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Equations for Computing Projected Observations, Constraints and their Cofactor Matrices from Original Observations, Constraints and their Cofactor Matrices

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Contents

Executive Summary

As part of the planned modernization of the National Spatial Reference System (NSRS), the National Geodetic Survey (NGS) is building new tools to improve their customer support. One such tool, SPROCCET (Software for PRojecting Observations, Constraints and Cofactor matrices/Errors through Time), will replace NGS's HTDP (Horizontal Time-Dependent Positioning) software. Overall, SPROCCET will perform much of the same function as HTDP in projecting geodetic data through time, but with certain improvements: SPROCCET will use vertical velocities and displacements, will project both geometric and orthometric quantities, and will also project errors through time.

In order to support these expansions to the HTDP functionality, a variety of papers were written, addressing the so-called multi-epoch least-squares adjustment (ME-LSA) problem (Smith et al. 2023a, Smith et al. 2023b, Smith 2023). However, those papers were highly generalized, without specifically applying their equations to particular observations or constraints. This memorandum builds upon those papers, using assumptions mentioned therein, specifically ignoring certain correlations (Smith 2023), and applying the equations to the observations that will be supported in the NGS Online Positioning User Service (OPUS) and reference epoch coordinate (REC) adjustments (NGS 2021), with a particular emphasis on ensuring that SPROCCET is able to create files usable by the NGS new least-squares adjustment suite, LASER (Least-squares Adjustments: Statistics, Estimates and Residuals)

This memorandum contains the equations needed to project specific geometric and orthometric values (observations or constraints, and their cofactor matrices) through time. The equations in this memorandum are suitable for immediate incorporation into SPROCCET or any other software that projects geodetic data through time.

1 SPROCCET and the purpose of this paper

As part of its efforts to modernize the National Spatial Reference System (NSRS), the National Geodetic Survey (NGS) is planning to use least-squares adjustment (LSA) techniques to estimate parameters (primarily geodetic coordinates) at passive control at specific epochs, based on observations that could span decades (NGS 2021). Those estimated coordinates[1] are called *reference epoch coordinates* (RECs).

The mathematical basis for combining observations and constraints from a variety of epochs into a least-squares adjustment of coordinates at a single epoch (which NGS has dubbed *the multi-epoch least-squares adjustment problem*, or ME-LSA) were detailed in a variety of recent papers (Smith et al. 2023a; Smith et al. 2023b; Smith 2023). One of the key ideas in those papers (excluding Smith et al. 2023b) is the transforming (or "projecting") of *observations* and *constraints* (at their original epochs) and their cofactor matrices through time into *projected observations* and *projected constraints* (at the adjustment epoch) and their *projected cofactor matrices*, as a necessary step prior to the LSA[2].

For many years, NGS has relied upon its Horizontal Time-Dependent Positioning (HTDP) software to compute projected quantities (Snay 1999), though the term *projected* was not used at the time. In fact, HTDP was the key pre-processor for all Global Navigation Satellite System (GNSS) data as NGS prepared for the national adjustment in 2011 (Dennis 2020). Unfortunately, HTDP suffers from a variety of shortcomings, including:

- It doesn't work with orthometric quantities
- It doesn't have very much vertical change information
- It doesn't propagate random errors through time
- It is in FORTRAN, making it suboptimal for maintenance and incorporating into online tools using more current programming languages
- It contains at least one known error in an earthquake model
- It mixes 14 parameter Helmert transformations with in-frame deformation models in a non-modularized way that is counter to the direction NGS is taking with new, complex software packages

For these reasons, HTDP does not have the necessary functionality to support the modernized NSRS. Therefore, NGS is engaged in a project to create a new software tool named SPROCCET (Software for PRojecting Observations, Constraints and Cofactor matrices/Errors through Time), which is intended to replace HTDP as the primary engine for computing projected geodetic values. The purpose of SPROCCET is to project observations and constraints and their cofactor

[1] Most estimated parameters will be coordinates. However, some will not be, such as orientations of unoriented horizontal directions. Such parameters are often called "nuisance parameters". For the sake of simplicity, we will generally use *parameters* and *coordinates* interchangeably unless it is necessary to be more specific.

[2] NGS is also planning to estimate coordinates from much shorter time-spans of observations (NGS 2021), called *survey epoch coordinates* (SECs), which do not need to be projected through time. As such, those adjustment projects will not need to rely upon the projection functionality within SPROCCET.

matrices through time into *projected* observations, *projected* constraints and their *projected* cofactor matrices. In addition to all of its other functionality, SPROCCET will incorporate three primary modules:

1) 14H — A newly developed subroutine whose sole purpose is to perform 14-parameter Helmert transformations
2) IFDM2022 — One of the two geodetic value change models (GVCM) within SPROCCET, which uses velocities and displacing events (i.e., earthquakes) to project geometric quantities through time within ITRF2020
3) DGEOID2022 — The second geodetic value change model (GVCM) within SPROCCET, which uses velocities to project geoid undulations through time

The ME-LSA papers (Smith et al. 2023a, Smith et al. 2023b, Smith 2023) were highly theoretical and generalized and did not contain explicit and easy-to-use equations for computing projected values. It was therefore seen as important to derive from those papers the exact, simple-to-use formulae needed to support the projecting of specific constraints and observations within SPROCCET. That is the ultimate purpose of this paper. The explicit equations to be coded up are found in sections 8 through 18, but a number of details must be clarified prior to presenting those equations.

2 Observations, constraints and the adjustment epoch

For the purposes of this paper, observations and constraints come in two types: geometric or orthometric. Constraints are further broken down into two sub-types: *Coordinate constraints* and *rank-invariant constraints.* The latter resemble *observations*, such as a slant distance, but are not true observations. However, because the equations in this paper that refer to projecting observations and those that will project rank-invariant constraints are identical, we will eschew the term rank-invariant constraint, and call them "*observational constraints*". *Coordinate constraints*, whether geometric or orthometric, refer to the coordinates of a single point. Geometrically, all least-squares adjustments performed by NGS using a newly-developed least-squares suite called LASER (Least-squares Adjustments: Statistics, Estimates and Residuals) will estimate Earth-centered, Earth-fixed (ECEF) global Cartesian coordinates (XYZ). As such, this is one of two permitted geometric coordinate constraint triads. The other is the triad of geodetic longitude, geodetic latitude, and ellipsoid height. Orthometrically the only *coordinate constraints* allowed are the orthometric heights (H) of points above the geoid.

Observations and *observational constraints* may involve one, two or three points, depending on the type of observation. A complete list of coordinate constraints and observations/observational constraints supported by SPROCCET (and ultimately to be supported in the products and services of the modernized NSRS) is found in Table 1.

Table 1: Supported values in SPROCCET, their categories and number of involved points

What?	Geom or Ortho?	Coordinate Constraint (CC) or Observation/ Observational Constraint (O/OC)	Number of points involved
ECEF Cartesian triad (XYZ)	Geometric	CC	1
Geodetic triad ($\lambda\phi h$)[3]	Geometric	CC	1
Orthometric height (H)	Orthometric	CC	1
Precise Point Positioning (PPP)	Geometric	O/OC	1
Slant distance	Geometric	O/OC	2
Zenith angle	Geometric	O/OC	2
Geodetic azimuth	Geometric	O/OC	2
Unoriented horizontal direction	Geometric	O/OC	2
GNSS measured baseline (GMB)	Geometric	O/OC	2
Horizontal angle	Geometric	O/OC	3
Differential orthometric height (DOH)	Orthometric or Geometric[4]	O/OC	2

Constraints may be fixed or stochastic. Although NGS policy toward REC projects is that such projects will never use *projected* constraints, the possibility exists that users will want to use projected constraints in OPUS.

2.1 The adjustment epoch

The single most important thing that SPROCCET will take as input is a unique date and time, t, called the *adjustment epoch*. SPROCCET will project observations or constraints and their cofactor matrices at a number of different epochs through time into *projected* observations or constraints, and their cofactor matrices, all at the adjustment epoch.

3 Coordinates vs coordinate constraints

The purpose of SPROCCET is to implement the equations that project observations and constraints through time, in preparation for their use in a least-squares adjustment, as outlined in the ME-LSA papers (Smith et al. 2023a, Smith 2023). This is a slight philosophical shift from HTDP which was used as a general crustal deformation model, able to project coordinates (without distinguishing them as "constraints") in general, through time (horizontally). In particular, the question of whether or not a coordinate is known without variance (fixed), with

[3] We note that listing these coordinates in this order is unconventional, as they are often listed with latitude first, as $\phi\lambda h$. However, as this paper will often rely upon quantities in the east, north, and up directions (in that order, to retain the right-handedness of the system), and these values have direct relationships to longitude, latitude, and ellipsoid height (respectively) it is best, for consistency, to list our geodetic coordinate constraints in $\lambda\phi h$.

[4] Differential orthometric heights are orthometric observations, but may sometimes be used in a geometric least-squares adjustment (when combined with differential geoid undulations). As such, they are listed both as an orthometric and a geometric observation.

variance (stochastic) or is not known well at all (unconstrained) is of critical importance. SPROCCET will only project the first two types, while leaving the third completely unmodified.

The difference between *unconstrained coordinates* and *coordinate constraints* is discussed below.

For some point A, whose coordinates are given, there are two possible scenarios.

1) The provided coordinates are *unconstrained coordinates*, sometimes called a-priori coordinates. Their sole function is to be "in the ballpark" of the true coordinates, and thus seed the least-squares adjustment engine in such a way as to allow it to work in residual values, reduce numbers of iterations and keep the LSA from finding an incorrect local minimum. Therefore, they might be inaccurate by up to tens of meters or more. *The epoch of such coordinates might or might not be known*. Such coordinates may or may not come with statistical information such as standard deviations. Examples of such coordinates might be those found in the 80 and 86 records of a Bfile (Yeager, 1980), or those flagged with "type=0" within a so-called "LASER Constraints[5] file" (Snow et al. 2023), whether geometric or orthometric. Considering the generally small differences between original coordinates and projected coordinates (likely a few centimeters to decimeters due to velocities and displacing events) and considering the lack of supporting information which may come with a-priori coordinates, *SPROCCET will not attempt to project unconstrained coordinates through time*. Nonetheless, such unconstrained coordinates have a very important function, both within least-squares adjustments in general, and in the projecting of observations in particular. That is, they serve as a-priori coordinates of points that are (a) adjusted in a LSA and (b) needed to properly compute the projecting equations later in this paper. Such a-priori coordinates will be represented with a "0" subscript in this document.

2) The provided coordinates are a *constraint* (either fixed or stochastic). These are *accurate* coordinates, with a *known* constraint epoch, and fall into a few sub-scenarios.
   a. The *constraint epoch is the same as the adjustment epoch*. If this is the case, then they do not need to be projected through time, and therefore, if fed into SPROCCET, the input constraint will be identical to the output projected constraint.
   b. The *constraint epoch is not the same as the adjustment epoch*, and therefore the constrained coordinates must be projected through time. In such a case, we must distinguish between two types of constraints: fixed or stochastic. The two types are listed below. The next paragraph will discuss what is needed for each, to project it through time.
      i. A fixed constraint, not at the adjustment epoch.
      ii. A stochastic constraint, not at the adjustment epoch.

[5] Despite its name, a LASER Constraints file can hold coordinates that are fixed constraints (type=2), stochastic constraints (type=1) or *unconstrained* (type=0).

When a coordinate constraint (fixed or stochastic) is provided to SPROCCET at some point, A, at some constraint epoch t_i[6] (that is not the adjustment epoch) the following must be provided to SPROCCET:

- If $X_{A,t_i}, Y_{A,t_i}, Z_{A,t_i}$ are *fixed* constraints, then only $X_{A,t_i}, Y_{A,t_i}, Z_{A,t_i}$ must be provided.
- If $X_{A,t_i}, Y_{A,t_i}, Z_{A,t_i}$ are *stochastic* constraints, then $X_{A,t_i}, Y_{A,t_i}, Z_{A,t_i}$, and their 3×3 cofactor matrix, Σ_{XYZ,A,t_i} must be provided (or related information, such as standard deviations in the east, north, and up directions, capable of computing the cofactor matrix is provided).
- If $\lambda_{A,t_i}, \phi_{A,t_i}, h_{A,t_i}$ are *fixed* constraints, then only $\lambda_{A,t_i}, \phi_{A,t_i}, h_{A,t_i}$ must be provided.
- If $\lambda_{A,t_i}, \phi_{A,t_i}, h_{A,t_i}$ are *stochastic* constraints, then $\lambda_{A,t_i}, \phi_{A,t_i}, h_{A,t_i}$, and their 3×3 cofactor matrix, $\Sigma_{\lambda\phi h,A,t_i}$ must be provided.
- If H_{A,t_i} is a *fixed* constraint, then H_{A,t_i}, as well as some a-priori longitude and latitude ($\lambda_{A,0}, \phi_{A,0}$), must be provided.
- If H_{A,t_j} is a *stochastic* constraint, then H_{A,t_i}, some a-priori longitude and latitude ($\lambda_{A,0}, \phi_{A,0}$) and the 1×1 cofactor matrix (scalar) for the orthometric height, Σ_{H,A,t_i} must be provided.

4 Observations not at the adjustment epoch

It would be exceedingly rare for an observation to occur exactly at the adjustment epoch. If so, however, then the input observation (and cofactor matrix) and the output *projected* observation (and *projected* cofactor matrix) would be identical.

Therefore, for any given observation with its given cofactor matrix, the following additional values must be *given*, prior to attempting to project the observation and its cofactor matrix through time:

1) A-priori[7] coordinates (longitude, latitude, and ellipsoid height: λ, ϕ, h) of all points involved in the observation(s):
    a. For **Precise Point Positioning (PPP)**, just point A: $\lambda_{A,0}, \phi_{A,0}, h_{A,0}$
    b. For **horizontal angles**, points A, B and C: $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}, \lambda_{C,0}, \phi_{C,0}, h_{C,0}$
    c. For **a single GNSS measured baseline (GMB)**, points A and B: $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}$

[6] In Smith et al. (2023a), variable t_i was used for observation epochs, while t_{p+j} and t_{p+q+k} were used for stochastic and fixed constraint epochs, respectively. This was necessary for the derivations within that paper. However, in this paper it is not necessary to maintain that level of separation, and we generalized any non-adjustment epoch (whether for an observation or a constraint) as t_i.

