\Delta h_{B,k}}\} + D\{e_{\Delta h_{B,k}}\}, \tag{52g}$$

$$\Sigma_{\bar{y},m_{k,21}} = 0, \tag{52h}$$

$$\Sigma_{\bar{y},m_{k,12}} = 0, \tag{52i}$$

$$\Sigma_{\bar{y},m_{k,22}} = 0. \tag{52j}$$

And third, for any $k \notin K(i)$ and $k \in K(j)$ we have:

$$\Sigma_{\bar{y},m_{k,11}} = 0, \tag{52k}$$

$$\Sigma_{\bar{y},m_{k,21}} = 0, \tag{52l}$$

$$\Sigma_{\bar{y},m_{k,12}} = 0, \tag{52m}$$

$$\Sigma_{\bar{y},m_{k,22}} = D\{e_{\Delta h_{C,k}}\} - 2C\{e_{\Delta h_{C,k}}, e_{\Delta h_{D,k}}\} + D\{e_{\Delta h_{D,k}}\}. \tag{52n}$$

As before, it has been assumed that covariances in the GVCM are known. If, however, there are no known covariances and we set them to zero, then the dispersion matrix $\Sigma_{\bar{y},m}$ simplifies dramatically to:

$$\Sigma_{\bar{y},m} = \left(\begin{bmatrix} \Delta t_i^{\,2}\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} + D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) & 0 \\ 0 & \Delta t_j^{\,2}\left(D\{e_{\dot{h}_C}\} + D\{e_{\dot{h}_D}\} + D\{e_{\dot{L}_C}\} + D\{e_{\dot{L}_D}\}\right) \end{bmatrix} \right.$$
$$+ \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} & 0 \\ 0 & D\{e_{\Delta h_{C,k}}\} + D\{e_{\Delta h_{D,k}}\} \end{bmatrix} + \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} & 0 \\ 0 & 0 \end{bmatrix} \tag{53}$$
$$\left. + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & D\{e_{\Delta h_{C,k}}\} + D\{e_{\Delta h_{D,k}}\} \end{bmatrix} \right)$$

The conclusions drawn from (52) and (53) are the same as in the geometric case: *Conclusion #1* for (52) and *Conclusion #2* for (53).

6.4 Orthometric example: Two height differences, one shared point

Next, consider one observation vector, containing two observations, one with an orthometric height difference from point A to point B at epoch i and another from point A to point C at epoch j. The observation equation is:

$$\boldsymbol{Y} = \begin{bmatrix} \Delta H_{AB,i} \\ \Delta H_{AC,j} \end{bmatrix} + \boldsymbol{e_y} \quad , \quad \boldsymbol{e_y} \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma_y}\right). \tag{54}$$

This expands to:

$$\boldsymbol{Y} = \begin{bmatrix} \Delta H_{AB,i} \\ \Delta H_{AC,j} \end{bmatrix} + \boldsymbol{e_y} = \begin{bmatrix} -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 \end{bmatrix} \begin{bmatrix} H_{A,i} \\ H_{B,i} \\ H_{C,i} \\ H_{A,j} \\ H_{B,j} \\ H_{C,j} \end{bmatrix} + \boldsymbol{e_y}. \tag{55}$$

Observations are again mapped into projected observations, yielding:

$$\bar{\boldsymbol{Y}} = \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{1,3} \\ 0_{1,3} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i \dot{h}_A \\ \Delta t_i \dot{h}_B \\ \Delta t_i \dot{h}_C \\ \Delta t_j \dot{h}_A \\ \Delta t_j \dot{h}_B \\ \Delta t_j \dot{h}_C \end{bmatrix} - \begin{bmatrix} \Delta t_i \dot{L}_A \\ \Delta t_i \dot{L}_B \\ \Delta t_i \dot{L}_C \\ \Delta t_j \dot{L}_A \\ \Delta t_j \dot{L}_B \\ \Delta t_j \dot{L}_C \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \\ \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \\ 0 \\ 0 \\ 0 \end{bmatrix} + q_j \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{C,k} \end{bmatrix} \right\}$$
$$- \begin{bmatrix} \bar{A}_i & 0_{1,3} \\ 0_{1,3} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i e_{\Delta \dot{h}_A} \\ \Delta t_i e_{\Delta \dot{h}_B} \\ \Delta t_j e_{\Delta \dot{h}_C} \\ \Delta t_j e_{\Delta \dot{h}_A} \\ \Delta t_j e_{\Delta \dot{h}_B} \\ \Delta t_j e_{\Delta \dot{h}_C} \end{bmatrix} - \begin{bmatrix} \Delta t_i e_{\Delta \dot{L}_A} \\ \Delta t_i e_{\Delta \dot{L}_B} \\ \Delta t_j e_{\Delta \dot{L}_C} \\ \Delta t_j e_{\Delta \dot{L}_A} \\ \Delta t_j e_{\Delta \dot{L}_B} \\ \Delta t_j e_{\Delta \dot{L}_C} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \\ 0 \\ 0 \\ 0 \end{bmatrix} + q_j \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{C,k}} \end{bmatrix} \right\}. \tag{56}$$

