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Storage and Use of GRS80 Ellipsoid Parameters in the Modernized NSRS

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Executive Summary

The Geodetic Reference System of 1980 (GRS80), a model of an ellipsoid of revolution and its implied normal gravity field, has been used in the geodetic community for decades. However, its defining parameters of semi-major axis (a), gravity-mass constant (GM), dynamic form factor (J_2) and angular velocity (ω), make the computation of derivative values, particularly concerning the geometric qualities of the ellipsoid, somewhat onerous. The National Geodetic Survey (NGS) will adopt a modified definition of GRS80 for use in the forthcoming modernized National Spatial Reference System (NSRS). In that modified definition, the value for the inverse flattening of the ellipsoid, f^{-1}, is given an exact value, rather than J_2.

Several tests were performed using this modified definition, and all conclusively showed that this modification only has a negligible impact. This memorandum describes the reasoning behind this decision, its implications, and its implementation.

1 Introduction and the need for exactly four ellipsoid parameters

When NGS initiated efforts toward a modernized NSRS, one of the earliest decisions was that GRS80[1] would be the reference ellipsoid for geometric and geopotential work (NGS 2021a, NGS 2021b). This memorandum outlines the policy and equations that NGS will adopt regarding the practical adoption of GRS80 in the modernized NSRS, including what values will be stored, how many digits, and what equations will be used to compute derivative values on the fly.

As per the original definition (Moritz, 1980), the GRS80 ellipsoid was defined by the International Association of Geodesy (IAG) through four parameters, considered *exact*, as shown in Table 1.

Table 1: The official defining parameters of GRS80

Name	Symbol	Value	Units
Semi-major axis	a	6378137	m
Gravity-mass constant	GM	3986005×10^{8}	m^3/s^2
Dynamic form factor	J_2	108263×10^{-8}	$unitless$
Angular velocity	ω	7292115×10^{-11}	s^{-1}

These four values contain the necessary information to calculate any other *derivative value* (geometric or geopotential) related to a reference ellipsoid. While these four are not the only set of four ellipsoid parameters for which this can be stated, they are the official parameters defined by the IAG for GRS80.

Storing only these four values and computing derivative values on the fly is the most rigorous way to adopt and use GRS80. The equations defining any derived value cannot yield a number with an exact value (that is, all digits are known). This means that if one adopted a fifth value as "known" without adopting that fifth value to an infinite number of digits, one would be setting up a non-uniqueness conflict in other downstream derived values.

By way of example, consider one of the most commonly adopted values outside of the original four: the (unitless) inverse flattening (f^{-1}) of the ellipsoid. When derived from a, GM, J_2 and ω, (see Appendices A and B) the value of the inverse flattening (to 25 of its infinite digits) is:

$$f^{-1} = 298.2572221008827112431628366 \ldots \ldots \tag{1}$$

In practice, the inverse flattening is often taken as the following *exact* value. We provide subscript a to differentiate between GRS80's true inverse flattening and this approximation.

$$(f^{-1})_a = 298.257222101 \tag{2}$$

[1] The authors acknowledge the various acronyms used for the "Geodetic Reference System of 1980" since its introduction. Although the International Association of Geodesy (IAG's) Geodesists' Handbook uses "GRS 80", the International Organization for Standardization (ISO) uses "GRS80". Because NGS has been increasingly complying with international standards of late, we will use "GRS80."

Now, let us compute the normal gravity potential on the ellipsoid's surface, U_0, two ways. First, from a, GM, J_2 and ω (see Appendix C) and then from a, GM, $(f^{-1})_a$ and ω (see Appendix D). The results are below. We continue to use subscript a to indicate a value different from GRS80.

From a, GM, J_2 and ω : $U_0 = 62636860.85004\mathbf{611865}\ldots \quad m^2/s^2$ (3a)

From a, GM, $(f^{-1})_a$ and ω : $(U_0)_a = 62636860.85004\mathbf{609111}\ldots \quad m^2/s^2$ (3b)

What (3) demonstrates is that adopting five, not four, parameters for a reference ellipsoid results in different derived values, depending on which four starting values are used. This is the non-uniqueness conflict mentioned above. Although the differences for the above values may be just beyond the limit of typical double precision (nominally 15 digits), they differ nonetheless.