[7] The ",0" subscript implies the a-priori nature of the coordinate value. It's critical to note that any given *observation* (involving points A and/or B and/or C) will almost certainly *disagree* with a so-called, and computable, "a-priori observation" that can be computed by using the a-priori coordinates at points A and/or B and/or C. The a-priori coordinates are not to be considered definitive, but are generally required to be "in the ballpark". They are needed for computations, but can be off by some fairly large amount from truth, without substantially impacting the computations in this paper.

d. For **two GNSS measured baselines (GMBs) in the same session, which share a single common point, A**, points A, B and C:
   $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}, \lambda_{C,0}, \phi_{C,0}, h_{C,0}$
e. For **all classical observations, aside from horizontal angles**, points A and B:
   $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}$
f. For **differential orthometric heights**, points A and B:
   $\lambda_{A,0}, \phi_{A,0}, H_{A,0}, \lambda_{B,0}, \phi_{B,0}, H_{B,0}$

2) Observation epoch t_i
3) IFDM2022 velocity grids and standard deviation grids in ENU
4) IFDM2022 displacement grids and standard deviation grids (and their respective epochs) in ENU
5) DGEOID2022 velocity grids and standard deviation grids
6) 14-parameter Helmert transformations
7) The a and e^2 values of the GRS80 ellipsoid

4.1 To be computed for each observation before performing any projections

Using the given values from above, the following should be computed prior to projection:

1) Given $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}, \lambda_{C,0}, \phi_{C,0}, h_{C,0}$ compute $X_{A,0}, Y_{A,0}, Z_{A,0}, X_{B,0}, Y_{B,0}, Z_{B,0}, X_{C,0}, Y_{C,0}, Z_{C,0}$
2) Given t and t_i , compute $\Delta t = t_i - t$

5 Constraints not at the adjustment epoch

As mentioned earlier, constraints in future NGS REC projects will be restricted to *coordinate* constraints *at the adjustment epoch*. There will be no observational constraints at all, nor any constraints not at the adjustment epoch. However, OPUS will be built to allow users to specify both coordinate constraints and observational constraints; furthermore, any given constraint might or might not be at the adjustment epoch. If not at the adjustment epoch, the constraint will be at the *constraint epoch*.

When the constraint is at the adjustment epoch the input constraint (and cofactor matrix, if the constraint is stochastic) and the output *projected* constraint (and *projected* cofactor matrix, if the constraint is stochastic) would be identical.

Therefore, for any given constraint with its given cofactor matrix (if stochastic), the following values must be *given*, prior to attempting to project the constraint and its cofactor matrix through time:

5.1 For Coordinate Constraints:

1) The coordinate constraints themselves. One of these three:

a. λ, ϕ, h[8]
b. XYZ
c. H

2) The cofactor matrix (or related information[9] capable of computing the cofactor matrix) of the coordinate constraints, if they are stochastic.
3) Constraint epoch t_i
4) IFDM2022 velocity grids and standard deviation grids in ENU
5) IFDM2022 displacement grids and standard deviation grids (and their respective epochs) in ENU
6) DGEOID2022 velocity grids and standard deviation grids
7) 14-parameter Helmert transformations
8) The a and e^2 values of the GRS80 ellipsoid

Using the given values from above, the following should be computed prior to projection:

1) Given $\lambda_A(t_i), \phi_A(t_i), h_A(t_i)$ compute $X_A(t_i), Y_A(t_i), Z_A(t_i)$
2) Given t and t_i , compute $\Delta t = t_i - t$

5.2 For Observational Constraints

1) The observational constraint value.
2) A-priori coordinates of the involved points:
   a. For **PPP**, just point A: $\lambda_{A,0}, \phi_{A,0}, h_{A,0}$
   b. For **horizontal angles**, points A, B and C: $\lambda_{A,0}\phi_{A,0},, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}, \lambda_{C,0}, \phi_{C,0}, h_{C,0}$
   c. For **a single GMB**, points A and B: $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}$
   d. For **two GMBs in the same session, which share a single common point, A**, points A, B and C: $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}, \lambda_{C,0}, \phi_{C,0}, h_{C,0}$
   e. For **all classical observations, aside from horizontal angles**, points A and B: $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}$
   f. For **differential orthometric heights**, points A and B: $\lambda_{A,0}, \phi_{A,0}, H_{A,0}, \lambda_{B,0}, \phi_{B,0}, H_{B,0}$
3) Constraint epoch t_i
4) IFDM2022 velocity grids and standard deviation grids in ENU
5) IFDM2022 displacement grids and standard deviation grids (and their respective epochs) in ENU
6) DGEOID2022 velocity grids and standard deviation grids

[8] Although the coordinates estimated in the NGS geometric adjustments will be strictly XYZ, it is possible that users will wish to constrain a point to its longitude, latitude, and ellipsoid height ($\lambda\phi h$) values. As such, equations to allow for this situation will be provided in this paper.
[9] For example, constrained coordinates in XYZ might be provided, but standard deviations in the east, north, and up directions available. These latter values could be used to compute the cofactor matrix of the constrained XYZ coordinates.

7) 14-parameter Helmert transformations
8) The a and e^2 values of the GRS80 ellipsoid

Using the given values from above, the following should be computed prior to projection:

1) Given $\lambda_{A,0}, \phi_{A,0}, h_{A,0}, \lambda_{B,0}, \phi_{B,0}, h_{B,0}, \lambda_{C,0}, \phi_{C,0}, h_{C,0}$ compute $X_{A,0}, Y_{A,0}, Z_{A,0}, X_{B,0}, Y_{B,0}, Z_{B,0}, X_{C,0}, Y_{C,0}, Z_{C,0}$.
2) Given t and t_i , compute $\Delta t = t_i - t$.

6 Relating a global ECEF Cartesian frame to a local geodetic horizon frame

We designate three rotation matrices M_1, M_2, and M_3 to represent a rotation of a Cartesian coordinate frame about the X, Y, or Z axes of that frame, respectively. These rotation matrices are consistent with a positive rotation in the counterclockwise direction of a right-handed coordinate system, when viewed down the axis from the viewpoint of its positive end (Leick and van Gelder, 1975). Each rotation matrix can be found in Appendix A of Leick (2004), but is repeated below. We purposefully avoid using "R" for these rotation matrices for reasons that will be clear soon.

$$M_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

$$M_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2}$$

$$M_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

In order to relate any vector (dx, dy, dz) in a local geodetic horizon (LGH) frame, with its origin at point A (on the surface of the Earth), where the x, y, and z axes point east, north, and up respectively, with the same vector but expressed in a global Cartesian (GC) frame (dX, dY, dZ), where the X, Y, and Z axes point to the prime meridian (X), the pole (Z), and the Y axis forms a right-handed system, we use the following general formula:

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} de \\ dn \\ du \end{bmatrix} = M_1\left(\frac{\pi}{2} - \phi_A\right) M_3\left(\frac{\pi}{2} + \lambda_A\right) \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} = M_A \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix}, \tag{4}$$

where we have introduced matrix M_A as:

$$M_A = M_1\left(\frac{\pi}{2} - \phi_A\right) M_3\left(\frac{\pi}{2} + \lambda_A\right) = \begin{bmatrix} -\sin\lambda_A & \cos\lambda_A & 0 \\ -\sin\phi_A \cos\lambda_A & -\sin\phi_A \sin\lambda_A & \cos\phi_A \\ \cos\phi_A \cos\lambda_A & \cos\phi_A \sin\lambda_A & \sin\phi_A \end{bmatrix}. \tag{5}$$

For ease of later notation, we now introduce matrix R_A as:

$$R_A = {M_A}^T = \begin{bmatrix} -\sin\lambda_A & -\sin\phi_A\cos\lambda_A & \cos\phi_A\cos\lambda_A \\ \cos\lambda_A & -\sin\phi_A\sin\lambda_A & \cos\phi_A\sin\lambda_A \\ 0 & \cos\phi_A & \sin\phi_A \end{bmatrix}. \tag{6}$$

The notation here matches that of the derivation of the multi-epoch least-squares adjustment (ME-LSA) in Smith et al. (2023a), therefore R_A is defined as the transpose of M_A.

If we have a-priori coordinates for point A (ϕ_A, λ_A) we define matrices $M_{A,0}$ and $R_{A,0}$ as:

$$M_{A,0} = \begin{bmatrix} -\sin\lambda_{A,0} & \cos\lambda_{A,0} & 0 \\ -\sin\phi_{A,0}\cos\lambda_{A,0} & -\sin\phi_{A,0}\sin\lambda_{A,0} & \cos\phi_{A,0} \\ \cos\phi_{A,0}\cos\lambda_{A,0} & \cos\phi_{A,0}\sin\lambda_{A,0} & \sin\phi_{A,0} \end{bmatrix}, \tag{7}$$

$$R_{A,0} = {M_{A,0}}^T = \begin{bmatrix} -\sin\lambda_{A,0} & -\sin\phi_{A,0}\cos\lambda_{A,0} & \cos\phi_{A,0}\cos\lambda_{A,0} \\ \cos\lambda_{A,0} & -\sin\phi_{A,0}\sin\lambda_{A,0} & \cos\phi_{A,0}\sin\lambda_{A,0} \\ 0 & \cos\phi_{A,0} & \sin\phi_{A,0} \end{bmatrix}. \tag{8}$$

Consider now the variety of classical observations that are generally collected in a local astronomic horizon (LAH) frame, and then through use of deflections of the vertical are reduced to the LGH frame. Generally speaking, they involve two points, which we will call A and B. In the case of horizonal angles, they involve three points, A, B and C. In every case the instrument is at point A, and sighting point B, and sometimes sighting point C. To see this more clearly, observe Figure 1 and Figure 2.

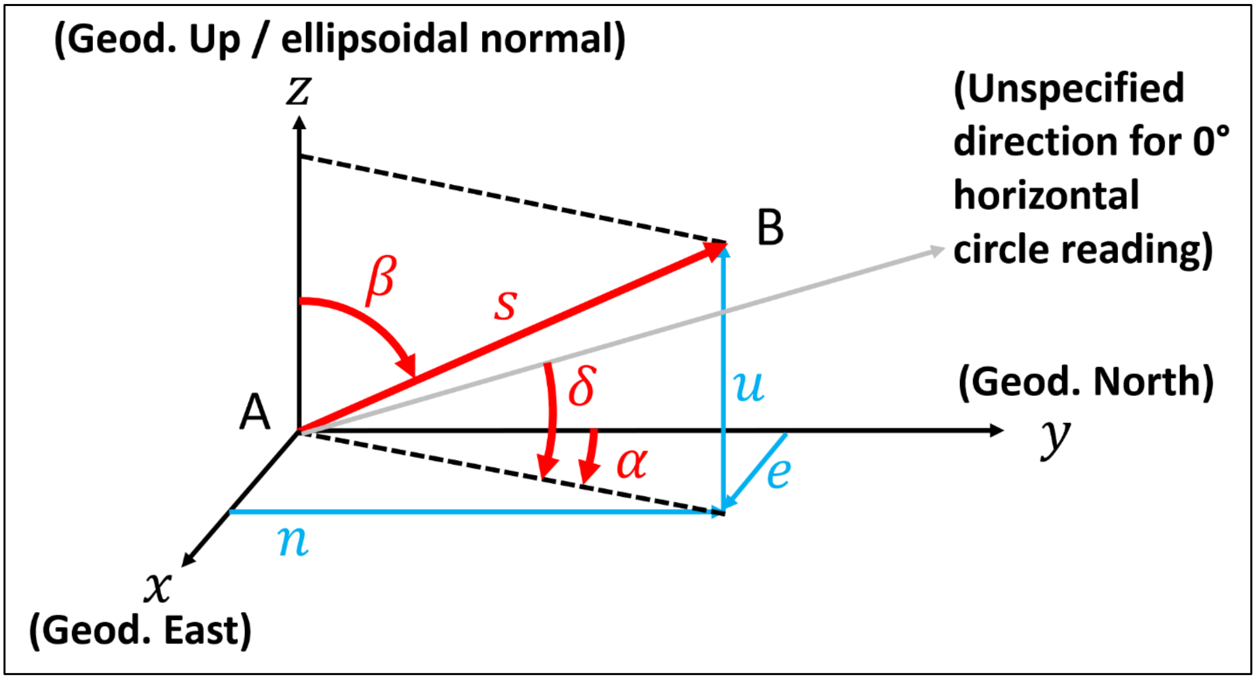

Figure 1: Relation between slant distance (s), zenith angle (β), geodetic azimuth (α) and unoriented direction (δ) and a local geodetic horizon ENU frame

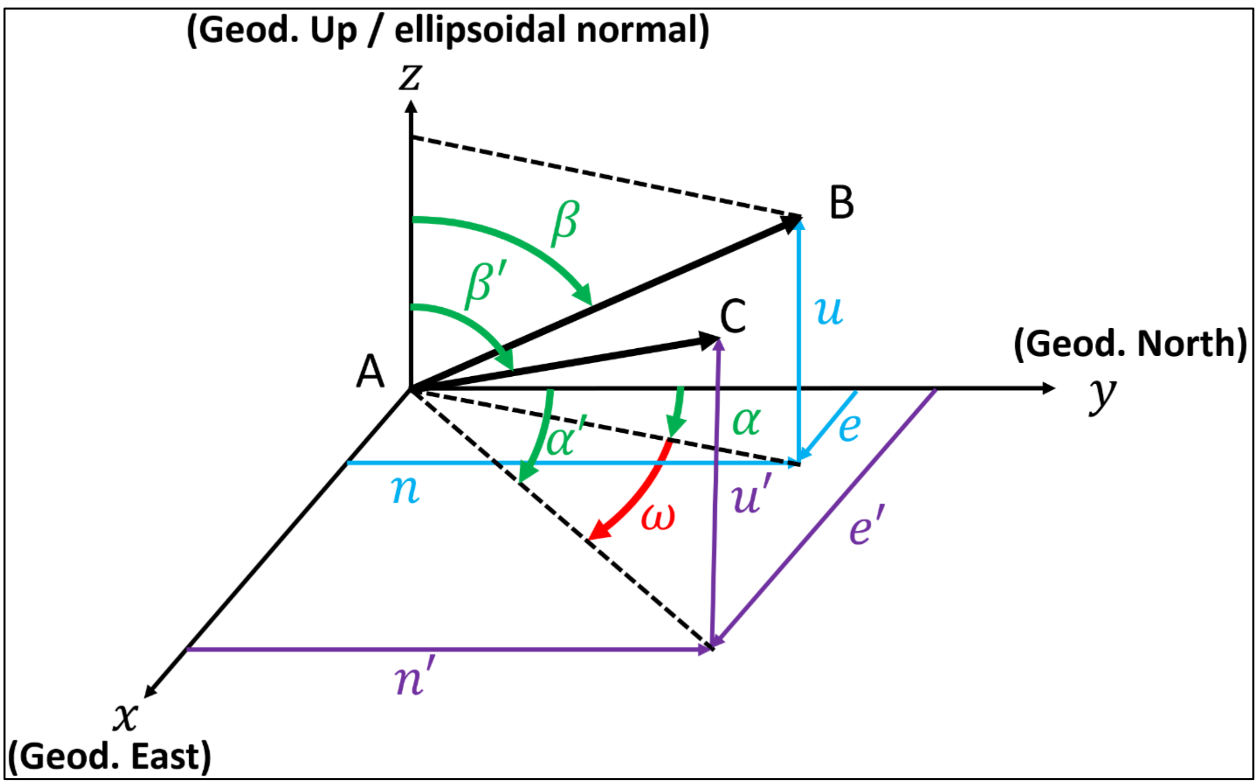

Figure 2: Relation between horizontal angle (ω) and a local geodetic horizon ENU frame