The error vector $\boldsymbol{e}_{\bar{y},m}$ of the MCPV contribution to projected observations is the last line in (56). As before, we ignore covariances in the GVCMs going forward, since they aren't expected to be available. In that case, the dispersion matrix $\Sigma_{\bar{y},m}$ simplifies to:

$$\Sigma_{\bar{y},m} = \left(\begin{bmatrix} \Delta t_i^{\,2}\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} + D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) & \Delta t_i \Delta t_j\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{L}_A}\}\right) \\ \Delta t_i \Delta t_j\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{L}_A}\}\right) & \Delta t_j^{\,2}\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} + D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) \end{bmatrix} \right.$$
$$+ \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} & D\{e_{\Delta h_{A,k}}\} \\ D\{e_{\Delta h_{A,k}}\} & D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} \end{bmatrix} + \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} & 0 \\ 0 & 0 \end{bmatrix} \tag{57}$$
$$\left. + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} \end{bmatrix} \right)$$

Comparing (57) to (32) shows a wide array of similarities, including *Conclusion #3*, and the potential problem of non-sparsity it raises. This potential problem will be discussed in section 7. As in the geometric case, we consider whether the potential problem still exists when the two observations are at the same epoch. In *this* (orthometric) case, the answer is now "*yes.*" (Recall that the answer was "probably not" for geometric adjustments.) The reason for this is that in most leveling adjustments, the dispersion matrix of the observations tends to be diagonal. Despite the obvious factors which would correlate two differential orthometric height observations in a single epoch (whether that be a single day or similar small period of time), the data are not processed *simultaneously* in a *session* the way that GNSS data are. Thus, we don't tend to have any covariance information between leveling observations, even if collected at a common epoch. With that being the case, even if the two leveling observations both occur at epoch i, we *still* expect a *diagonal* matrix Σ_y. Thus, the addition of the non-zero off-diagonal elements seen in (57) above, will *remain* a problem by creating non-zero off diagonal elements in $\Sigma_{\bar{y}}$, even if the observations occur at the same epoch. More details are found in section 7.

6.5 Orthometric example: Two height differences, two shared points

Finally, consider one observation vector, containing two orthometric observations, one with a height difference from point A to point B at epoch i and another from point A to point B at epoch j. The observation equation is:

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{\Delta X_{AB,i}} \\ \boldsymbol{\Delta X_{AB,j}} \end{bmatrix} + \boldsymbol{e_y} \quad , \quad \boldsymbol{e_y} \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma_y}\right). \tag{58}$$

This expands to:

$$\boldsymbol{Y} = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} H_{A,i} \\ H_{B,i} \\ H_{A,j} \\ H_{B,j} \end{bmatrix} + \boldsymbol{e_y}. \tag{59}$$

As usual we map observations into projected observations, as such:

$$\begin{aligned} \overline{\boldsymbol{Y}} = \boldsymbol{Y} - \begin{bmatrix} \bar{A}_i & 0_{1,2} \\ 0_{1,2} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i \dot{h}_A \\ \Delta t_i \dot{h}_B \\ \Delta t_j \dot{h}_A \\ \Delta t_j \dot{h}_B \end{bmatrix} - \begin{bmatrix} \Delta t_i \dot{L}_A \\ \Delta t_i \dot{L}_B \\ \Delta t_j \dot{L}_A \\ \Delta t_j \dot{L}_B \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} \Delta h_{A,k} \\ \Delta h_{B,k} \\ \Delta h_{A,k} \\ \Delta h_{B,k} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} \Delta h_{A,k} \\ \Delta h_{B,k} \\ 0 \\ 0 \end{bmatrix} + q_j \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 \\ 0 \\ \Delta h_{A,k} \\ \Delta h_{B,k} \end{bmatrix} \right\} \\ - \begin{bmatrix} \bar{A}_i & 0_{1,2} \\ 0_{1,2} & \bar{A}_j \end{bmatrix} \left\{ \begin{bmatrix} \Delta t_i e_{\Delta \dot{h}_A} \\ \Delta t_i e_{\Delta \dot{h}_B} \\ \Delta t_j e_{\Delta \dot{h}_A} \\ \Delta t_j e_{\Delta \dot{h}_B} \end{bmatrix} - \begin{bmatrix} \Delta t_i e_{\Delta \dot{L}_A} \\ \Delta t_i e_{\Delta \dot{L}_B} \\ \Delta t_j e_{\Delta \dot{L}_A} \\ \Delta t_j e_{\Delta \dot{L}_B} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \end{bmatrix} + q_i \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \\ 0 \\ 0 \end{bmatrix} + q_j \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 \\ 0 \\ e_{\Delta h_{A,k}} \\ e_{\Delta h_{B,k}} \end{bmatrix} \right\}. \end{aligned} \tag{60}$$