Whether or not this non-uniqueness is an issue is a matter of accuracy needs. In the U_0 example above, the difference is on the order of $10^{-8}\ m^2/s^2$, which translates (via Bruns' equation[2]) to approximately $10^{-7}\ m$ (0.0001 mm) in the determination of a normal geopotential surface in space, which is likely to be negligible for most applications.

However, just because one example does not yield any numeric issues is not a good reason to adopt five ellipsoid parameters and set oneself up for non-uniqueness issues. Consider, for example, the definition of the WGS 84 ellipsoid by the National Geospatial-Intelligence Agency (NGA, 2014). Originally, it was defined by adopting exact values for a, GM, $\bar{C}^*_{2,0}$ and ω. (Although NGA uses the notation $\bar{C}_{2,0}$, this is technically incorrect and should be $\bar{C}^*_{2,0}$; see Smith (1998)). In the late 1990s, a decision was made (NIMA, 2004) to adopt a, GM, f^{-1} and ω, dropping $\bar{C}^*_{2,0}$ as a defining parameter of the WGS 84 ellipsoid and thus acknowledging the importance of defining only *four* ellipsoid parameters.

In NGA (2014), there is an unfortunately confusing sentence: "*Additionally, there are now distinct values for the $\bar{C}_{2,0}$ term, one dynamically derived as part of the WGS 84 Earth Gravitational Models and the other geometric, implied by the defining parameters [of the reference ellipsoid].*" This is misleading because it equates two terms that are not equal: the 2nd degree zonal harmonic coefficient of Earth's external gravitational potential ($\bar{C}_{2,0}$) and the 2nd degree zonal harmonic coefficient of the normal gravitational field of the reference ellipsoid ($\bar{C}^*_{2,0}$) that is related to J_2 through the equation $\bar{C}^*_{2,0} = -J_2/\sqrt{5}$. Using one term, $\bar{C}_{2,0}$, to mean both coefficients is confusing, and more importantly, incorrect; see Smith (1998). Further, the NGA discussion of EGM96 (NGA, 2014), near this sentence was irrelevant, since ultimately the $\bar{C}^*_{2,0}$ value of the updated WGS 84 ellipsoid was (as stated in that paragraph) derived from a, GM, f^{-1} and ω, not related to the EGM96 model at all.

Unfortunately, the non-uniqueness problem is often ignored in practice. Instead, storing and using one or more *pre-computed* derivative values is expected, rather than returning to first

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[2] Brun's equation, $N = (W_0 - U_0)/\gamma$, shows the distance from the reference ellipsoid ($U = U_0$) to the geoid ($W = W_0$). If the U_0 value is changed by some small amount (δU_0) without changing the W_0 value, then this implies a change in that distance. That is, it implies that the reference ellipsoid is in a different location. Specifically, it is vertically displaced by $\delta U_0/\gamma$, where γ is approximately 9.8 m/s^2.

principles every time one is interested in them. The number of such derivative values is extensive. However, some values are needed more frequently than others, such as the semi-minor axis (b), flattening (f), inverse flattening (f^{-1}), first eccentricity squared (e^2), and normal gravity potential on the ellipsoid (U_0). Part of this is simply practical: the e^2 value is used (with a) in one of the most common geodetic equations, relating geocentric Cartesian (XYZ) coordinates to geodetic latitude, longitude, and ellipsoid height ($\phi\lambda h$). To re-compute e^2 from a, GM, J_2 and ω (see Appendix B) is much more burdensome than simply storing and using some rounded e^2 or f^{-1} value.

We recognize that storing pre-computed derivative values is both widely done in practice and (if one of those is the inverse flattening) makes computing geometric quantities easier. This is what led to the change mentioned above to the definition of the WGS 84 ellipsoid, as well as to a modified definition of GRS80 at the International Organization for Standardization (ISO, 2018), described in the next section.