Note that in the above figures, the sighting vector from A to B originates at point A and ends at some point B. Because A is the origin of the LGH frame, it has coordinates $(x_A, y_A, z_A) = (0,0,0)$. If we then designate the vector components from A to B as (e, n, u) we can simplify and expand (4) as follows:

$$\begin{bmatrix} e \\ n \\ u \end{bmatrix} = M_A \begin{bmatrix} X_B - X_A \\ Y_B - Y_A \\ Z_B - Z_A \end{bmatrix}$$
$$= \begin{bmatrix} -(X_B - X_A)\sin\lambda_A + (Y_B - Y_A)\cos\lambda_A \\ -(X_B - X_A)\cos\lambda_A\sin\phi_A - (Y_B - Y_A)\sin\lambda_A\sin\phi_A + (Z_B - Z_A)\cos\phi_A \\ (X_B - X_A)\cos\lambda_A\cos\phi_A + (Y_B - Y_A)\sin\lambda_A\cos\phi_A + (Z_B - Z_A)\sin\phi_A \end{bmatrix}. \quad (9)$$

When a third point, C, is involved (generally only when a horizontal angle is the observation), we use the following equation for the second set of east, north, and up vector elements:

$$\begin{bmatrix} e' \\ n' \\ u' \end{bmatrix} = M_A \begin{bmatrix} X_C - X_A \\ Y_C - Y_A \\ Z_C - Z_A \end{bmatrix}$$
$$= \begin{bmatrix} -(X_C - X_A)\sin\lambda_A + (Y_C - Y_A)\cos\lambda_A \\ -(X_C - X_A)\cos\lambda_A\sin\phi_A - (Y_C - Y_A)\sin\lambda_A\sin\phi_A + (Z_C - Z_A)\cos\phi_A \\ (X_C - X_A)\cos\lambda_A\cos\phi_A + (Y_C - Y_A)\sin\lambda_A\cos\phi_A + (Z_C - Z_A)\sin\phi_A \end{bmatrix}. \quad (10)$$

When we are provided with a-priori coordinates for points A, B or C, then we can generate a-priori versions of the east, north, and up vector elements as:

$$\begin{bmatrix} e_0 \\ n_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} -(X_{B,0} - X_{A,0})\sin\lambda_{A,0} + (Y_{B,0} - Y_{A,0})\cos\lambda_{A,0} \\ -(X_{B,0} - X_{A,0})\cos\lambda_{A,0}\sin\phi_{A,0} - (Y_{B,0} - Y_{A,0})\sin\lambda_{A,0}\sin\phi_{A,0} + (Z_{B,0} - Z_{A,0})\cos\phi_{A,0} \\ (X_{B,0} - X_{A,0})\cos\lambda_{A,0}\cos\phi_{A,0} + (Y_{B,0} - Y_{A,0})\sin\lambda_{A,0}\cos\phi_{A,0} + (Z_{B,0} - Z_{A,0})\sin\phi_{A,0} \end{bmatrix}, \quad (11)$$

and

$$\begin{bmatrix} e'_0 \\ n'_0 \\ u'_0 \end{bmatrix} = \begin{bmatrix} -(X_{C,0} - X_{A,0})\sin\lambda_{A,0} + (Y_{C,0} - Y_{A,0})\cos\lambda_{A,0} \\ -(X_{C,0} - X_{A,0})\cos\lambda_{A,0}\sin\phi_{A,0} - (Y_{C,0} - Y_{A,0})\sin\lambda_{A,0}\sin\phi_{A,0} + (Z_{C,0} - Z_{A,0})\cos\phi_{A,0} \\ (X_{C,0} - X_{A,0})\cos\lambda_{A,0}\cos\phi_{A,0} + (Y_{C,0} - Y_{A,0})\sin\lambda_{A,0}\cos\phi_{A,0} + (Z_{C,0} - Z_{A,0})\sin\phi_{A,0} \end{bmatrix}. \quad (12)$$

The above formulae will be referenced extensively in the next few sections. Note in the figures below that x, y, and z (lower case) are the names of the axes in the east, north, and up direction in a LGH frame. They should not be confused with X, Y, and Z (upper case), which are coordinates in the GC frame.

7 Commonly needed vectors, matrices and scalars

Projecting observations or constraints through time requires the use of certain scalars, vectors and matrices, whose sizes depend on whether one is projecting something that refers to one, two, or three points. See Table 1 for a summary. This section will discuss the content and sizes of those commonly used items.

The first thing needed is the observation or constraint epoch t_i. From this, a difference with the adjustment epoch, t, is formed as:

$$\Delta t = t_i - t. \tag{13a}$$

Also needed will be the sign of Δt as noted in Smith (2023):

$$q = sgn(\Delta t) = \begin{cases} +1 & \text{if } \Delta t > 0 \\ -1 & \text{if } \Delta t < 0 \\ 0 & \text{if } \Delta t = 0. \end{cases} \tag{13b}$$

Most observations/observational constraints, with the exception of PPP and horizontal angles, involve two points, which we will call A and B. Coordinate constraints and PPP observations/observational constraints involve only one point, A, while horizontal angles involve three points, A, B, and C. Additionally complicating the situation is that GNSS measured baselines (GMBs) are frequently processed together in a session, and while any one GMB involves two points, the session processing means that we must consider the correlations between two GMBs. More on that is found in in section 15.

For all observations or constraints, certain values will be extracted from the GVCMs themselves (IFDM2022 and DGEOID2022), and used to populate certain vectors and matrices needed to perform the projection of observations or constraints through time. This section provides those details.

First, for any given point, say point A, there will be an a-priori longitude and latitude. This value is used to interpolate a value off of numerous grids.

Second, from IFDM2022 (crustal deformation and velocity model), there will be three grids of crustal velocity (one each for east, north, and up) and three companion grids of standard deviation of crustal velocity (one each for east, north, and up). Then, for each event (earthquake) that impacts the observation or constraint, there will be six grids (three for crustal displacements in east, north, and up and three for the standard deviations of those crustal displacements in east, north, and up).

Next, from DGEOID2022 (velocity model of GEOID2022), there will be one grid of geoid undulation velocity (in the up direction) and one companion grid of standard deviation of geoid undulation velocity (in the up direction). Current plans do not call for DGEOID2022 to contain any displacement grids to account for earthquakes or other episodic events.

After interpolation from the various grids, we have the following quantities at point A.

- ENU velocities: $\dot{E}_A, \dot{N}_A, \dot{U}_A$
- Standard deviation of ENU velocities: $\sigma_{\dot{E}_A}, \sigma_{\dot{N}_A}, \sigma_{\dot{U}_A}$
- ENU displacement for each event "k": $\Delta E_{A,k}, \Delta N_{A,k}, \Delta U_{A,k}$
- Standard deviation of ENU displacement for each event "k": $\sigma_{\Delta E_{A,k}}, \sigma_{\Delta N_{A,k}}, \sigma_{\Delta U_{A,k}}$

- Geoid undulation velocities[10]: $\dot{L}_A$
- Standard deviation of geoid undulation velocities: $\sigma_{\dot{L}_A}$

We will refer to this set of values as *the interpolated values* at point A. A similar set of *interpolated values* at point B or at point C will exist for those points, if needed. These values are then used to populate specific vectors and matrices, as outlined below. When discussing whether or not an event impacts an observation or constraint, we will say that the observation or constraint epoch is i, and events (k) which impact this observation or constraint will fall between epoch i and the adjustment epoch and will be designated by $k \in K(i)$.

7.1 For coordinate constraints, *single-point* observations/observational constraints or a common point between two GNSS measured baselines within the same GNSS session

In this case, only one point, A, is involved. The interpolated values will populate three vectors and three dispersion matrices. Further, the a-priori coordinates at point A will be used to populate a useful rotation matrix, R. We define vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and $\boldsymbol{w}$ and matrices Σ_v, Σ_d, Σ_w and R as:

$$\boldsymbol{v} = \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{U}_A \end{bmatrix}, \tag{14a}$$

$$\boldsymbol{d} = \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta U_{A,k} \end{bmatrix}, \tag{14b}$$

$$\Sigma_v = \begin{bmatrix} \sigma^2_{\dot{E}_A} & 0 & 0 \\ 0 & \sigma^2_{\dot{N}_A} & 0 \\ 0 & 0 & \sigma^2_{\dot{U}_A} \end{bmatrix}, \tag{14c}$$

$$\Sigma_d = \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \sigma^2_{\Delta E_{A,k}} & 0 & 0 \\ 0 & \sigma^2_{\Delta N_{A,k}} & 0 \\ 0 & 0 & \sigma^2_{\Delta U_{A,k}} \end{bmatrix}, \tag{14d}$$

[10] The conventional variable for geoid undulation in the geodetic literature is N. However, as N is being used exclusively to mean "north" in the ENU system, we adopt the variable L for geoid undulation for this paper, which matches the notion used in Smith (2023).

$$\boldsymbol{w} = [\dot{L}_A], \tag{14e}$$

$$\Sigma_w = [\sigma^2_{\dot{L}_A}], \tag{14f}$$

$$R = R_{A,0} = \begin{bmatrix} -\sin\lambda_{A,0} & -\cos\lambda_{A,0}\sin\phi_{A,0} & \cos\lambda_{A,0}\cos\phi_{A,0} \\ \cos\lambda_{A,0} & -\sin\lambda_{A,0}\sin\phi_{A,0} & \sin\lambda_{A,0}\cos\phi_{A,0} \\ 0 & \cos\phi_{A,0} & \sin\phi_{A,0} \end{bmatrix}. \tag{14g}$$

7.1.1 Special case for orthometric height constraints

Orthometric heights (H) are the coordinates at points in an orthometric adjustment, but projecting them through time does not have any dependence upon the velocities and displacements from IFDM2022 in the east (longitude) nor north (latitude) directions. As such, for this specific coordinate constraint, there are some special vectors and matrices, which are used in conjunction with (14e) and (14f):

$$\boldsymbol{v} = [\dot{U}_A], \tag{15a}$$

$$\boldsymbol{d} = \sum_{\substack{k \\ k \in K(i)}} [\Delta U_{A,k}], \tag{15b}$$

$$\Sigma_v = [\sigma^2_{\dot{U}_A}], \tag{15c}$$

$$\Sigma_d = \sum_{\substack{k \\ k \in K(i)}} [\sigma^2_{\Delta U_{A,k}}]. \tag{15d}$$

7.2 For *two-point* observations/observational constraints (slant distances, etc.)

In this case, the observation/observational constraint involves two points, A and B. The interpolated values will populate three vectors and three dispersion matrices. Further, the a-priori coordinates at points A and B will be used to populate a useful rotation matrix, R, extrapolating from the definition of $R_{A,0}$ in (8) to make $R_{B,0}$, below. We define vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and $\boldsymbol{w}$ and matrices Σ_v, Σ_d, Σ_w and R as:

$$\boldsymbol{v} = \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{U}_A \\ \dot{E}_B \\ \dot{N}_B \\ \dot{U}_B \end{bmatrix}, \tag{16a}$$

$$\boldsymbol{d} = \sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta U_{A,k} \\ \Delta E_{B,k} \\ \Delta N_{B,k} \\ \Delta U_{B,k} \end{bmatrix}, \tag{16b}$$

$$\Sigma_v = \begin{bmatrix} \sigma^2_{\dot{E}_A} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{\dot{N}_A} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{\dot{U}_A} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{\dot{E}_B} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{\dot{N}_B} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{\dot{U}_B} \end{bmatrix}, \tag{16c}$$

$$\Sigma_d = \sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \sigma^2_{\Delta E_{A,k}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{\Delta N_{A,k}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{\Delta U_{A,k}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{\Delta E_{B,k}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{\Delta N_{B,k}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{\Delta U_{B,k}} \end{bmatrix}, \tag{16d}$$

$$\boldsymbol{w} = \begin{bmatrix} \dot{L}_A \\ \dot{L}_B \end{bmatrix}, \tag{16e}$$

$$\Sigma_w = \begin{bmatrix} \sigma^2_{\dot{L}_A} & 0 \\ 0 & \sigma^2_{\dot{L}_B} \end{bmatrix}, \tag{16f}$$

$$R = \begin{bmatrix} R_{A,0} & 0 \\ 0 & R_{B,0} \end{bmatrix} =$$
$$= \begin{bmatrix} -\sin\lambda_{A,0} & -\cos\lambda_{A,0}\sin\phi_{A,0} & \cos\lambda_{A,0}\cos\phi_{A,0} & 0 & 0 & 0 \\ \cos\lambda_{A,0} & -\sin\lambda_{A,0}\sin\phi_{A,0} & \sin\lambda_{A,0}\cos\phi_{A,0} & 0 & 0 & 0 \\ 0 & \cos\phi_{A,0} & \sin\phi_{A,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\lambda_{B,0} & -\cos\lambda_{B,0}\sin\phi_{B,0} & \cos\lambda_{B,0}\cos\phi_{B,0} \\ 0 & 0 & 0 & \cos\lambda_{B,0} & -\sin\lambda_{B,0}\sin\phi_{B,0} & \sin\lambda_{B,0}\cos\phi_{B,0} \\ 0 & 0 & 0 & 0 & \cos\phi_{B,0} & \sin\phi_{B,0} \end{bmatrix}. \quad (16g)$$