The error vector $\boldsymbol{e_{\bar{y},m}}$ of the MCPV contribution to projected observations is the last line in (60). Finally, and again ignoring covariances in the GVCMs, we compute matrix $\Sigma_{\bar{y},m}$ as:

$$\begin{aligned} \Sigma_{\bar{y},m} = \Bigg(& \begin{bmatrix} \Delta t_i^{\,2}\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} + D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) & \Delta t_i \Delta t_j \left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} + D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) \\ \Delta t_i \Delta t_j \left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} + D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) & \Delta t_j^{\,2}\left(D\{e_{\dot{h}_A}\} + D\{e_{\dot{h}_B}\} + D\{e_{\dot{L}_A}\} + D\{e_{\dot{L}_B}\}\right) \end{bmatrix} \\ & + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} & D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} \\ D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} & D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} \end{bmatrix} + \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} & 0 \\ 0 & 0 \end{bmatrix} \\ & + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & D\{e_{\Delta h_{A,k}}\} + D\{e_{\Delta h_{B,k}}\} \end{bmatrix} \Bigg) \end{aligned} \tag{61}$$

Compare (61) to (36) to see that (61) also takes the general form seen in (37). As such, the same conclusions are drawn as in the geometric adjustment. That is, there are some very specific cases in the orthometric adjustments where $\Sigma_{\bar{y},m}$ will be singular, raising the concern that other cases exist. Further details are found in the appendix.

7 Practical considerations

There were three driving questions which motivated this paper:

1) If covariances within the GVCMs are *known*, how do we use them to compute the dispersion matrix $\Sigma_{\bar{y},m}$ of the contribution of the MCPV to projected observations in the ME-LSA problem?
2) If covariances within the GVCMs are *known*, which off-diagonal, or off-*block*-diagonal, elements, that are *zero* in the dispersion matrix for the observations Σ_y, are going to be *non-zero* in the dispersion matrix for the *projected* observations $\Sigma_{\bar{y}}$?

3) If the covariances in the GVCMs are *not* known, what conditions will cause the previously-zero covariance elements in Σ_y to be non-zero in $\Sigma_{\bar{y}}$?

The first question has been answered with equations found in Table 2.

Table 2: Summary of equations which compute elements of the dispersion matrix for the contribution of the MCPV to the projected observations.

Adjustment Type	Number of observations	GVCM covariances known?	Number of points in common	Equation number
Geometric	1	Yes	N/A	18
Geometric	1	No	N/A	19,20,21
Geometric	2	Yes	0	27
Geometric	2	No	0	28
Geometric	2	No[19]	1	32
Geometric	2	No	2	36
Orthometric	1	Yes	N/A	46
Orthometric	1	No	N/A	47
Orthometric	2	Yes	0	52
Orthometric	2	No	0	53
Orthometric	2	No[19]	1	57
Orthometric	2	No	2	61

The second question can be answered by examination of the equations just listed, where this conclusion becomes obvious:

> *Conclusion 5: Knowledge of the covariances in the GVCM introduces covariances between the errors of* every pair *of projected observations.*

In simple terms, it means that the dispersion matrix $\Sigma_{\bar{y},m}$ of the contribution of the MCPV to projected observations will *always* be full, and thus so will the dispersion matrix $\Sigma_{\bar{y}}$ of the projected observations. By extension, assuming the Gauss-Markov Model is used, the cofactor matrix $Q_{\bar{y}}$ will also *always* be full. The implications of a full cofactor matrix are daunting, and will be discussed in subsection 7.1.