2 ISO standard for GRS80

The International Organization for Standardization (ISO) defines GRS80 (ISO, 2018) as a geometric ellipsoid with two values, as seen in Table 2.

Table 2: The official defining parameters of GRS80 in the ISO geodetic registry

Name	Symbol	Value	Units
Semi-major axis	a	6378137	m
Inverse flattening	$(f^{-1})_{GRS80(ISO)}$	298.257222101	$unitless$

While the ISO acknowledges (in remarks) that three other values are associated with GRS80 (GM, J_2 and ω), *they are not part of the ISO standard*. Note that $(f^{-1})_{GRS80(ISO)}$ is identical to the oft-used $(f^{-1})_a$ value mentioned earlier.

Also, as mentioned earlier, this value of $(f^{-1})_{GRS80(ISO)}$ does cause some numerical non-uniqueness when compared to values relying upon a, GM, J_2 and ω. To provide further insight, NGS performed some additional numeric tests using $(f^{-1})_{GRS80(ISO)}$ and f^{-1}. Specifically, NGS compared the conversion from latitude, longitude, and ellipsoid height to X, Y, and Z coordinates, using the two versions of f^{-1}. To perform this test, a computer program was written which computed X, Y, and Z in quadruple precision at every combination of latitude, longitude, and ellipsoid height in these domains: $-90° \leq \phi \leq +90°$, $0° \leq \lambda \leq +359°$, and $-1000\ m \leq h \leq +10{,}000\ m$, in 1-degree and 10-meter increments. The two values, $(f^{-1})_{GRS80(ISO)}$ and f^{-1} (computed from a, GM, J_2, and ω in quadruple precision), were used and the differences in X, Y, and Z computed for each. The RMS differences in X, Y, and Z, between the two different versions of f^{-1}, were 1.5×10^{-9}, 1.5×10^{-9}, and 7.6×10^{-9}meters, respectively. The maximum differences in X, Y, and Z were 3.2×10^{-9}, 3.2×10^{-9}, and 9.2×10^{-9} meters, with each maximum at latitude $-55°$ and ellipsoid height around $-1000\ m$. Errors of similar magnitude (10^{-6} millimeters) were also found when computing State Plane coordinates. As before, this is only important for certain applications.

3 NGS policy for GRS80 in the modernized NSRS

As part of NSRS modernization, NGS has been moving steadily toward adopting various international standards. For instance, the four planned reference frames of NATRF2022, PATRF2022, CATRF2022, and MATRF2022 will all be derived from ITRF2020, the internationally recognized standard for global positioning (UN-GGIM 2015). To continue this trend, and after careful consideration of the pros and cons of such a decision, NGS has chosen to adopt *an alternate form of GRS80* by adopting exactly the a, GM and ω values as seen in Table 1, and further adopting f^{-1} using the ISO standard value as seen in Table 2. This exact set of four defining parameters has no name. Officially, it is not GRS80, but from a numerical standpoint, it will be identical for all practical purposes. Still, to be as rigorous as possible, we will refer to this ellipsoid as GRS80(NGS2022) in this document. Note that although we use GRS80(NGS2022) here to identify this ellipsoid, we do not propose it be added to the ISO standard, since the geometric quantities are identical to the existing ISO definition of GRS80. To formalize our defined ellipsoid, we place its values in Table 3.

Table 3: The defining parameters of the GRS80(NGS2022) ellipsoid to be used in the modernized NSRS

Name	Symbol	Value	Units
Semi-major axis	$a_{GRS80(NGS2022)}$	6378137	m
Gravity-mass constant	$GM_{GRS80(NGS2022)}$	3986005×10^{8}	m^3/s^2
Inverse flattening	$(f^{-1})_{GRS80(NGS2022)}$	298.257222101	$unitless$
Angular velocity	$\omega_{GRS80(NGS2022)}$	7292115×10^{-11}	s^{-1}