7.2.1 Special case for differential orthometric height observations/observational constraints

Differential orthometric heights (DOHs) are a two-point observation, but projecting them through time does not have any dependence upon the velocities and displacements from the IFDM in the east (longitude) nor north (latitude) directions. As such, for this specific observation/observational constraint, there are some special vectors and matrices, which are used in conjunction with (16e) and (16f):

$$\boldsymbol{v} = \begin{bmatrix} \dot{U}_A \\ \dot{U}_B \end{bmatrix}, \quad (17a)$$

$$\boldsymbol{d} = \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta U_{A,k} \\ \Delta U_{B,k} \end{bmatrix}, \quad (17b)$$

$$\Sigma_v = \begin{bmatrix} \sigma^2_{\dot{U}_A} & 0 \\ 0 & \sigma^2_{\dot{U}_B} \end{bmatrix}, \quad (17c)$$

$$\Sigma_d = \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \sigma^2_{\Delta U_{A,k}} & 0 \\ 0 & \sigma^2_{\Delta U_{B,k}} \end{bmatrix}. \quad (17d)$$

7.3 For *three-point* observations (horizontal angles)

In this case, the observation involves three points, A, B and C. The interpolated values will populate three vectors and three dispersion matrices. Further, the a-priori coordinates at points A and B will be used to populate a useful rotation matrix, R, extrapolating from the definition of $R_{A,0}$ in (8) to make $R_{B,0}$ and $R_{C,0}$ below. We define vectors $\boldsymbol{v}$, $\boldsymbol{d}$, and $\boldsymbol{w}$ and matrices Σ_v, Σ_d, Σ_w, and R as:

$$\boldsymbol{v} = \begin{bmatrix} \dot{E}_A \\ \dot{N}_A \\ \dot{U}_A \\ \dot{E}_B \\ \dot{N}_B \\ \dot{U}_B \\ \dot{E}_C \\ \dot{N}_C \\ \dot{U}_C \end{bmatrix}, \tag{18a}$$

$$\boldsymbol{d} = \sum_{\substack{k \\ k \in K(i)}} \begin{bmatrix} \Delta E_{A,k} \\ \Delta N_{A,k} \\ \Delta U_{A,k} \\ \Delta E_{B,k} \\ \Delta N_{B,k} \\ \Delta U_{B,k} \\ \Delta E_{C,k} \\ \Delta N_{C,k} \\ \Delta U_{C,k} \end{bmatrix}, \tag{18b}$$

$$\Sigma_v = \begin{bmatrix} \sigma_{\dot{E}_A}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{\dot{N}_A}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{\dot{U}_A}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{\dot{E}_B}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\dot{N}_B}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\dot{U}_B}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{\dot{E}_C}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{\dot{N}_C}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{\dot{U}_C}^2 \end{bmatrix}, \tag{18c}$$

$$\Sigma_d = \sum_{\substack{k \\ k\in K(i)}} \begin{bmatrix} \sigma^2_{\Delta E_{A,k}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{\Delta N_{A,k}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{\Delta U_{A,k}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{\Delta E_{B,k}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{\Delta N_{B,k}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{\Delta U_{B,k}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{\Delta E_{C,k}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{\Delta N_{C,k}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{\Delta U_{C,k}} \end{bmatrix}, \tag{18d}$$

$$\boldsymbol{w} = \begin{bmatrix} \dot{L}_A \\ \dot{L}_B \\ \dot{L}_C \end{bmatrix}, \tag{18e}$$

$$\Sigma_w = \begin{bmatrix} \sigma^2_{\dot{L}_A} & 0 & 0 \\ 0 & \sigma^2_{\dot{L}_B} & 0 \\ 0 & 0 & \sigma^2_{\dot{L}_C} \end{bmatrix}, \tag{18f}$$

$$R = \begin{bmatrix} R_{A,0} & 0 & 0 \\ 0 & R_{B,0} & 0 \\ 0 & 0 & R_{C,0} \end{bmatrix} = \begin{bmatrix} -\sin\lambda_{A,0} & -\cos\lambda_{A,0}\sin\phi_{A,0} & \cos\lambda_{A,0}\cos\phi_{A,0} & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\lambda_{A,0} & -\sin\lambda_{A,0}\sin\phi_{A,0} & \sin\lambda_{A,0}\cos\phi_{A,0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\phi_{A,0} & \sin\phi_{A,0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\lambda_{B,0} & -\cos\lambda_{B,0}\sin\phi_{B,0} & \cos\lambda_{B,0}\cos\phi_{B,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\lambda_{B,0} & -\sin\lambda_{B,0}\sin\phi_{B,0} & \sin\lambda_{B,0}\cos\phi_{B,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi_{B,0} & \sin\phi_{B,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin\lambda_{C,0} & -\cos\lambda_{C,0}\sin\phi_{C,0} & \cos\lambda_{C,0}\cos\phi_{C,0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\lambda_{C,0} & -\sin\lambda_{C,0}\sin\phi_{C,0} & \sin\lambda_{C,0}\cos\phi_{C,0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\phi_{C,0} & \sin\phi_{C,0} \end{bmatrix}. \tag{18g}$$

Although horizontal angles (the only 3-point observation in this paper) are purely geometric, we have included (18e) and (18f), which deal with orthometric quantities, just for completeness.

The remainder of this paper is broken into sections, one for each observation/observational constraint type first, with coordinate constraints in the sections after that. In each section (except for GMBs, section 15) are two primary equations, outlined in a box, one for converting a value into a projected value, and the second for converting the cofactor matrix into the projected cofactor matrix. For GMBs, because off-diagonal values must be computed, and two different scenarios of point-sharing considered, there are additional primary equations, all outlined in a box. All primary equations are derived directly from Smith et al. (2023a), but section-by-section, are derived for the particular observation/observational constraint or coordinate constraint being discussed.

8 Observation/observational constraint: Slant distances

Slant distances are uncorrelated with one another. As such, one slant distance will be a scalar, and its cofactor matrix, being a 1×1 matrix, will also be a scalar, which we will call its variance.

The slant distance, when standing at point A and sighting point B will be denoted by s and its variance by σ_s^2. The *projected* slant distance and the variance of the *projected* slant distance will be designated by $\bar{s}$ and $\sigma_{\bar{s}}^2$ respectively.

The equations relating the original values to their projected values is [see equations 22, A25 and A31 in Smith et al. (2023a) and equations 14 and 19 of Smith (2023)] are seen in the box below.

$$\bar{s} = s - AR(\Delta t\boldsymbol{v} + q\boldsymbol{d}) \tag{19}$$

$$\sigma_{\bar{s}}^2 = \sigma_s^2 + AR(\Delta t^2\Sigma_v + \Sigma_d)R^TA^T \tag{20}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (16). Δt and q are found in (13).

To derive matrix A, we begin with the equation relating the slant distance to the geometric coordinates of its involved points is (note this does not require use of the e/n/u values):

$$s(X_A, Y_A, Z_A, X_B, Y_B, Z_B) = \sqrt{(X_B - X_A)^2 + (Y_B - Y_A)^2 + (Z_B - Z_A)^2}. \tag{21}$$

Matrix A is then derived from (21) as per equations 10, 37 or 60[11] of Smith et al. (2023a):

$$A = [A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6] = \left[\frac{\partial s}{\partial X_A} \quad \frac{\partial s}{\partial Y_A} \quad \frac{\partial s}{\partial Z_A} \quad \frac{\partial s}{\partial X_B} \quad \frac{\partial s}{\partial Y_B} \quad \frac{\partial s}{\partial Z_B}\right]\Bigg|_{\substack{X_A=X_{A,0},Y_A=Y_{A,0},Z_A=Z_{A,0},\\ X_B=X_{B,0},Y_B=Y_{B,0},Z_B=Z_{B,0}}}$$
$$= \left[\frac{-(X_{B,0} - X_{A,0})}{\sqrt{s_0}} \quad \frac{-(Y_{B,0} - Y_{A,0})}{\sqrt{s_0}} \quad \frac{-(Z_{B,0} - Z_{A,0})}{\sqrt{s_0}} \quad \frac{(X_{B,0} - X_{A,0})}{\sqrt{s_0}} \quad \frac{(Y_{B,0} - Y_{A,0})}{\sqrt{s_0}} \quad \frac{(Z_{B,0} - Z_{A,0})}{\sqrt{s_0}}\right]. \tag{22}$$

In (22), the s_0 value is computed from a-priori coordinates as:

[11] If the slant distance is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

$$s_0(X_{A,0}, Y_{A.0}, Z_{A,0}, X_{B,0}, Y_{B,0}, Z_{B,0}) = \sqrt{(X_{B,0} - X_{A,0})^2 + (Y_{B,0} - Y_{A,0})^2 + (Z_{B,0} - Z_{A,0})^2}. \quad (23)$$

9 Observation/observational constraint: Geodetic zenith angles

Geodetic zenith angles are uncorrelated with one another. As such, one geodetic zenith angle will be a scalar, and its cofactor matrix, being a 1×1 matrix, will also be a scalar, which we will call its variance.

The geodetic zenith angle, when standing at point A and sighting point B, will be denoted by β and its variance by σ_β^2. See Figure 1. The *projected* geodetic zenith angle and the variance of the *projected* geodetic zenith angle will be designated by $\bar{\beta}$ and $\sigma_{\bar{\beta}}^2$ respectively.

The equations relating the original values to their projected values are seen in the box below.

$$\bar{\beta} = \beta - AR(\Delta t\boldsymbol{v} + q\boldsymbol{d}) \quad (24)$$

$$\sigma_{\bar{\beta}}^2 = \sigma_\beta^2 + AR(\Delta t^2\Sigma_v + \Sigma_d)R^T A^T \quad (25)$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (16). Δt and q are found in (13).

To derive matrix A, we begin with the equation relating the geodetic zenith angle to the vector components in the LGH frame:

$$\beta(X_A, Y_A, Z_A, X_B, Y_B, Z_B) = \arctan\left[\frac{\sqrt{e^2 + n^2}}{u}\right]. \quad (26)$$

Substituting (9) into (26) will yield the relationship between the geodetic zenith angle and the GC coordinates[12], allowing us to take the derivatives of β with respect to X, Y and Z.

Matrix A is then derived from (26) using (9), as per equations 10, 37 or 60[13] of Smith et al. (2023a):

$$A = [A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6] = \left[\frac{\partial\beta}{\partial X_A} \quad \frac{\partial\beta}{\partial Y_A} \quad \frac{\partial\beta}{\partial Z_A} \quad \frac{\partial\beta}{\partial X_B} \quad \frac{\partial\beta}{\partial Y_B} \quad \frac{\partial\beta}{\partial Z_B}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0}}}. \quad (27a)$$

where:

$$A_1 = -A_4 = \frac{s_0 \cos\phi_{A,0} \cos\lambda_{A,0} - \cos\beta_0 \left(X_{B,0} - X_{A,0}\right)}{s_0^2 \sin\beta_0}, \quad (27b)$$

---

[12] Because they are lengthy, those expanded equations are not shown here, but can be inferred from Leick (2004) and Wolf (1963).

[13] If the zenith angle is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

$$A_2 = -A_5 = \frac{s_0 \cos\phi_{A,0} \sin\lambda_{A,0} - \cos\beta_0 \left(Y_{B,0} - Y_{A,0}\right)}{s_0^2 \sin\beta_0}, \tag{27c}$$

$$A_3 = -A_6 = \frac{s_0 \sin\phi_{A,0} - \cos\beta_0 \left(Z_{B,0} - Z_{A,0}\right)}{s_0^2 \sin\beta_0}, \tag{27d}$$

In (27), the s_0 value comes from (23) and the β_0 value is computed from a-priori coordinates as:

$$\beta_0\left(X_{A,0}, Y_{A.0}, Z_{A,0}, X_{B,0}, Y_{B,0}, Z_{B,0}\right) = \arctan\left[\frac{\sqrt{e_0^2 + n_0^2}}{u_0}\right], \tag{28}$$

where the e_0, n_0 and u_0 values can be computed from (11), using the a-priori coordinates of points A and B.

10 Observation/observational constraint: Geodetic azimuths

Geodetic azimuths are uncorrelated with one another. As such, one geodetic azimuth will be a scalar, and its cofactor matrix, being a 1×1 matrix, will also be a scalar, which we will call its variance.

The geodetic azimuth, when standing at point A and sighting point B, will be denoted by α and its variance by σ_α^2. (See Figure 1.) The *projected* geodetic azimuth and the variance of the *projected* geodetic azimuth will be designated by $\bar{\alpha}$ and $\sigma_{\bar{\alpha}}^2$ respectively.

The equations relating the original values to their projected values are seen in the box below.

$$\bar{\alpha} = \alpha - AR(\Delta t\boldsymbol{v} + q\boldsymbol{d}) \tag{29}$$
$$\sigma_{\bar{\alpha}}^2 = \sigma_\alpha^2 + AR(\Delta t^2 \Sigma_v + \Sigma_d)R^T A^T \tag{30}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (16). Δt and q are found in (13).

To derive matrix A, we begin with the equation relating the geodetic azimuth to the vector components in the LGH frame:

$$\alpha(X_A, Y_A, Z_A, X_B, Y_B, Z_B) = \arctan\left[\frac{e}{n}\right]. \tag{31}$$

Substituting (9) into (31) will yield the relationship between the geodetic azimuth and the GC coordinates[14], allowing us to take the derivatives of α with respect to X, Y and Z.

Matrix A is then derived from (31), using (9), as per equations 10, 37 or 60[15] of Smith et al. (2023a):

[14] Because they are lengthy, those expanded equations are not shown here, but can be inferred from Leick (2004) and Wolf (1963).

[15] If the geodetic azimuth is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

$$A = [A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6] = \left[\frac{\partial \alpha}{\partial X_A} \quad \frac{\partial \alpha}{\partial Y_A} \quad \frac{\partial \alpha}{\partial Z_A} \quad \frac{\partial \alpha}{\partial X_B} \quad \frac{\partial \alpha}{\partial Y_B} \quad \frac{\partial \alpha}{\partial Z_B}\right]\Bigg|_{\substack{X_A=X_{A,0},Y_A=Y_{A,0},Z_A=Z_{A,0}, \\ X_B=X_{B,0},Y_B=Y_{B,0},Z_B=Z_{B,0}}} \tag{32a}$$

where:

$$A_1 = -A_4 = \frac{-\sin\phi_{A,0}\cos\lambda_{A,0}\sin\alpha_0 + \sin\lambda_{A,0}\cos\alpha_0}{s_0 \sin\beta_0}, \tag{32b}$$

$$A_2 = -A_5 = \frac{-\sin\phi_{A,0}\sin\lambda_{A,0}\sin\alpha_0 - \cos\lambda_{A,0}\cos\alpha_0}{s_0 \sin\beta_0}, \tag{32c}$$

$$A_3 = -A_6 = \frac{\cos\phi_{A,0}\sin\alpha_0}{s_0 \sin\beta_0}. \tag{32d}$$

In (32), the s_0 value comes from (23), the β_0 value comes from (28) and the α_0 value is computed from a-priori coordinates as:

$$\alpha_0\left(X_{A,0}, Y_{A.0}, Z_{A,0}, X_{B,0}, Y_{B,0}, Z_{B,0}\right) = \arctan\left[\frac{e_0}{n_0}\right]. \tag{33}$$

where the e_0, n_0 and u_0 values can be computed from (11), using the a-priori coordinates of points A and B.