The third question was answered in equations 32, 36, 57 and 61. In those equations it was seen that two observations which share at least one common point, even at different epochs, will transform from two *uncorrelated* observations into two *correlated* projected observations, even in the absence of GVCM covariances. While this will not create a full $\Sigma_{\bar{y},m}$ or $\Sigma_{\bar{y}}$ matrix, it will create non-zero elements where none existed in Σ_y. This loss of sparsity is a concern discussed next.

[19] Note that the addition of GVCM covariances was ignored for later examples, since (a) they aren't expected to be known and (b) they only complicated and distracted from the work.

7.1 Dealing with a dispersion matrix of projected observations that is less sparse than the dispersion matrix of observations

In the preceding section, two scenarios were discussed where non-zero off-diagonal elements would occur in the $\Sigma_{\bar{y},m}$ matrix, which also means that these same elements will be non-zero in the $\Sigma_{\bar{y}}$ matrix. Those two scenarios are:

1) Covariances are known in the GVCM(s)
2) Covariances are not known in the GVCM, but two observations share one or more common points

The first scenario is more problematic than the second, since the first yields a *full* matrix $\Sigma_{\bar{y}}$, while the second only adds *some* non-zero off-diagonal elements to $\Sigma_{\bar{y}}$ where there were zeroes in Σ_y. But in both cases, we will likely begin with a diagonal or block-diagonal matrix Σ_y, which is one of the simplest sparse matrices, but will end with a much less sparse, possibly full, matrix $\Sigma_{\bar{y}}$.

In order to understand the implications of this less-sparse matrix $\Sigma_{\bar{y}}$, we will work through a thought experiment. Let us begin by assuming that we are working with the Gauss-Markov Model, so that the relationship between the dispersion matrix, the cofactor matrix and the weight matrix of the observations is:

$$\Sigma_y = \sigma_0^2 Q_y = \sigma_0^2 {P_y}^{-1}. \quad (62)$$

The dispersion matrix Σ_y is generally unavailable. The cofactor matrix Q_y generally *is* available, and the weight matrix P_y is needed, and usually computed by *inverting the cofactor matrix*.

Now, consider a large LSA problem, such as the 2011 National Adjustment (Dennis, 2020). In that adjustment, the input cofactor matrix of the observations Q_y was entirely block-diagonal. In some cases, the blocks were merely 3×3, being a single measured baseline within a GNSS session. In others a block might have been a few hundred elements on a side, being hundreds of measured baselines processed in a single GNSS session simultaneously. Despite such large blocks on the diagonal, this nonetheless left vast quantities of off-diagonal zeros in the cofactor matrix. With some 400,000 GNSS measured baselines, there were about 1,200,000 observations, and thus the cofactor matrix was of size $1{,}200{,}000 \times 1{,}200{,}000$. Although an exact estimate of the sparsity of Q_y in this adjustment was not provided, there was an average of 5.43 measured baselines per session. Let us round that up to 6. A simultaneously processed session with 6 measured baselines has 18 observations, so it would fill an 18×18 block on the diagonal of matrix Q_y. If the entire matrix Q_y were populated with such blocks, there would be about 66,667 such blocks, for 21,600,000 non-zero elements in Q_y, out of a total of 1,440,000,000,000 elements. That would imply 99.9985% of the elements in Q_y were equal to zero. That value is only a rough estimate. The reality is that some blocks were much larger, and some smaller, but we do know that Q_y *was* block diagonal, and can therefore confidently assume its sparsity percentage is well into the high 90s if not above 99%.

Now consider that same adjustment, but treating HTDP, the GVCM in that adjustment, as stochastic. If covariances within the GVCM were known, then every single element in matrix $Q_{\bar{y},m}$ and thus $Q_{\bar{y}}$ will be non-zero: its sparsity will be 0%.

Only when we have $Q_{\bar{y}}$ can we determine the weight matrix of the projected observations $P_{\bar{y}}$, by inverting $Q_{\bar{y}}$; and we *must* have $P_{\bar{y}}$ to solve the LSA. However, there is a difference, in fact a colossal difference, between inverting a matrix of size $1{,}200{,}000 \times 1{,}200{,}000$ with a sparsity of 99.9985% and one of sparsity 0%. This was mentioned in Smith, et al. (2023, Appendix A), but bears a brief repeat here. If inverting a full matrix requires $O(n^3)$ operations, and inverting a block-diagonal matrix takes between $O(n)$ and $O(n^2)$ operations (where n is the dimension of one side of a square matrix) then inverting a full cofactor matrix $Q_{\bar{y}}$ for this example would take, at *best*, *thousands of times longer* than inverting a block-diagonal cofactor matrix, and possibly as much as *millions of times longer*, depending on how many CPUs were available and how parallel one could make the inversion process.