To avoid the non-uniqueness problem, NGS will not adopt a J_2 value, but will derive it as needed from a, GM, f^{-1} and ω (see Appendix E). To be clear, the difference is minuscule, as seen below:

As defined by GRS80:	$J_2 = 0.00108263\mathbf{0000000000}\ldots$	(4a)
From $a_{GRS80(NGS2022)}$, $GM_{GRS80(NGS2022)}$, $(f^{-1})_{GRS80(NGS2022)}$ and $\omega_{GRS80(NGS2022)}$:	$(J_2)_{GRS80(NGS2022)} = 0.00108262\mathbf{9999999122}\ldots$	(4b)

The differences are small and effectively negligible. This approach aligns with the procedures for the State Plane Coordinate System of 2022 (Dennis, 2023).

NGS intends to implement the above-defined version of GRS80 in all modernized NSRS products and services. However, because "GRS80(NGS2022)" is a lengthy acronym, and the small, subtle, and effectively negligible difference with GRS80 may not be significant to most users, NGS will label the ellipsoid in our modernized NSRS products and services as "GRS80". Users interested in the difference with GRS80(NGS2022) will be directed to this document.

When NGS software needs to compute derivative values from a, GM, f^{-1}, and ω, it will do so using equations found in Appendices A, D and E. Specifically, geometric values (f, e^2, b, E, and e'^2) are first computed from (11) in Appendix A. After that, U_0 and J_2 come from Appendices D and E respectively.

Beyond those, only a few other terms of common interest will be mentioned. The value of q_0' is needed for some additional complex geopotential terms:

$$q_0' = 3\left(1+\frac{b^2}{E^2}\right)\left(1-\frac{b}{E}\arctan\frac{E}{b}\right)-1 \tag{5}$$

Each zonal harmonic of the normal gravity potential field beyond J_2 may be computed through (6), and the normal gravity values at equator and pole come from (7) and (8).

$$J_{2n} = (-1)^{n+1}\frac{3(e^2)^n}{(2n+1)(2n+3)}\left(1-n+5n\frac{J_2}{e^2}\right) \quad \forall\, n \geq 2 \tag{6}$$

Normal gravity on the surface of the ellipsoid at the equator

$$\gamma_a = \frac{GM}{ab}\left(1-m-\frac{m}{6}\frac{e'q_0'}{q_0}\right) \tag{7}$$

Normal gravity on the surface of the ellipsoid at the poles

$$\gamma_b = \frac{GM}{a^2}\left(1+\frac{m}{3}\frac{e'q_0'}{q_0}\right) \tag{8}$$

In (6) through (8), the values m and q_0 come from Appendix E and Appendix B, respectively. The above list of equations is meant to be partial. As other related quantities are needed, NGS software can be expanded without expanding this document.

4 Summary

NGS is modernizing the NSRS and, as part of that, has relied upon the GRS80 ellipsoid for all geometric and geopotential computations requiring a reference ellipsoid. However, GRS80 was defined using four parameters (upon a, GM, J_2, and ω) that do not easily lend themselves to the vast majority of geodetic computations, namely geometric ones. NGS frequently needs the inverse flattening (f^{-1}) or the first eccentricity squared (e^2) for geometric computations. Still, these values require a not-insignificant computational burden to correctly re-compute from a, GM, J_2, and ω. To avoid that burden, and avoid a problem of non-uniqueness, and align with the ISO standard, NGS has chosen to adopt an alternative form of GRS80, which we have called GRS80(NGS2022), whose four defining parameters are a, GM, f^{-1} and ω, where a, GM, and ω are identical to the original definition of GRS80, and f^{-1} has been set exactly to the widely-used value of 298.257222101, matching the ISO standard. Derived geometric values will therefore be easily computed from a and f^{-1}, without the more cumbersome approach of returning to first principles with a, GM, J_2 and ω.