11 Observation/observational constraint: Unoriented horizontal directions

Unoriented horizontal directions (UHDs) are simply geodetic azimuths plus some constant, but unknown, orientation parameter, θ (being the angle from the horizontal-circle “zero reading” to geodetic north). As θ is *not* projected through time, the equations for projecting UHDs follow very closely those of geodetic azimuths. For completeness, the equations are shown below.

UHDs are uncorrelated with one another. As such, one UHD will be a scalar, and its cofactor matrix, being a 1×1 matrix, will also be a scalar, which we will call its variance.

The UHD, when standing at point A and sighting point B, will be denoted by δ and its variance by σ_δ^2. (See Figure 1.) The *projected* UHD and the variance of the *projected* UHD will be designated by $\bar{\delta}$ and $\sigma_{\bar{\delta}}^2$ respectively.

The equations relating the original values to their projected values are seen in the box below.

$$\bar{\delta} = \delta - AR(\Delta t \boldsymbol{v} + q\boldsymbol{d}) \tag{34}$$

$$\sigma_{\bar{\delta}}^2 = \sigma_\delta^2 + AR(\Delta t^2 \Sigma_v + \Sigma_d)R^T A^T \tag{35}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (16). Δt and q are found in (13).

To derive matrix A, we begin with the equation relating the UHD to the vector components in the LGH frame:

$$\delta(X_A, Y_A, Z_A, X_B, Y_B, Z_B) = \alpha + \theta = \arctan\left[\frac{e}{n}\right] + \theta. \tag{36}$$

Substituting (9) into (36) will yield the relationship between the unoriented horizontal direction and the GC coordinates[16], allowing us to take the derivatives of δ with respect to X, Y, and Z.

Matrix A is then derived from (36) using (9), as per equations 10, 37, or 60[17] of Smith et al. (2023a):

$$\begin{aligned}
A &= [A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6] = \left[\frac{\partial\delta}{\partial X_A} \quad \frac{\partial\delta}{\partial Y_A} \quad \frac{\partial\delta}{\partial Z_A} \quad \frac{\partial\delta}{\partial X_B} \quad \frac{\partial\delta}{\partial Y_B} \quad \frac{\partial\delta}{\partial Z_B}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0}}} \\
&= \left[\frac{\partial(\alpha+\theta)}{\partial X_A} \quad \frac{\partial(\alpha+\theta)}{\partial Y_A} \quad \frac{\partial(\alpha+\theta)}{\partial Z_A} \quad \frac{\partial(\alpha+\theta)}{\partial X_B} \quad \frac{\partial(\alpha+\theta)}{\partial Y_B} \quad \frac{\partial(\alpha+\theta)}{\partial Z_B}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0}}} \\
&= \left[\frac{\partial\alpha}{\partial X_A} + \frac{\partial\theta}{\partial X_A} \quad \frac{\partial\alpha}{\partial Y_A} + \frac{\partial\theta}{\partial Y_A} \quad \frac{\partial\alpha}{\partial Z_A} + \frac{\partial\theta}{\partial Z_A} \quad \frac{\partial\alpha}{\partial X_B} + \frac{\partial\theta}{\partial X_B} \quad \frac{\partial\alpha}{\partial Y_B} + \frac{\partial\theta}{\partial Y_B} \quad \frac{\partial\alpha}{\partial Z_B} + \frac{\partial\theta}{\partial Z_B}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0}}} \\
&= \left[\frac{\partial\alpha}{\partial X_A} + 0 \quad \frac{\partial\alpha}{\partial Y_A} + 0 \quad \frac{\partial\alpha}{\partial Z_A} + 0 \quad \frac{\partial\alpha}{\partial X_B} + 0 \quad \frac{\partial\alpha}{\partial Y_B} + 0 \quad \frac{\partial\alpha}{\partial Z_B} + 0\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0}}} \\
&= \left[\frac{\partial\alpha}{\partial X_A} \quad \frac{\partial\alpha}{\partial Y_A} \quad \frac{\partial\alpha}{\partial Z_A} \quad \frac{\partial\alpha}{\partial X_B} \quad \frac{\partial\alpha}{\partial Y_B} \quad \frac{\partial\alpha}{\partial Z_B}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0}}}
\end{aligned} \tag{37}$$

Note that matrix A in (37) is identical to that in (32), so all of the elements of the A matrix in (37) can be found in (32b)-(32d).

12 Observation/observational constraint: Horizontal angles

Horizontal angles are uncorrelated with one another. As such, one horizontal angle will be a scalar, and its cofactor matrix, being a 1×1 matrix, will also be a scalar, which we will call its variance.

The horizontal angle, when standing at point A and sighting point B, and then turning clockwise to sight point C, will be denoted by ω and its variance by σ_ω^2. (See Figure 2.) The *projected* horizontal angle and the variance of the *projected* horizontal angle will be designated by $\bar{\omega}$ and $\sigma_{\bar{\omega}}^2$ respectively.

The equations relating the original values to their projected values are seen in the box below.

$$\bar{\omega} = \omega - AR(\Delta t\boldsymbol{v} + q\boldsymbol{d}) \tag{38}$$
$$\sigma_{\bar{\omega}}^2 = \sigma_\omega^2 + AR(\Delta t^2\Sigma_v + \Sigma_d)R^TA^T \tag{39}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (18). Δt and q are found in (13).

[16] Because they are lengthy, those expanded equations are not shown here, but can be inferred from Leick (2004) and Wolf (1963).

[17] If the unoriented horizontal direction is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

To derive matrix A, we begin with the equation relating the horizontal angle to the vector components in the LGH frame:

$$\omega(X_A, Y_A, Z_A, X_B, Y_B, Z_B, X_C, Y_C, Z_C) = \alpha' - \alpha = \arctan\left[\frac{e'}{n'}\right] - \arctan\left[\frac{e}{n}\right]. \tag{40}$$

In (40) we apply the fact that the horizontal angle is simply the difference of two geodetic azimuths. See Figure 2.

Substituting (9) and (10) into (40) will yield the relationship between the geodetic azimuth and the GC coordinates[18], allowing us to take the derivatives of ω with respect to X, Y, and Z.

Matrix A is then derived from (40) using (9) and (10), as per equations 10, 37 or 60[19] of Smith et al. (2023a):

$$\begin{aligned}
A &= \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 \end{bmatrix} \\
&= \left[\frac{\partial\omega}{\partial X_A} \;\; \frac{\partial\omega}{\partial Y_A} \;\; \frac{\partial\omega}{\partial Z_A} \;\; \frac{\partial\omega}{\partial X_B} \;\; \frac{\partial\omega}{\partial Y_B} \;\; \frac{\partial\omega}{\partial Z_B} \;\; \frac{\partial\omega}{\partial X_C} \;\; \frac{\partial\omega}{\partial Y_C} \;\; \frac{\partial\omega}{\partial Z_C}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0},\\ X_C=X_{C,0}, Y_C=Y_{C,0}, Z_C=Z_{C,0}}} \\
&= \left[\frac{\partial(\alpha'-\alpha)}{\partial X_A} \;\; \frac{\partial(\alpha'-\alpha)}{\partial Y_A} \;\; \frac{\partial(\alpha'-\alpha)}{\partial Z_A} \;\; \frac{\partial(\alpha'-\alpha)}{\partial X_B} \;\; \frac{\partial(\alpha'-\alpha)}{\partial Y_B} \;\; \frac{\partial(\alpha'-\alpha)}{\partial Z_B}\right. \\
&\qquad \left.\frac{\partial(\alpha'-\alpha)}{\partial X_C} \;\; \frac{\partial(\alpha'-\alpha)}{\partial Y_C} \;\; \frac{\partial(\alpha'-\alpha)}{\partial Z_C}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0},\\ X_C=X_{C,0}, Y_C=Y_{C,0}, Z_C=Z_{C,0}}} \\
&= \left[\frac{\partial\alpha'}{\partial X_A} \;\; \frac{\partial\alpha'}{\partial Y_A} \;\; \frac{\partial\alpha'}{\partial Z_A} \;\; \frac{\partial\alpha'}{\partial X_B} \;\; \frac{\partial\alpha'}{\partial Y_B} \;\; \frac{\partial\alpha'}{\partial Z_B} \;\; \frac{\partial\alpha'}{\partial X_C} \;\; \frac{\partial\alpha'}{\partial Y_C} \;\; \frac{\partial\alpha'}{\partial Z_C}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0},\\ X_C=X_{C,0}, Y_C=Y_{C,0}, Z_C=Z_{C,0}}} \\
&\quad - \left[\frac{\partial\alpha}{\partial X_A} \;\; \frac{\partial\alpha}{\partial Y_A} \;\; \frac{\partial\alpha}{\partial Z_A} \;\; \frac{\partial\alpha}{\partial X_B} \;\; \frac{\partial\alpha}{\partial Y_B} \;\; \frac{\partial\alpha}{\partial Z_B} \;\; \frac{\partial\alpha}{\partial X_C} \;\; \frac{\partial\alpha}{\partial Y_C} \;\; \frac{\partial\alpha}{\partial Z_C}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0},\\ X_C=X_{C,0}, Y_C=Y_{C,0}, Z_C=Z_{C,0}}} \\
&= \left[\frac{\partial\alpha'}{\partial X_A} \;\; \frac{\partial\alpha'}{\partial Y_A} \;\; \frac{\partial\alpha'}{\partial Z_A} \;\; 0 \;\; 0 \;\; 0 \;\; \frac{\partial\alpha'}{\partial X_C} \;\; \frac{\partial\alpha'}{\partial Y_C} \;\; \frac{\partial\alpha'}{\partial Z_C}\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0},\\ X_C=X_{C,0}, Y_C=Y_{C,0}, Z_C=Z_{C,0}}} \\
&\quad - \left[\frac{\partial\alpha}{\partial X_A} \;\; \frac{\partial\alpha}{\partial Y_A} \;\; \frac{\partial\alpha}{\partial Z_A} \;\; \frac{\partial\alpha}{\partial X_B} \;\; \frac{\partial\alpha}{\partial Y_B} \;\; \frac{\partial\alpha}{\partial Z_B} \;\; 0 \;\; 0 \;\; 0\right]\Bigg|_{\substack{X_A=X_{A,0}, Y_A=Y_{A,0}, Z_A=Z_{A,0},\\ X_B=X_{B,0}, Y_B=Y_{B,0}, Z_B=Z_{B,0},\\ X_C=X_{C,0}, Y_C=Y_{C,0}, Z_C=Z_{C,0}}}
\end{aligned} \tag{41a}$$

Note that a similar reverse-sign parallelism between the derivatives of an azimuth with respect to point A and point B, as seen in section 10 occurs here, only it is complicated by the fact that we are dealing with two azimuths and three points. Specifically, that:

$$\left[\frac{\partial\alpha}{\partial X_B} \;\; \frac{\partial\alpha}{\partial Y_B} \;\; \frac{\partial\alpha}{\partial Z_B}\right] = \left[-\frac{\partial\alpha}{\partial X_A} \;\; -\frac{\partial\alpha}{\partial Y_A} \;\; -\frac{\partial\alpha}{\partial Z_A}\right], \tag{41b}$$

[18] Because they are lengthy, those expanded equations are not shown here, but can be inferred from Leick (2004) and Wolf (1963).

[19] If the horizontal angle is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

$$\begin{bmatrix}\frac{\partial\alpha'}{\partial X_C} & \frac{\partial\alpha'}{\partial Y_C} & \frac{\partial\alpha'}{\partial Z_C}\end{bmatrix} = \begin{bmatrix}-\frac{\partial\alpha'}{\partial X_A} & -\frac{\partial\alpha'}{\partial Y_A} & -\frac{\partial\alpha'}{\partial Z_A}\end{bmatrix}. \tag{41c}$$

This lets us write (41a) entirely as derivatives with respect to point A, creating (41d), which is useful when keeping track of so many equations and sign changes.