We therefore reach our next conclusion:

Conclusion 6: NGS cannot use all covariances in the GVCM, even if they are known.

Let us therefore consider the second scenario, where we do not know the covariances in the GVCM, but still covariances between projected observations will occur if two observations share one or more common points.

In this scenario, the sparsity of $Q_{\bar{y}}$ will only be worse than the sparsity of Q_y by the addition of covariances between observations that share points. This raises a few questions if we are to allow these additional non-zero elements in matrix $Q_{\bar{y}}$:

1) Will $Q_{\bar{y}}$ remain sparse enough to be inverted (in a reasonable amount of time using reasonable resources) for the computation of $P_{\bar{y}}$?
2) If $P_{\bar{y}}$ is not sparse, as sometimes happens when a sparse matrix is inverted, will this be a problem?

The answer to the first question depends upon the situation. In recent experiments with a national geometric adjustment, there were 420,328 instances of two GNSS measured baselines sharing a common point (out of 630,492 total GNSS measured baselines). This would yield 420,328 3×3 non-zero off-diagonal blocks in the upper triangular portion of matrix $Q_{\bar{y}}$, and an identical number in the lower triangular portion. As mentioned earlier, that matrix began with about 21,600,000 non-zero on-block-diagonal elements. This common-point situation would add 7,565,904 new non-zero elements, changing the sparsity from 99.9985% to 99.9980%, a trivial change in *the value of sparsity*, but not a trivial change in the *complexity* of the matrix. This is because all new additions are *off-diagonal*, and this may cause a non-trivial change in the computational time needed to invert the matrix.

Let us assume for a moment that inversion of this sparse, but not block-diagonal matrix $Q_{\bar{y}}$ remains logistically possible. If we are lucky, matrix $P_{\bar{y}}$ will also be sparse, but this is by no means guaranteed. Examples can be found which show how the inverse of a sparse matrix is itself sparse, while other examples may be found to show how it can be nearly or even completely full. In order to cover all scenarios, we will assume the worst case, that matrix $P_{\bar{y}}$ will be full.

This brings up the need to answer the second question, above. It turns out that having a full matrix $P_{\bar{y}}$ *does* cause a problem. The problem is not due to inverting $P_{\bar{y}}$, as that never happens (and besides, we already know its inverse). Rather, it is about the impact of a full $P_{\bar{y}}$ upon the normal matrix $\bar{A}^T P_{\bar{y}} \bar{A}$, since that matrix *is* inverted. Consider the matrix computations needed to find a least-squares solution within the GMM. As per Snow (2021), but retaining the notation used herein, the estimated incremental parameters are:

$$\hat{\boldsymbol{\xi}} = \left(\bar{A}^T P_{\bar{y}} \bar{A}\right)^{-1} \bar{A}^T P_{\bar{y}} \bar{\mathbf{y}}. \tag{63}$$

It is not particularly onerous to perform the multiplication $\bar{A}^T P_{\bar{y}} \bar{A}$ nor $\bar{A}^T P_{\bar{y}} \bar{\mathbf{y}}$, especially since matrix $\bar{A}$ is itself sparse. In fact, it is worth recalling that $\bar{A}$ is *extraordinarily* sparse in most geodetic applications, as each row will not contain more than about six non-zero elements (being the coefficients relating observations to the very few parameters of which that observation is a function), no matter whether there are thousands or millions of observations or parameters. But the sparseness of $\bar{A}^T P_{\bar{y}} \bar{A}$ is the main question, since that matrix must be inverted. As it turns out, the sparseness of $\bar{A}^T P_{\bar{y}} \bar{A}$is highly dependent upon the sparseness of $P_{\bar{y}}$. Consider the following two conditions for any given row/column element of $\bar{A}^T P_{\bar{y}} \bar{A}$:

1) If $P_{\bar{y}}$ is *diagonal*, then off-diagonal element (r, c) of $\bar{A}^T P_{\bar{y}} \bar{A}$ will be *zero* unless there is at least one observation related to both unknown parameter r and unknown parameter c.
2) If $P_{\bar{y}}$ is *full*, then off-diagonal element (r, c) of $\bar{A}^T P_{\bar{y}} \bar{A}$ will be *non-zero* unless there are no observations related to unknown parameter r and also no observations related to unknown parameter c.

These rules can easily be checked through some simple linear algebra. Let us consider what they mean, and draw some conclusions about the sparsity of $\bar{A}^T P_{\bar{y}} \bar{A}$.