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6 Appendices

6.1 Appendix A: Converting common geometric values between one another

The geometric qualities (size and shape) of an ellipsoid of revolution can be uniquely described by giving just two terms: one to provide scale and one to provide shape. However, there are numerous geometric terms of interest, all interrelated. For this memorandum, we restrict ourselves to the semi-major axis (a), semi-minor axis (b), flattening (f), inverse flattening (f^{-1}), first eccentricity squared (e^2), linear eccentricity (E) and second eccentricity squared (e'^2).

We provide below the equations necessary to compute any of the aforementioned geometric terms from any two given terms, under the assumption that the semi-minor axis (a) will always be one of the two given terms.

If a and b are given:	If a and f are given:	If a and f^{-1} are given:
$b = b$ (9a)	$b = a(1-f)$ (10a)	$b = a(1 - \frac{1}{f^{-1}})$ (11a)
$f = \frac{a-b}{a}$ (9b)	$f = f$ (10b)	$f = \frac{1}{f^{-1}}$ (11b)
$f^{-1} = \frac{a}{a-b}$ (9c)	$f^{-1} = \frac{1}{f}$ (10c)	$f^{-1} = f^{-1}$ (11c)
$e^2 = \frac{a^2-b^2}{a^2}$ (9d)	$e^2 = 2f - f^2$ (10d)	$e^2 = \frac{2f^{-1}-1}{(f^{-1})^2}$ (11d)
$E = \sqrt{a^2-b^2}$ (9e)	$E = a\sqrt{2f-f^2}$ (10e)	$E = a\sqrt{\frac{2f^{-1}-1}{(f^{-1})^2}}$ (11e)
$e'^2 = \frac{a^2-b^2}{b^2}$ (9f)	$e'^2 = \frac{1}{(1-f)^2} - 1$ (10f)	$e'^2 = \frac{2f^{-1}-1}{(f^{-1}-1)^2}$ (11f)

If a and e^2 are given:	If a and E are given:	If a and e'^2 are given:
$b = a\sqrt{1-e^2}$ (12a)	$b = a\sqrt{1-\frac{E^2}{a^2}}$ (13a)	$b = \frac{a}{\sqrt{1+e'^2}}$ (14a)
$f = 1 - \sqrt{1-e^2}$ (12b)	$f = 1 - \sqrt{1-\frac{E^2}{a^2}}$ (13b)	$f = 1 - \frac{1}{\sqrt{1+e'^2}}$ (14b)
$f^{-1} = \frac{1}{1-\sqrt{1-e^2}}$ (12c)	$f^{-1} = \frac{1}{1-\sqrt{1-\frac{E^2}{a^2}}}$ (13c)	$f^{-1} = \frac{1+e'^2+\sqrt{1+e'^2}}{e'^2}$ (14c)

$e^2 = e^2$ (12d)	$e^2 = \frac{E^2}{a^2}$ (13d)	$e^2 = \frac{e'^2}{1+e'^2}$ (14d)
$E = a\sqrt{e^2}$ (12e)	$E = E$ (13e)	$E = a\sqrt{\frac{e'^2}{1+e'^2}}$ (14e)
$e'^2 = \frac{e^2}{1-e^2}$ (12f)	$e'^2 = \frac{E^2}{a^2 - E^2}$ (13f)	$e'^2 = e'^2$ (14f)

6.2 Appendix B: Computing geometric values from a, GM, J_2 and ω

In order to derive any of the geometric terms from the previous section (b, f, f^{-1}, e^2, E or e'^2), from a, GM, J_2 and ω we follow the guidance found in Moritz (1980), where the first step is computing the first eccentricity (e^2). In that paper, the recommendation is to solve for e^2 by iterating on (15), below. Note we have expressed e^3 as $(e^2)^{3/2}$ in order to maintain use of e^2 as the iterated variable in (15).

$$e^2 = 3J_2 + \frac{4}{15}\frac{\omega^2 a^3}{GM}\frac{(e^2)^{3/2}}{2q_0} \tag{15}$$

The common formula for q_0 (Heiskanen and Moritz, 1967, eq. 2-58) is found in (16).