$$\begin{aligned}
&\Rightarrow A = \begin{bmatrix}A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9\end{bmatrix}\\
&= \begin{bmatrix}\frac{\partial\alpha'}{\partial X_A} & \frac{\partial\alpha'}{\partial Y_A} & \frac{\partial\alpha'}{\partial Z_A} & 0 & 0 & 0 & -\frac{\partial\alpha'}{\partial X_A} & -\frac{\partial\alpha'}{\partial Y_A} & -\frac{\partial\alpha'}{\partial Z_A}\end{bmatrix}\Bigg|_{\substack{X_A=X_{A,0},Y_A=Y_{A,0},Z_A=Z_{A,0},\\ X_B=X_{B,0},Y_B=Y_{B,0},Z_B=Z_{B,0},\\ X_C=X_{C,0},Y_C=Y_{C,0},Z_C=Z_{C,0}}}\\
&\quad - \begin{bmatrix}\frac{\partial\alpha}{\partial X_A} & \frac{\partial\alpha}{\partial Y_A} & \frac{\partial\alpha}{\partial Z_A} & -\frac{\partial\alpha}{\partial X_A} & -\frac{\partial\alpha}{\partial Y_A} & -\frac{\partial\alpha}{\partial Z_A} & 0 & 0 & 0\end{bmatrix}\Bigg|_{\substack{X_A=X_{A,0},Y_A=Y_{A,0},Z_A=Z_{A,0},\\ X_B=X_{B,0},Y_B=Y_{B,0},Z_B=Z_{B,0},\\ X_C=X_{C,0},Y_C=Y_{C,0},Z_C=Z_{C,0}}}\\
&= \begin{bmatrix}\frac{\partial\alpha'}{\partial X_A} & \frac{\partial\alpha'}{\partial Y_A} & \frac{\partial\alpha'}{\partial Z_A} & 0 & 0 & 0 & -\frac{\partial\alpha'}{\partial X_A} & -\frac{\partial\alpha'}{\partial Y_A} & -\frac{\partial\alpha'}{\partial Z_A}\end{bmatrix}\Bigg|_{\substack{X_A=X_{A,0},Y_A=Y_{A,0},Z_A=Z_{A,0},\\ X_B=X_{B,0},Y_B=Y_{B,0},Z_B=Z_{B,0},\\ X_C=X_{C,0},Y_C=Y_{C,0},Z_C=Z_{C,0}}}\\
&\quad + \begin{bmatrix}-\frac{\partial\alpha}{\partial X_A} & -\frac{\partial\alpha}{\partial Y_A} & -\frac{\partial\alpha}{\partial Z_A} & \frac{\partial\alpha}{\partial X_A} & \frac{\partial\alpha}{\partial Y_A} & \frac{\partial\alpha}{\partial Z_A} & 0 & 0 & 0\end{bmatrix}\Bigg|_{\substack{X_A=X_{A,0},Y_A=Y_{A,0},Z_A=Z_{A,0},\\ X_B=X_{B,0},Y_B=Y_{B,0},Z_B=Z_{B,0},\\ X_C=X_{C,0},Y_C=Y_{C,0},Z_C=Z_{C,0}}}\\
&= \Bigg[\left(\frac{\partial\alpha'}{\partial X_A} - \frac{\partial\alpha}{\partial X_A}\right) \quad \left(\frac{\partial\alpha'}{\partial Y_A} - \frac{\partial\alpha}{\partial Y_A}\right) \quad \left(\frac{\partial\alpha'}{\partial Z_A} - \frac{\partial\alpha}{\partial Z_A}\right)\\
&\quad \left(\frac{\partial\alpha}{\partial X_A}\right) \quad \left(\frac{\partial\alpha}{\partial Y_A}\right) \quad \left(\frac{\partial\alpha}{\partial Z_A}\right) \quad -\left(\frac{\partial\alpha'}{\partial X_A}\right) \quad -\left(\frac{\partial\alpha'}{\partial Y_A}\right) \quad -\left(\frac{\partial\alpha'}{\partial Z_A}\right)\Bigg]\Bigg|_{\substack{X_A=X_{A,0},Y_A=Y_{A,0},Z_A=Z_{A,0},\\ X_B=X_{B,0},Y_B=Y_{B,0},Z_B=Z_{B,0},\\ X_C=X_{C,0},Y_C=Y_{C,0},Z_C=Z_{C,0}}}
\end{aligned} \tag{41d}$$

And thus, we have:

$$\begin{aligned}A_1 = &\left(\frac{-\sin\phi_{A,0}\cos\lambda_{A,0}\sin\alpha_0' + \sin\lambda_{A,0}\cos\alpha_0'}{s_0'\sin\beta_0'}\right)\\ &- \left(\frac{-\sin\phi_{A,0}\cos\lambda_{A,0}\sin\alpha_0 + \sin\lambda_{A,0}\cos\alpha_0}{s_0\sin\beta_0}\right)\end{aligned} \tag{41e}$$

$$\begin{aligned}A_2 = &\left(\frac{-\sin\phi_{A,0}\sin\lambda_{A,0}\sin\alpha_0' - \cos\lambda_{A,0}\cos\alpha_0'}{s_0'\sin\beta_0'}\right)\\ &- \left(\frac{-\sin\phi_{A,0}\sin\lambda_{A,0}\sin\alpha_0 - \cos\lambda_{A,0}\cos\alpha_0}{s_0'\sin\beta_0}\right)\end{aligned} \tag{41f}$$

$$A_3 = \left(\frac{\cos\phi_{A,0}\sin\alpha_0'}{s_0'\sin\beta_0'}\right) - \left(\frac{\cos\phi_{A,0}\sin\alpha_0}{s_0\sin\beta_0}\right) \tag{41g}$$

$$A_4 = \left(\frac{-\sin\phi_{A,0}\cos\lambda_{A,0}\sin\alpha_0 + \sin\lambda_{A,0}\cos\alpha_0}{s_0\sin\beta_0}\right) \tag{41h}$$

$$A_5 = \left(\frac{-\sin\phi_{A,0}\sin\lambda_{A,0}\sin\alpha_0 - \cos\lambda_{A,0}\cos\alpha_0}{s_0\sin\beta_0}\right) \tag{41i}$$

$$A_6 = \left(\frac{\cos\phi_{A,0}\sin\alpha_0}{s_0\sin\beta_0}\right) \tag{41j}$$

$$A_7 = -\left(\frac{-\sin\phi_{A,0}\cos\lambda_{A,0}\sin\alpha_0' + \sin\lambda_{A,0}\cos\alpha_0'}{s_0'\sin\beta_0'}\right) \tag{41k}$$

$$A_8 = -\left(\frac{-\sin\phi_{A,0}\sin\lambda_{A,0}\sin\alpha'_0 - \cos\lambda_{A,0}\cos\alpha'_0}{s'_0\sin\beta'_0}\right) \tag{41l}$$

$$A_9 = -\left(\frac{\cos\phi_{A,0}\sin\alpha'_0}{s'_0\sin\beta'_0}\right). \tag{41m}$$

Where s_0, β_0 and α_0 come from (23), (28), and (33) respectively. In (41), the s'_0, value is computed from a-priori coordinates as:

$$s'_0(X_{A,0}, Y_{A.0}, Z_{A,0}, X_{C,0}, Y_{C,0}, Z_{C,0}) = \sqrt{(X_{C,0} - X_{A,0})^2 + (Y_{C,0} - Y_{A,0})^2 + (Z_{C,0} - Z_{A,0})^2}, \tag{42}$$

the β'_0 value is computed as:

$$\beta'_0(X_{A,0}, Y_{A.0}, Z_{A,0}, X_{C,0}, Y_{C,0}, Z_{C,0}) = \arctan\left[\frac{\sqrt{{e'_0}^2 + {n'_0}^2}}{u'_0}\right], \tag{43}$$

and the α'_0 value is computed as:

$$\alpha'_0(X_{A,0}, Y_{A.0}, Z_{A,0}, X_{C,0}, Y_{C,0}, Z_{C,0}) = \arctan\left[\frac{e'_0}{n'_0}\right], \tag{44}$$

where the e'_0, n'_0 and u'_0 values can be computed from (12), using the a-priori coordinates of points A and C.

13 Observation/observational constraint: PPP

Precise point positioning (PPP) observations are uncorrelated with one another, but they come as a triad of GC coordinates, rather than single values. As such, one PPP observation will be a 3×1 vector, and its cofactor matrix will be a 3×3 matrix.

The PPP observation/observational constraint at point A, will be denoted[20] by $\boldsymbol{X} = [X_A, Y_A, Z_A]^T$ and its cofactor matrix by $\Sigma_{\boldsymbol{X}}$. The *projected* PPP observation and the cofactor matrix of the *projected* PPP observation will be designated by $\overline{\boldsymbol{X}}$ and $\Sigma_{\overline{\boldsymbol{X}}}$ respectively.

The equations relating the original values to their projected values are seen in the box below.

$$\overline{\boldsymbol{X}} = \boldsymbol{X} - AR(\Delta t\boldsymbol{v} + q\boldsymbol{d}) \tag{45}$$

$$\Sigma_{\overline{\boldsymbol{X}}} = \Sigma_{\boldsymbol{X}} + AR(\Delta t^2\Sigma_v + \Sigma_d)R^TA^T \tag{46}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (14). Δt and q are found in (13).

To derive matrix A, we begin with the equation relating the PPP observation to GC coordinates:

[20] Using the notation that a bold value indicates a vector.

$$\boldsymbol{X}(X_A, Y_A, Z_A) = \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}. \tag{47}$$

Matrix A, which is 3×3, is then derived from (47) as per equations 10, 37 or 60[21] of Smith et al. (2023a):

$$A = [\boldsymbol{A_1} \quad \boldsymbol{A_2} \quad \boldsymbol{A_3}] = \left[\frac{\partial \boldsymbol{X}}{\partial X_A} \quad \frac{\partial \boldsymbol{X}}{\partial Y_A} \quad \frac{\partial \boldsymbol{X}}{\partial Z_A}\right]\Bigg|_{X_A = X_{A,0}, Y_A = Y_{A,0}, Z_A = Z_{A,0}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \tag{48}$$

Note that matrix A is entirely filled with constants, so there is no reliance of matrix A upon a-priori coordinates.

14 Observation/observational constraint: Differential Orthometric Heights

Differential orthometric heights (DOHs) may be an observation/observational constraint in either an orthometric adjustment[22] (estimating orthometric heights, H) or in a geometric adjustment[23] (estimating ECEF global Cartesian coordinates, XYZ). However, the equations for projecting them are identical in each case, and are covered in this section.

DOHs are uncorrelated with one another. As such, one DOH will be a scalar, and its cofactor matrix, being a 1×1 matrix, will also be a scalar, which we will call its variance.

The DOH is the difference in orthometric heights between points A and B, and will be denoted by ΔH and its variance by $\sigma^2_{\Delta H}$. The *projected* DOH and the variance of the *projected* DOH will be designated by $\overline{\Delta H}$ and $\sigma^2_{\overline{\Delta H}}$ respectively.

The equations relating the original values to their projected values are seen in the box below.

$$\overline{\Delta H} = \Delta H - A(\Delta t \boldsymbol{v} + q\boldsymbol{d} - \Delta t \boldsymbol{w}) \tag{49}$$
$$\sigma^2_{\overline{\Delta H}} = \sigma^2_{\Delta H} + A(\Delta t^2 \Sigma_v + \Sigma_d + \Delta t^2 \Sigma_w) A^T \tag{50}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v and Σ_d are found in (17). Note that these are 2×1 vectors and 2×2 matrices, different from those in (16). Δt and q are found in (13). Note the addition of vector $\boldsymbol{w}$ and matrix Σ_w, found in (16), which did not appear in earlier equations dealing with purely geometric quantities. Also note that no rotation matrix, R, is needed, since orthometric heights are parallel with the direction of the up velocities and displacements in IFDM2022 and geoid velocities in DGEOID2022.

[21] If the PPP value is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

[22] In NGS parlance, these observations will usually be found in "41* and 42* records in an "R6 or R7 file".

[23] In NGS parlance, these observations will usually be found in "*45* records" in a "B-file".

To derive matrix A, we begin with the equation relating a DOH to the orthometric coordinates of its involved points:

$$\Delta H(H_A, H_B) = H_B - H_A. \tag{51}$$

Matrix A is then derived from (51) as per equations 10, 37 or 60[24] of Smith et al. (2023a):

$$A = [A_1 \quad A_2] = \left.\begin{bmatrix} \frac{\partial \Delta H}{\partial H_A} & \frac{\partial \Delta H}{\partial H_B} \end{bmatrix}\right|_{H_A = H_{A,0}, H_B = H_{B,0}} = [-1 \quad +1]. \tag{52}$$

Note that matrix A is entirely filled with constants, so there is no reliance of matrix A upon a-priori coordinates.

15 Observation/observational constraint: GNSS measured baselines

GNSS measured baselines[25] (GMBs) are *correlated* with one another, processed together in one *session*. These correlations will cause a significantly different approach to computing projected observations/observational constraints and cofactor matrices than was seen with previous sections.

Let us begin with some assumptions. First, though a session is often hours (sometimes days) long, there is a mean epoch associated with each session. It will be assumed that all GMBs in that one session have the same observation epoch, which will be the mean epoch for the session. Second, it is assumed that because the GMBs in one session are correlated, that the cofactor matrix for all GMBs will be *full*. Finally, based on (Smith 2023), the projected cofactor matrix will only differ from the original cofactor matrix in the following ways:

- The on-diagonal 3×3 blocks for each GMB will change
- The off-diagonal 3×3 blocks relating two GMBs that share one point will change

Aside from these, no changes to the cofactor matrix for the session will occur. Due to this special situation, the equations relating *original* cofactor matrices to *projected* cofactor matrices will be more complex than in previous cases.

Before proceeding, we note that the variables i and j will be used in this section as an index for two different GMBs. Thus, in this section, i will not refer to an observation epoch. In fact, as mentioned above, the entire GNSS session is assumed to have a single observation epoch common to every GMB in the session. Thus Δt is computed as the difference between the adjustment epoch and that one GNSS session epoch.

One GMB will be stored in a 3×1 vector. Let us assume that a session contains r GMBs. The i^{th} GMB, where $1 \leq i \leq r$, from point F_i to point T_i, will be denoted by $\boldsymbol{\Delta X_i} =$

[24] If the differential orthometric height is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

[25] We avoid the word "vector" here, to reduce confusion with vectors in the linear algebra sense.

$\left[X_{T,i} - X_{F,i}, Y_{T,i} - Y_{F,i}, Z_{T,i} - Z_{F,i}\right]^T$. The j^{th} GMB, where $1 \leq j \leq r$, from point F_j and T_j, will be denoted by $\boldsymbol{\Delta X_j} = \left[X_{T,j} - X_{F,j}, Y_{T,j} - Y_{F,j}, Z_{T,j} - Z_{F,j}\right]^T$. The complete vector containing all r GMBs will be sized $3r \times 1$ and be denoted by $\boldsymbol{\Delta X} = \left[\boldsymbol{\Delta X_1}^{\boldsymbol{T}}, \boldsymbol{\Delta X_2}^{\boldsymbol{T}}, \dots, \boldsymbol{\Delta X_r}^{\boldsymbol{T}}\right]^T$.

The cofactor matrix for all r GMBs will be sized $3r \times 3r$. Each 3×3 *on*-diagonal block will correspond to one GMB. Each 3×3 *off*-diagonal block will correspond to the pairing of two GMBs. The full cofactor matrix will be designated $\Sigma_{\boldsymbol{X}}$. The i^{th} on-diagonal block, corresponding to the i^{th} GMB, will be designated $\Sigma_{\boldsymbol{X_i}}$. The j^{th} on-diagonal block, corresponding to the j^{th} GMB, will be designated $\Sigma_{\boldsymbol{X_j}}$. The off-diagonal block corresponding to the i^{th} and j^{th} GMBs, where $1 \leq i < j \leq r$ will be designated $\Sigma_{\boldsymbol{X_{i,j}}}$.

The vector of all projected GMBs will be designated $\overline{\boldsymbol{\Delta X}}$, and its projected cofactor matrix will be designated as $\Sigma_{\overline{\boldsymbol{X}}}$. Sub-vectors and sub-matrices designated $\overline{\boldsymbol{\Delta X_i}}$, $\overline{\boldsymbol{\Delta X_j}}$, $\Sigma_{\overline{\boldsymbol{X}}_i}$, $\Sigma_{\overline{\boldsymbol{X}}_j}$ and $\Sigma_{\overline{\boldsymbol{X}}_{i,j}}$ can be understood by the context of their subscripts.