First, for the diagonal case of $P_{\bar{y}}$, the sparsity of $\bar{A}^T P_{\bar{y}} \bar{A}$ will grow quickly as the size of the geodetic network grows. To exemplify this, consider the simple case of GNSS measured baselines only. If the network consists of, say, only four points, it is a simple matter to collect an observation between every possible pair of those points (that is, six observations). This would yield a full matrix $\bar{A}^T P_{\bar{y}} \bar{A}$, e.g., with a sparsity of 0%. However, if the network consists of 100,000 points, it would be practically impossible to collect an observation connecting every possible pair of points (4,999,950,000 observations), and thus the matrix $\bar{A}^T P_{\bar{y}} \bar{A}$ would be very sparse.

However, when $P_{\bar{y}}$ is *full* the sparsity of $\bar{A}^T P_{\bar{y}} \bar{A}$ will *always* be 0%. The only time an element of $\bar{A}^T P_{\bar{y}} \bar{A}$ could be non-zero is if there existed two unknown parameters which both had zero observations relating to them. As one cannot estimate unknown parameters unless at least one observation relates to that parameter, such a situation should never arise.

What this means is that it is risky to construct matrix $Q_{\bar{y}}$ into any form besides block-diagonal. We can guarantee that if $Q_{\bar{y}}$ is not singular and is block-diagonal, that $P_{\bar{y}}$ will also be block-diagonal with the same distribution of non-zero elements as $Q_{\bar{y}}$. But the same cannot be said for a sparse, but not block-diagonal $Q_{\bar{y}}$. This brings us to another conclusion:

> *Conclusion 7: For large least-squares adjustments, unless it can be guaranteed that $P_{\bar{y}}$ is sparse, one should not construct $Q_{\bar{y}}$ into any form besides block-diagonal.*

The term "large" is left purposefully vague, since it is dependent upon the amount of computational power and memory available. However, even if there were unlimited computational resources, there remains one other issue: most legacy LSA software packages at NGS presume matrix $P_{\bar{y}}$ will be diagonal or at least block-diagonal, due to lack of known correlations between observations, as reasons mentioned earlier. This is the case for NGS's original ADJUST (Milbert and Kass, 1987), ASTA[20], BIGADJUST[20] and NETSTAT (Pursell and Potterfield, 2008) software packages, and is the case for the new LSA package being developed, LASER. Even if we can guarantee that a sparse, but not-block diagonal $Q_{\bar{y}}$ would yield a sparse, but not-block diagonal $P_{\bar{y}}$, there remains the issue that the LSA software must be prepared to read and use such an unusually formed $P_{\bar{y}}$ matrix. As that is not going to immediately be the case at NGS, we arrive at a modification of the above conclusion:

> *Conclusion 7, revision 1: For large least-squares adjustments, NGS should not construct $Q_{\bar{y}}$ into any form besides block-diagonal.*

However even this conclusion is not quite satisfactory yet, as it doesn't specify a way to prevent $Q_{\bar{y}}$ from using blocks so large that they become a computational burden.

Consider that each on-diagonal block in the cofactor matrix of the original observations, Q_y, contains covariance information between observations at a single observational epoch representing a single GNSS session (covariance information being generally unavailable between other observation types), but that between any two observational epochs we assume no covariances. Consider an example of, say, two observational epochs containing two and three GNSS measured baselines respectively. Their on-diagonal blocks in matrix Q_y would be 6×6 and 9×9 in size, and the off-diagonal 6×9 and 9×6 blocks would be zeroes. Now consider that one or more of the points observed in the first epoch were also observed in the second epoch. As these two sets of observations become two sets of *projected* observations, the 15×15 block representing these two epochs in $Q_{\bar{y}}$ would contain more non-zero elements than just the 6×6 and 9×9 on-diagonal blocks, and could be completely full if all points were common between

[20] These programs have never been formally documented.

the epochs. Taken to its logical conclusion, we will end up with a matrix $Q_{\bar{y}}$ that could either be described as "sparse, but not block-diagonal", or one that is "block-diagonal, with large blocks that are sparse". Neither case is acceptable for large networks, as mentioned earlier. As such, we revise again conclusion 7 as:

> *Conclusion 7, revision 2: For large least-squares adjustments, NGS will only construct non-zero elements in $Q_{\bar{y}}$ if the corresponding element in Q_y is also non-zero.*

The corollary to this, since $Q_{\bar{y}}$ is the sum Q_y and $\Sigma_{\bar{y},m}$ is:

> *Conclusion 8: For large least-squares adjustments, NGS will only construct non-zero elements in $\Sigma_{\bar{y},m}$ if the corresponding element in Q_y is also non-zero.*

The only place the above two conclusions will be invoked will be within one GNSS session, with multiple GNSS measured baselines processed simultaneously. In that instance, and that instance alone, NGS will invoke equations 32 or 36 to compute $\Sigma_{\bar{y},m}$, for any two GNSS measured baselines in a single session that already have a known covariance in matrix Q_y. In all other cases, equations 19, 20 and 21, (for geometric adjustments) and equation 47 (for orthometric adjustments) will be used. In this way, NGS may assure that diagonal and/or block-diagonal structure of $Q_{\bar{y}}$ will match that of Q_y.