$$q_0 = \frac{1}{2}\left[\left(1 + 3\frac{b^2}{E^2}\right)\arctan\frac{E}{b} - 3\frac{b}{E}\right] \tag{16}$$

To iterate on e^2 in (15) it will be useful to write q_0 in terms of e^2. To do so, we rely on the relationship between b and E and e^2 as found in (12a) and (12e). Plugging (12a) and (12e) into (16) yields the alternate version of q_0 seen in (17).

$$q_0 = \frac{1}{2}\left[\left(1 + 3\frac{1-e^2}{e^2}\right)\arctan\sqrt{\frac{e^2}{1-e^2}} - 3\sqrt{\frac{1-e^2}{e^2}}\right] \tag{17}$$

The iteration proceeds by choosing an initial value of e^2, plugging that into (17) to get an initial value of q_0, which is then plugged, with the initial value of e^2, into (15) to arrive at an updated value of e^2. The iteration proceeds until the desired accuracy of e^2 is achieved.

Once e^2 is known to the desired accuracy, other geometric values may be computed using (12). Note that this iterative solution may not yield values at double precision accuracy (nominally 15 digits) even when double precision computations are used. For example, f^{-1} = 298.257222100883 to 15 digits, but a double precision iterative solution of e^2 will typically not yield this value when computed using (12). The last three digits can vary depending on when and how iteration is terminated. However, double precision iteration will reliably yield f^{-1} = 298.257222101, which is likely the reason this 12-digit value was widely adopted in the first place. To reliably get more digits requires quadruple precision or other high-precision computation methods. Because most algorithms are limited to double precision, it is impractical to routinely compute e^2 and its derived values to full double precision in most software. This limitation is one of the main reasons f^{-1} = 298.257222101 was adopted as an exact defining value for GRS80 (together with its long history of usage in the geospatial community).

6.3 Appendix C: Computing U_0 from a, GM, J_2 and ω

To derive the normal gravity potential on the ellipsoid's surface (U_0) from a, GM, J_2 and ω we follow the guidance found in Moritz (1980), which is to first derive e^2 using the method in Appendix B.

Once we have e^2, we need to derive E and b, as shown in (12). Then, the formula for U_0, which happens to be exact (Moritz 1980; see also Heiskanen and Moritz, equation 2-61, noting that $e' = E/b$), is:

$$U_0 = \frac{GM}{E}\arctan\frac{E}{b} + \frac{1}{3}\omega^2 a^2 \tag{18}$$

6.4 Appendix D: Computing U_0 from a, GM, f^{-1} and ω

To derive the normal gravity potential on the ellipsoid's surface (U_0) from a, GM, f^{-1} and ω we begin by computing E and b from a and f^{-1}, using (11), above. Note that there is no iteration involved. Once we have E and b, we apply (18), above.

6.5 Appendix E: Computing J_2 from a, GM, f^{-1} and ω

To show how to compute the dynamic form factor of the reference ellipsoid (J_2) from $a, GM,$ f^{-1} and ω we must first turn to Heiskanen and Moritz (1967), and perform a few derivations. We begin by comparing equation (2-88) with the unnumbered equation between (2-91) and (2-92), which allows us to write:

$$A_2\frac{P_2(\cos\theta)}{r^3} = -\frac{GM}{r}J_2\left(\frac{a}{r}\right)^2 P_2(\cos\theta) \tag{19}$$

This leads to the following equation:

$$J_2 = \frac{-A_2}{GMa^2} \tag{20}$$

The equation for A_2 is provided in (ibid) above equation (2-90):

$$A_2 = -\frac{1}{3}GME^2\left(1 - \frac{2}{15}\frac{me'}{q_0}\right) \tag{21}$$

Where the term $q_0$is found earlier in (16) and m is:

$$m = \frac{\omega^2 a^2 b}{GM} \tag{22}$$

Therefore, to compute J_2 from a, GM, f^{-1} and ω we first compute b, E and e' from a and f^{-1} using (11). These are then used to compute m and q_0 in (22) and (16), which are used to compute A_2 from (21) which finally is used to compute J_2 from (20).