Because GMBs are correlated with one another, they require slightly more equations than in previous examples.

The first equation is much like previous ones for projected observations/observational constraints, in this case relating a single GMB to its projected version, and is in the box below.

$$\overline{\boldsymbol{\Delta X}}_i = \boldsymbol{\Delta X_i} - A_i R_i (\Delta t \boldsymbol{v_i} + q \boldsymbol{d_i}) \tag{53}$$

The vectors $\boldsymbol{v_i}$, $\boldsymbol{d_i}$ and matrix R_i are found in (16), but using F_i for point A with and T_i for point B. Δt and q are found in (13).

To derive matrix A_i, we begin with the equation relating the GMB to the geometric coordinates of its involved points:

$$\boldsymbol{\Delta X_i}\left(X_{F_i}, Y_{F_i}, Z_{F_i}, X_{T_i}, Y_{T_i}, Z_{T_i}\right) = \begin{bmatrix} X_{T_i} - X_{F_i} \\ Y_{T_i} - Y_{F_i} \\ Z_{T_i} - Z_{F_i} \end{bmatrix}. \tag{54}$$

Matrix A_i, which is 3×6, is then derived from (54) as per equations 10, 37 or 60[26] of Smith et al. (2023a):

[26] If the GMB is an observation, equation 10 is the relevant equation. If it is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

$$A_i = [\boldsymbol{A}_{1,i} \quad \boldsymbol{A}_{2,i} \quad \boldsymbol{A}_{3,i} \quad \boldsymbol{A}_{4,i} \quad \boldsymbol{A}_{5,i} \quad \boldsymbol{A}_{6,i}] =$$

$$= \left[\frac{\partial \Delta \boldsymbol{X}_i}{\partial X_{F_i}} \quad \frac{\partial \Delta \boldsymbol{X}_i}{\partial Y_{F_i}} \quad \frac{\partial \Delta \boldsymbol{X}_i}{\partial Z_{F_i}} \quad \frac{\partial \Delta \boldsymbol{X}_i}{\partial X_{T_i}} \quad \frac{\partial \Delta \boldsymbol{X}_i}{\partial Y_{T_i}} \quad \frac{\partial \Delta \boldsymbol{X}_i}{\partial Z_{T_i}}\right]\Bigg|_{\substack{X_{T_i}=X_{T_{i,0}}, Y_{T_i}=Y_{T_{i,0}}, Z_{T_i}=Z_{T_{i,0}}, \\ X_{F_i}=X_{F_{i,0}}, Y_{F_i}=Y_{F_{i,0}}, Z_{F_i}=Z_{F_{i,0}}}}$$

$$= \begin{bmatrix} -1 & 0 & 0 & +1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \end{bmatrix} = [-I_3 \quad I_3]. \tag{55}$$

Note that matrix A is entirely filled with constants, so there is no reliance of matrix A upon a-priori coordinates.

The projected cofactor matrix is more complicated to form than in previous sections. This is because we will need to compute *off*-diagonal blocks, something not seen in any of the previous observations. The projected cofactor matrix will therefore be built one 3×3 block at a time, using two equations. The first is the equation for the i^{th} 3×3 *on-diagonal* block, corresponding to GMB "i", seen in the box below.

$$\boxed{\Sigma_{\overline{X}_i} = \Sigma_{X_i} + A_i R_i\left(\Delta t^2 \Sigma_{v_i} + \Sigma_{d_i}\right) {R_i}^T {A_i}^T} \tag{56}$$

The matrices Σ_{v_i}, Σ_{d_i} and R_i are found in (16), but using F_i for point A with and T_i for point B. Δt is found in (13). Matrix A_i is found in (55). Note that (56) is identical to the upper *left* 3×3 matrix in equation 32 of Smith (2023).

The second equation is for the 3×3 *off-diagonal* block in the upper triangular portion of the projected cofactor matrix that corresponds to GMBs i and j ($i < j$), and which only happens if GMBs i and j share a common point. However, there are four scenarios where this happens, and they do impact the equation to use. Recall that GMB i runs from F_i to T_i and GMB j runs from F_j to T_j. The four scenarios are:

a) Their "from" points are the same (e.g. F_i and F_j are the same point)
b) Their "to" points are the same (e.g. T_i and T_j are the same point)
c) The "from" point of GMB i is the "to" point of GMB j (e.g. F_i and T_j are the same point)
d) The "to" point of GMB i is the "from" point of GMB j (e.g. T_i and F_j are the same point)

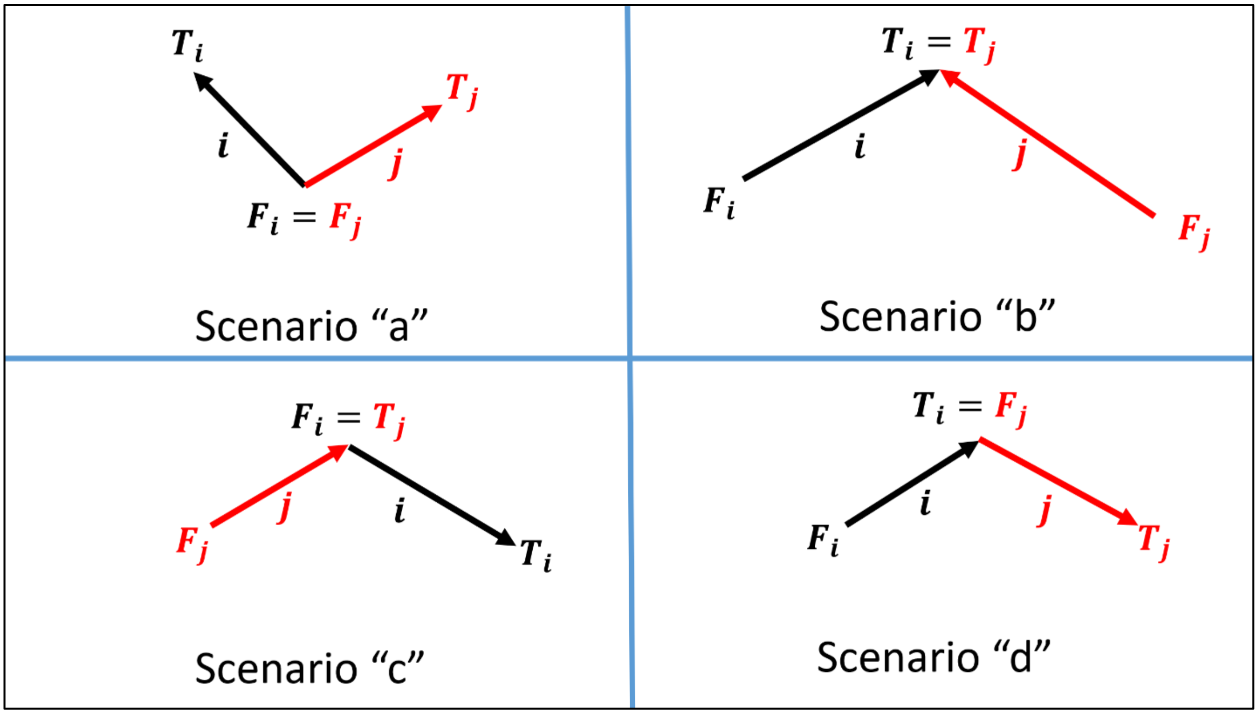

Figure 3: Four scenarios for two GMBs that share a common point

Under scenarios a and b, we have:

$$\Sigma_{\bar{X}_{i,j}} = \Sigma_{X_{i,j}} + R_c\left(\Delta t^2 \Sigma_{v_c} + \Sigma_{d_c}\right) R_c{}^T. \tag{57a}$$

Under scenarios c and d, we have:

$$\Sigma_{\bar{X}_{i,j}} = \Sigma_{X_{i,j}} - R_c\left(\Delta t^2 \Sigma_{v_c} + \Sigma_{d_c}\right) R_c{}^T. \tag{57b}$$

Both (57a) and (57b) represent the upper *right* 3×3 matrix in equation32 of Smith (2023). In (57a) and (57b), the subscript "c" means "common point between the two GMBs". To be explicit, we would do the following:

For scenario a: We use (57a). The "c" means point F_i (or F_j, being the same). Matrices Σ_{v_c}, Σ_{d_c} and R_c come from (14), but using F_i (or F_j) for point A.

For scenario b: We use (57a). The "c" means point T_i (or T_j, being the same). Matrices Σ_{v_c}, Σ_{d_c} and R_c come from (14), but using T_i (or T_j) for point A.

For scenario c: We use (57b). The "c" means point F_i (or T_j, being the same). Matrices Σ_{v_c}, Σ_{d_c} and R_c come from (14), but using F_i (or T_j) for point A.

For scenario d: We use (57b). The "c" means point T_i (or F_j, being the same). Matrices Σ_{v_c}, Σ_{d_c} and R_c come from (14), but using T_i (or F_j) for point A.

To clarify, there is no matrix A in (57a) or (57b), because each scenario has a different A matrix, (full of positive and negative identity matrices and zeroes) which has already been multiplied through to yield (57a) and (57b). This is why there is a plus sign in (57a), but a minus sign in (57b).

16 Coordinate Constraint: XYZ

Stochastically constrained global ECEF Cartesian coordinates, XYZ, might be provided with correlations between points. This might occur when an earlier LSA has been performed, and the full estimated dispersion matrix of estimated XYZ coordinates was computed. In such a case, that full matrix could enter the ME-LSA problem as inverse weight matrix ${P_w}^{-1}$ (Smith et al. 2023a). However, allowing a full inverse weight matrix for stochastic constraints to enter the LSA problem is beyond the current capabilities of LASER. As such, we will restrict ourselves to the case where stochastically constrained XYZ coordinates will not have correlations between points. Obviously, this is a non-issue if the coordinates are *fixed* constraints.

Still, even if correlations between points are ignored, stochastically constrained coordinates may have correlations between any of the three coordinates at the point itself. Thus, stochastically constrained global ECEF Cartesian coordinates, XYZ, must be given with a 3×3 cofactor matrix that may or may not contain covariances with one another (between X and Y, for example). SPROCCET will be built with the assumption that, at most, covariances between X and Y, X, and Z and Y and Z are given at one point, but that no covariances are given between the coordinates of different points.

As such, one geometric coordinate triad will be a 3×1 vector, and its cofactor matrix will be a (possibly full) 3×3 matrix.

The XYZ coordinates at point A, will be denoted by $\boldsymbol{X} = [X_A, Y_A, Z_A]^T$ and its cofactor matrix by $\Sigma_{\boldsymbol{X}}$. The *projected* coordinates and the cofactor matrix of the *projected* coordinates will be designated by $\overline{\boldsymbol{X}}$ and $\Sigma_{\overline{\boldsymbol{X}}}$ respectively.

The equations relating the original values to their projected values are seen in the box below.

$$\overline{\boldsymbol{X}} = \boldsymbol{X} - AR(\Delta t\boldsymbol{v} + q\boldsymbol{d}) \tag{58}$$

$$\Sigma_{\overline{\boldsymbol{X}}} = \Sigma_{\boldsymbol{X}} + AR(\Delta t^2\Sigma_v + \Sigma_d)R^T A^T \tag{59}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (14). Δt and q are found in (13).

To derive matrix A, we begin with the trivial equation relating the XYZ coordinates to themselves:

$$\boldsymbol{X}(X_A, Y_A, Z_A) = \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}. \tag{60}$$

Matrix A, which is 3×3, is then derived from (60) as per equations 37 or 60[27] of Smith et al. (2023a):

$$A = [\boldsymbol{A_1} \quad \boldsymbol{A_2} \quad \boldsymbol{A_3}] = \left[\frac{\partial \boldsymbol{X}}{\partial X_A} \quad \frac{\partial \boldsymbol{X}}{\partial Y_A} \quad \frac{\partial \boldsymbol{X}}{\partial Z_A}\right]\Bigg|_{X_A = X_{A,0}, Y_A = Y_{A,0}, Z_A = Z_{A,0}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \tag{61}$$

Note that matrix A is entirely filled with constants, so there is no reliance of matrix A upon a-priori coordinates.

16.1 Special case: Constrained XYZ with standard deviations in east, north and up

There is a possibility that a coordinate constraint will be provided in XYZ while the only information about its cofactor matrix is in the form of standard deviations in the east, north, and up directions (σ_E, σ_N and σ_U). In this case, it is assumed that such standard deviations are in linear (e.g., meters) units, and not curvilinear (e.g., arcseconds). In such a case we must show how to compute the projected vector in XYZ, while providing a projected cofactor matrix either in ENU or in XYZ.

We begin by defining the cofactor matrix, which we call, Σ_{ENU}, based on the provided east, north and up standard deviations as:

$$\Sigma_{ENU} = \begin{bmatrix} \sigma_{E_A}^2 & 0 & 0 \\ 0 & \sigma_{N_A}^2 & 0 \\ 0 & 0 & \sigma_{U_A}^2 \end{bmatrix}. \tag{62}$$

Using the information in section 6 we may write:

$$\Sigma_{\boldsymbol{X}} = R_{A,0}\Sigma_{ENU}, \tag{63}$$

where $R_{A,0}$ comes from (8). We use $R_{A,0}$, rather than R_A, since we only have a-priori coordinates at our point A. Note that, due to the full nature of the $R_{A,0}$ matrix, that matrix $\Sigma_{\boldsymbol{X}}$ will be full, rather than diagonal.

Now we insert (63) into (59) to arrive at $\Sigma_{\overline{\boldsymbol{X}}}$. If this was the goal, we are done. If, however, we wish to revert our cofactor matrix back to ENU, then we must apply matrix $R_{A,0}$ again, this time in transpose, to arrive at:

$$\Sigma_{\overline{ENU}} = R_{A,0}^T \Sigma_{\overline{\boldsymbol{X}}}. \tag{64}$$

It is possible that matrix $\Sigma_{\overline{ENU}}$ will be full, so if one is only interested in the projected standard deviations in the east, north and up directions, then taking the square roots of the diagonal elements will achieve this.

[27] If the XYZ coordinate triad is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

17 Coordinate Constraint: λϕh

Despite NGS policy to perform geometric adjustments in a purely XYZ system, it is possible that constrained coordinates might be provided as geodetic longitude, geodetic latitude, and ellipsoid height ($\lambda\phi h$).