8 Summary

The multi-epoch least-squares adjustment (ME-LSA) problem was documented, in very general terms, in Smith et al. (2023). Key to solving the ME-LSA is the existence of one or more geodetic value change models (GVCMs), which will yield a single model of changes to parameter values (MCPV), which in its own turn is used to relate observations to *projected* observations. In (ibid) the derivations allowed for the GVCM to be fixed or stochastic. However, the existence of, and impact of, covariances within the GVCM was not fully detailed in that paper, nor was the topic of correlations arising between projected observations due to the use of a common velocity and/or displacement field. The need to properly understand those two topics is what motivated this paper.

Within this paper, three questions were asked and answered: 1) How should we convert GVCM covariances, if they exist, into MCPV covariances; 2) How will covariances within the GVCM impact the sparsity of the cofactor matrix of the projected observations $Q_{\bar{y}}$; and 3) Even without covariances in the GVCMs, how will common velocities or displacing events impact the sparsity of $Q_{\bar{y}}$?

The first question was answered in equations 18 (geometric) and 46 (orthometric), though in each case we have assumed certain covariances will never be available, such as those between velocities and displacing events or between ellipsoid height velocities and geoid height velocities.

The second question was answered thus: such covariances will yield a full cofactor matrix of the projected observations $Q_{\bar{y}}$ which, in large networks, would be impractical, if not impossible, to invert for the needed weight matrix of projected observations, $P_{\bar{y}}$.

The answer to the third question was that such covariances will yield a cofactor matrix of the projected observations $Q_{\bar{y}}$ which is sparse, but not block-diagonal, which comes with a variety of complications. We therefore concluded that an element in matrix $Q_{\bar{y}}$ should be non-zero only if that same element was also non-zero in the cofactor matrix of the original observations Q_y.

NGS will compute reference epoch coordinates (RECs) using the strategies and conclusions found in this paper. This means that, generally speaking, all non-zero off-diagonal blocks in matrix $\Sigma_{\bar{y},m}$ will be ignored, except in a single case: when two GNSS vectors occur at the same epoch, were simultaneously processed, and they share one or more points. Then, and only then, will off-diagonal elements be computed and used in the construction of $\Sigma_{\bar{y},m}$.

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10 Appendix A: Dealing with the singularity arising from one baseline with two common points

Section 5.5 showed that matrix $\Sigma_{\bar{y},m}$ takes a special form when two GNSS measured baselines share the same two points. That pattern was seen in (37). That equation is repeated here, for ease of reference.

$$\Sigma_{\bar{y},m} = \left(\begin{bmatrix} a^2H & abH \\ abH & b^2H \end{bmatrix} + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} \begin{bmatrix} H_k & H_k \\ H_k & H_k \end{bmatrix} + \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} \begin{bmatrix} H_k & 0 \\ 0 & 0 \end{bmatrix} + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} \begin{bmatrix} 0 & 0 \\ 0 & H_k \end{bmatrix} \right)$$

The question we now raise is whether or not a matrix of this form is invertible. However, note that (37) was a special circumstance. There were only two observations and they shared both points. What if there were multiple observations, so that the pattern in (37) would remain but only represent blocks of a larger matrix $\Sigma_{\bar{y},m}$?

First, we consider solely to the two-observation situation. We will show that each of the four types of contributing matrices in (37) are *singular*. We begin by noticing that sub-matrices H and

H_k are both positive-definite, and thus invertible. Also, since H is invertible, then so must be any non-zero scalar multiple of it. Thus, a^2H, abH and b^2H are also all invertible. Going forward, we will rely on the following matrix inversion lemma:

> **<u>Lemma:</u>** If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and both A and D are square, then M is invertible if and only if both D and the Schur complement of D are invertible.

Let us apply this Lemma to each of the four types of matrices seen in (37).