Unlike in the previous section, we must be meticulous in our use of units, since longitude and latitude are expected to be provided in curvilinear units (e.g., degrees), while their standard deviations may be in curvilinear units or in linear units (e.g., meters). To distinguish among these various possibilities, we introduce a bracketed sub-script for units only, where $[d]$ means "degrees", $[m]$ means "meters" and "$[d,m]$" will mean a mix of degrees and meters (which happens when values related to longitude, latitude, and ellipsoid height all fall in the same vector or matrix). Thus $\sigma_{\lambda,[d]}$ and $\sigma_{\phi,[d]}$ are the standard deviations of longitude and latitude in degrees and $\sigma_{\lambda,[m]}$ and $\sigma_{\phi,[m]}$ are their standard deviations in meters, while, for example, $\Sigma_{\boldsymbol{v},[d,m]}$ is the cofactor matrix of velocities (expressed in degrees/year for longitude and latitude, but in meters/year for ellipsoid height) in IFDM2022.

As in the previous section, we will allow for correlations between coordinates at a point, but not between points. Thus, stochastically constrained geodetic coordinates, $\lambda\phi h$, must be given with a 3×3 cofactor matrix that may or may not contain covariances with one another (between λ and ϕ for example). SPROCCET will be built with the assumption that, at most, covariances between λ and ϕ and, λ and h and ϕ and h are given at one point, but that no covariances are given between the coordinates of different points.

As such, one geodetic coordinate triad will be a 3×1 vector, and its cofactor matrix will be a (possibly full) 3×3 matrix.

The $\lambda\phi h$ coordinates at point A, will be denoted by $\boldsymbol{\Lambda} = [\lambda_A, \phi_A, h_A]^T$ and its cofactor matrix by $\Sigma_{\boldsymbol{\Lambda}}$. The *projected* coordinates and their *projected* cofactor matrix will be designated by $\overline{\boldsymbol{\Lambda}}$ and $\Sigma_{\overline{\boldsymbol{\Lambda}}}$ respectively.

Due to the non-linear relationship between XYZ and $\lambda\phi h$, and due to the happy circumstance of $\lambda\phi h$ aligning with the ENU storage in IFDM2022 and DGEOID2022, we will directly relate the original and projected quantities within the $\lambda\phi h$ system, without translating into or out of the XYZ system.

The equations relating the original values to their projected values are seen in the box below.

$$\overline{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} - A\left(\Delta t \boldsymbol{v}_{[\boldsymbol{d},\boldsymbol{m}]} + q \boldsymbol{d}_{[\boldsymbol{d},\boldsymbol{m}]}\right) \tag{65}$$

$$\Sigma_{\overline{\boldsymbol{\Lambda}}} = \Sigma_{\boldsymbol{\Lambda}} + A\left(\Delta t^2 \Sigma_{v,[d,m]} + \Sigma_{d,[d,m]}\right) A^T \tag{66}$$

where:

$$\boldsymbol{v}_{[\boldsymbol{d},\boldsymbol{m}]} = \begin{bmatrix} \frac{180}{\pi R_\lambda} & 0 & 0 \\ 0 & \frac{180}{\pi R_\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{v}, \tag{67a}$$

$$\boldsymbol{d}_{[\boldsymbol{d},\boldsymbol{m}]} = \begin{bmatrix} \frac{180}{\pi R_\lambda} & 0 & 0 \\ 0 & \frac{180}{\pi R_\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{d}, \tag{67b}$$

$$\Sigma_{v,[d,m]} = \begin{bmatrix} \left(\frac{180}{\pi R_\lambda}\right)^2 & 0 & 0 \\ 0 & \left(\frac{180}{\pi R_\phi}\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Sigma_v, \tag{67c}$$

$$\Sigma_{d,[d,m]} = \begin{bmatrix} \left(\frac{180}{\pi R_\lambda}\right)^2 & 0 & 0 \\ 0 & \left(\frac{180}{\pi R_\phi}\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Sigma_d, \tag{67d}$$

and

$$R_\lambda = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}}, \tag{68a}$$

$$R_\phi = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}}. \tag{68b}$$

Equations 65 and 66 provide a projected coordinate constraint, and its projected cofactor matrix, in $\lambda\phi h$. The NGS LASER software is capable of incorporating that projected constraint, despite the current policy that geometric adjustments are performed solely in the XYZ system. However, if an LSA software suite is not capable of using $\lambda\phi h$ constraints in an XYZ adjustment, then the $\overline{\boldsymbol{\Lambda}}$ vector should be converted to an $\overline{\mathbf{X}}$ vector, and matrix $\Sigma_{\overline{\Lambda}}$ converted to matrix $\Sigma_{\overline{\mathbf{X}}}$. Methods for doing so are not included in this memorandum.

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v, Σ_d and R are found in (14). Δt and q are found in (13).

To derive matrix A, we begin with the trivial equation relating the $\lambda\phi h$ coordinates to themselves:

$$\boldsymbol{\Lambda}(\lambda_A, \phi_A, h_A) = \begin{bmatrix} \lambda_A \\ \phi_A \\ h_A \end{bmatrix}. \tag{69}$$

Matrix A, which is 3×3, is then derived from (69) as per equations 37 or 60[28] of Smith et al. (2023a):

$$A = [\boldsymbol{A_1} \quad \boldsymbol{A_2} \quad \boldsymbol{A_3}] = \left[\frac{\partial \boldsymbol{\Lambda}}{\partial \lambda_A} \quad \frac{\partial \boldsymbol{\Lambda}}{\partial \phi_A} \quad \frac{\partial \boldsymbol{\Lambda}}{\partial h_A}\right]\Bigg|_{\lambda=\lambda_{A,0}, \phi_A=\phi_{A,0}, h_A=h_{A,0}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \tag{70}$$

Note that matrix A is entirely filled with constants, so there is no reliance of matrix A upon a-priori coordinates.

18 Coordinate Constraint: Orthometric Height (H)

Orthometric heights ("coordinates"), H, might be provided with correlations between points. This might occur when an earlier LSA has been performed, and the full estimated dispersion matrix of estimated H coordinates was computed. In such a case, that full matrix could enter the ME-LSA problem as inverse weight matrix ${P_w}^{-1}$ (Smith et al. 2023a). However, allowing a full inverse weight matrix for stochastic constraints to enter the LSA problem is beyond the current capabilities of LASER. As such, we will restrict ourselves solely to the case where stochastically constrained H coordinates will *not* have correlations between points. Obviously, this is a non-issue if the coordinates are *fixed* constraints.

In general, SPROCCET will be built with the assumption that no point-to-point correlations are given. As such, one orthometric coordinate will be a scalar, and its cofactor matrix will also be a scalar, which we will call its variance.

The orthometric coordinate at point A will be denoted by H and its cofactor matrix by Σ_H. The *projected* coordinate and the cofactor matrix of the *projected* coordinate will be designated by $\overline{H}$ and $\sigma_{\overline{H}}^2$ respectively. The equations relating the original values to their projected values are seen in the box below.

$$\overline{H} = H - A(\Delta t\boldsymbol{v} + q\boldsymbol{d} - \Delta t\boldsymbol{w}) \tag{71}$$
$$\sigma_{\overline{H}}^2 = \sigma_H^2 + A(\Delta t^2\Sigma_v + \Sigma_d + \Delta t^2\Sigma_w)A^T \tag{72}$$

The vectors $\boldsymbol{v}$, $\boldsymbol{d}$ and matrices Σ_v and Σ_d are found in (14). Δt and q are found in (13).

To derive matrix A, we begin with the trivial equation relating the orthometric height to itself:

$$H = H_A. \tag{73}$$

[28] If the $\phi\lambda h$ coordinate triad is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

Matrix A is then derived from (73) as per equations 37 or 60[29] of Smith et al. (2023a):

$$A = [A_1] = \left[\frac{\partial H}{\partial H_A}\right] = +1. \tag{74}$$

Note that matrix A is composed of a single constant, so there is no reliance of matrix A upon a-priori coordinates.

19 Implications of quantities that can be fixed or stochastic

As noted earlier, constraints may be fixed or stochastic. Additionally, as mentioned in Smith et al. (2023a), the GVCMs (IFDM2022 and DGEOID2022) may be treated as known with variance (stochastic) or known without variance (fixed). Finally, observations themselves (not observational constraints) always have random observation error, making them stochastic by nature.

Each stochastic value has some random error that contributes to the projected cofactor matrix in its own way. Essentially, their roles come down to a few simple rules:

1. If you are projecting an observation, the observational error is always used.
2. If you are projecting an observational constraint, and it is a stochastic constraint, then the observational constraint error is used. Otherwise, the observational constraint error is set to zero.
3. If you are projecting a coordinate constraint, and it is a stochastic constraint, then the coordinate constraint error is used. Otherwise, the coordinate constraint error is set to zero.
4. No matter what is being projected, if IFDM2022 is being treated as fixed, then set the Σ_v and Σ_d matrices to zero. Otherwise use them as is.
5. No matter what is being projected, if GEOID2022 is being treated as fixed, then set the Σ_w matrix to zero. Otherwise use it as is.

See Figure 4 for an example of how these rules are applied. Although this is for the projected cofactor matrix of a differential orthometric height, it can be extrapolated to other observations or constraints.

[29] If the H coordinate is a stochastic constraint, equation 37 is relevant, substituting matrix S for matrix A in this paper. If it is a fixed constraint, equation 60 is relevant, substituting matrix F for matrix A in this paper.

$$\sigma^2_{\overline{\Delta H}} = \sigma^2_{\Delta H} + A(\Delta t^2 \Sigma_v + \Sigma_d + \Delta t^2 \Sigma_w)A^T$$

This is always non-zero for ***observations***.

These are zero if IFDM2022 is treated as ***fixed***.

If ΔH is an ***observational constraint***, then this value is set to zero if it is a ***fixed*** constraint.

This is zero if DGEOID2022 is treated as ***fixed***.

Figure 4: How to apply various stochastic values in the equation for the projected cofactor matrix of a differential orthometric height.

20 Summary

The multi-epoch least squares adjustment problem, ME-LSA, (Smith et al. 2023a), will be a cornerstone of both NGS adjustment projects as well as OPUS adjustment projects in the modernized NSRS (NGS 2021). The ME-LSA involves some complicated mathematical equations that center around the concept of projecting geodetic quantities through time, turning observations, constraints and their cofactor matrices into projected observations, projected constraints and their projected cofactor matrices. Both in the original derivation (Smith et al. 2023a) as well as in an expansive discussion of covariances within the ME-LSA (Smith 2023), generalized equations covering a variety of observations, etc., were derived. But such generalized equations do not lend themselves to the problem of coding up working algorithms on a case-by-case basis for each specific type of observation or constraint.

This paper takes the foundational work in Smith et al. (2023a) and Smith (2023), and, on a case-by-case basis, derives the equations needed to code up an algorithm for projecting quantities through time. These equations will be the heart of a new piece of software called SPROCCET, which will replace NGS's current software HTDP. When complete, SPROCCET will be capable of projecting all supported observations and constraints through time, as well as propagating their random errors so that a projected cofactor matrix will also be computed. Once SPROCCET is complete, NGS plans to use it as a pre-processor for all NGS REC adjustment projects, all OPUS adjustments and to integrate its functionality into NGS's growing toolkit of integrated products.

21 Bibliography

Dennis, M., 2020: The National Adjustment of 2011, Alignment of Passive GNSS Control with the Three Frames of the North American Datum of 1983 at Epoch 2010.00: NAD83 (2011), NAD83 (PA11), and NAD83 (MA11), *NOAA Technical Report NOS NGS 65*, July 2020. URL https://geodesy.noaa.gov/library/pdfs/NOAA_TR_NOS_NGS_0065.pdf

Leick, A, 2004: *GPS Satellite Surveying*, John Wiley & Sons, Inc., Hoboken, NJ.

Leick, A. and B.H.W. van Gelder, 1975: On Similarity Transformations and Geodetic Network Distortions Based on Doppler Satellite Observations, *Reports of the Department of Geodetic Science, No. 235*, The Ohio State University, Columbus, Ohio. https://earthsciences.osu.edu/sites/earthsciences.osu.edu/files/report-235.pdf

Milbert, D. and W. Kass, 1987: ADJUST: THE HORIZONTAL OBSERVATION ADJUSTMENT PROGRAM. *NOAA Technical Memorandum NOS NGS 47*. URL https://geodesy.noaa.gov/library/pdfs/NOAA_TM_NOS_NGS_0047.pdf

National Geodetic Survey, 2021: Blueprint for the Modernized NSRS, Part 3: Working in the Modernized NSRS, *NOAA Technical Report NOS NGS 67*, February 2021. URL https://geodesy.noaa.gov/library/pdfs/NOAA_TR_NOS_NGS_0067.pdf

Smith, D.A., 2023: Covariances within the multi-epoch least-squares adjustment problem and their impact on estimating reference epoch coordinates in the modernized National Spatial Reference System. *NOAA Technical Memorandum NOS NGS 95*. URL https://geodesy.noaa.gov/library/pdfs/NOAA_TM_NOS_NGS_0095.pdf

Smith, D.A., D. Gillins, J. Heck, D. Roman, 2023a: The multi-epoch least-squares adjustment problem: General equations relating estimable coordinates at a single epoch to geodetic observations, stochastic constraints and fixed constraints at multiple other epochs. Volume I: The Projection Method. *NOAA Technical Report NOS NGS 79*. URL https://geodesy.noaa.gov/library/pdfs/NOAA_TR_NOS_NGS_0079.pdf

Smith, D.A., D. Gillins, J. Heck, D. Roman, 2023b: The multi-epoch least-squares adjustment problem: General equations relating estimable coordinates at a single epoch to geodetic observations, stochastic constraints and fixed constraints at multiple other epochs. Volume II: The Direct Method. *NOAA Technical Report NOS NGS 80*. URL https://geodesy.noaa.gov/library/pdfs/NOAA_TR_NOS_NGS_0080.pdf

Snay, R., 1999: Using the HTDP Software to Transform Spatial Coordinates Across Time and Between Reference Frames, *Surveying and Land Information Systems*, V. 59, No. 1, pp 15-25.

Snow, K., F. Wang and A. Whitacre, 2023: A Guide to LASER: LSA Libraries and Programs for LASER 2.2, *NGS Internal Document*.

Wolf, H., 1963: Die Grundgleichungen der Dreidimensionalen Geodäsie in elementarer Darstellung, Zeitschrift für Vermessungswesn, *Heft 6*, 88. Jahrgang, Juni 1963.

Yeager, J.A, 1980: Input Formats and Specifications of the National Geodetic Survey Data Base (Revised 1994), *NOAA Manual NOS NGS 2*, https://geodesy.noaa.gov/library/pdfs/NOAA_Manual_NOS_NGS_0002.pdf