We begin with $\begin{bmatrix} a^2H & abH \\ abH & b^2H \end{bmatrix}$. We have already shown that b^2H is invertible. We therefore examine the Schur complement of b^2H:

$$\begin{aligned} S(b^2H) &= (a^2H) - (abH)(b^2H)^{-1}(abH) = (a^2H) - (abb^{-2}ab)(H)(H)^{-1}(H) \\ &= (a^2H) - (a^2H) = 0 \end{aligned} \tag{64}$$

Being equal to 0, the Schur complement of b^2H is singular, therefore we conclude that any matrix of the form $\begin{bmatrix} a^2H & abH \\ abH & b^2H \end{bmatrix}$ is singular, even if H is invertible.

Looking at the second matrix (inside the summation), $\begin{bmatrix} H_k & H_k \\ H_k & H_k \end{bmatrix}$, and assuming H_k is invertible, then the Schur compliment of H_k is:

$$S(H_k) = (H_k) - (H_k)(H_k)^{-1}(H_k) = H_k - H_k = 0$$

Thus, we see that any matrix of the form $\begin{bmatrix} H_k & H_k \\ H_k & H_k \end{bmatrix}$ is singular, even if H_k is invertible.

The singularity of the third and fourth matrices (also inside summations), $\begin{bmatrix} H_k & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & H_k \end{bmatrix}$, can be seen by simple inspection: they both have a block-column (and block-row) full of zeroes, a condition that always results in singularity.

We may therefore conclude that every matrix in the sum in (37) is singular. We now ask whether the *total* matrix is singular.

We begin by writing (37) in its combined form, as seen in (38).

$$\Sigma_{\bar{y},m} = \begin{bmatrix} a^2H + \sum\limits_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k + \sum\limits_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} H_k & abH + \sum\limits_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k \\ abH + \sum\limits_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k & b^2H + \sum\limits_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k + \sum\limits_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} H_k \end{bmatrix} \tag{65}$$

In (65) we know that each 3×3 block must be invertible, since each component matrix (a^2H, b^2H, abH or H_k) is positive-definite, and the sum of two or more positive definite matrices remains positive definite (and thus invertible). As such, we can attempt to apply the matrix inverse lemma from earlier, examining the Schur complement of the bottom right block:

$$
\begin{aligned}
S\left(b^2H + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} H_k\right)
&= \left(a^2H + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k + \sum_{\substack{k \\ k\in K(i) \\ k\notin K(j)}} H_k\right) \\
&\quad - \left(abH + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k\right)\left(b^2H + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k + \sum_{\substack{k \\ k\notin K(i) \\ k\in K(j)}} H_k\right)^{-1}\left(abH\right. \\
&\quad \left. + \sum_{\substack{k \\ k\in K(i) \\ k\in K(j)}} H_k\right)
\end{aligned}
\tag{66}
$$

This is where things could get complicated. To avoid that, let us first draw an immediate conclusion: if there are no *displacements* which contribute to $\Sigma_{\bar{y},m}$, then (66) reduces to (64) and we may conclude that, *in the absence of displacements, $\Sigma_{\bar{y},m}$ is always singular for two GNSS measured baselines that share the same endpoints*.[21]

Rather than attempting to prove under which circumstances a larger $\Sigma_{\bar{y},m}$ might be singular (say with certain combinations of point-sharing observations), it is enough to have proven that it is *sometimes* singular, even in this very small, yet possible example. However, a number of simple examples of matrices of the form $\begin{bmatrix} a^2H & x & abH \\ x & y & z \\ abH & z & b^2H \end{bmatrix}$ were constructed and tested, where *x*, *y* and *z* were blocks of the appropriate size, and each constructed matrix was either (a) singular or (b) else

[21] Though a similar conclusion could be made by only allowing displacements that impact both i and j, and no velocities, such a case is not relevant, since the GVCM (IFDM2022) is, for now, a model consisting of velocities that *always* impact *all* projected observations, and displacements that *might* impact *some* projected observations.

their inverse had at least one eigenvalue equal to zero. Knowing that $\Sigma_{\bar{y},m}$ (whether for two observations or more) is *occasionally* singular is reason enough to completely avoid any attempt to invert the matrix when modeling the multi-epoch least-squares adjustment problem.

Thus, as mentioned in Smith et al. (2023), the occasional, though possibly rare, singular nature of $\Sigma_{\bar{y},m}$ was kept in mind, and a careful reading of that paper will show that, while inverses of both Σ_y and $\Sigma_{\bar{y}}$ are used in the ME-LSA problem, the inverse of $\Sigma_{\bar{y},m}$ is never used.