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Earth's Gravity Field and Station Coordinates From Doppler Data, Satellite Triangulation, and Gravity Anomalies

KARL-RUDOLF KOCH

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NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
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EARTH'S GRAVITY FIELD AND STATION COORDINATES FROM DOPPLER DATA, SATELLITE TRIANGULATION, AND GRAVITY ANOMALIES

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ABSTRACT. A solution for the Earth's gravitational field represented by the potential of a simple layer is obtained by means of the analysis of 108,491 observations from 21 weeks of Doppler data from 41 stations to seven satellites. In a combined solution gravity anomalies are added to satellite observations. In addition the results for 17 stations of the worldwide satellite triangulation tied by survey to corresponding Doppler stations are introduced into the combined solution so that coordinates of 41 Doppler stations and 17 stations of the satellite triangulation are obtained. The semimajor axis of the best fitting ellipsoid of the combined solution equals 6,378,134 m and its flattening 1/298.254.

1. INTRODUCTION

Data for geometric satellite geodesy have been collected extensively in the worldwide satellite triangulation [Schmid 1974a], while for dynamic satellite geodesy large amounts of Doppler data have been acquired by the U.S. Navy Doppler network [Anderle 1971]. A number of stations of both networks lie close together and are connected by survey ties, so there is the unique possibility of combining the data of the two methods. While the results of the worldwide satellite triangulation are available [Schmid 1974a], only part of the Doppler data can be obtained. However, as already shown [Koch and Witte 1971], the analysis of the Doppler data then available gave excellent results for the gravity field.

Following this last analysis, additional Doppler observations were obtained so that a new dynamic solution could be prepared. The results of the satellite triangulation for those stations which lie close to Doppler tracking sites were added. Furthermore, gravity anomalies were also introduced into the combination of the results. However, no attempt was made to increase the resolution of the dynamic solution with respect to the gravity

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field by means of the gravity anomalies. The model of the geopotential is based on the simple-layer potential, in contrast to the expansion into spherical harmonics.

2. COMPUTATIONAL PROCEDURES

2.1 Representation of the Gravity Field

To represent the gravity field, the potential of a simple layer is chosen [Koch 1968]. Hence, the potential W of the Earth is divided into the potential U, which is assumed to be known and is expressed by an expansion into spherical harmonics, and into the potential T, which is unknown and is represented by the potential of a simple layer distributed over the surface of the Earth. Therefore

$$W = U + T \tag{1}$$

with

$$U = \frac{kM}{r}\left[1 + \sum_{n=2}^{n_c}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n \bar{P}_{nm}(\sin\phi)\left(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right)\right] + \tfrac{1}{2}\left(\omega^2 r^2 \cos^2\phi\right) \tag{2}$$

and

$$T = \iint_{\Sigma} \frac{\Phi}{\ell}\, d\Sigma\ , \tag{3}$$

where r, ϕ, λ are geocentric coordinates in the usual Earth-fixed system, k is the gravitational constant, M the mass of the Earth, a the mean equatorial radius, $\bar{P}_{nm}$ the normalized associated Legendre function of degree n and order m, and ω the angular velocity of the Earth. Outside the Earth, the term containing ω equals zero. $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are normalized harmonic coefficients which are taken up to degree n_c from available results. Φ is the unknown density of the simple layer distributed over the surface Σ of the Earth. ℓ denotes the distance between the fixed point at which the potential is computed and the moving point on Σ. Σ is obtained by adding the topographic heights to the reference surface computed from (2) by setting $U = U_0$ and solving for the radius r. U_0 is defined as the potential of a level ellipsoid, whose constants kM, a, $\bar{C}_{2,0}$ and ω are taken from (2) [Koch 1968].

By introducing the auxiliary density χ and assuming constant densities χ_i for the p surface elements ΔE_i into which the surface of the Earth is divided, we obtain instead of (3) [Koch 1972],

$$T = \sum_{i=1}^{p} x_i \iint_{\Delta E_i} \frac{dE}{\ell} . \tag{4}$$

The surface elements ΔE_i are bordered by meridians and parallels and are approximately of equal size. The integral over ΔE_i is solved numerically by subdividing ΔE_i into four elements at whose midpoints the kernel of the integral is assumed to be constant. The method of defining the surface elements and their midpoints is given by Koch [1971].

The density values x_i can be converted into normalized spherical harmonics [Koch 1972]:

$$\bar{C}_{nm} = \bar{C}_{nmu} + \frac{1}{(2n+1)kMa^n} \sum_{i=1}^{p} x_i \iint_{\Delta E_i} r^n \, \bar{P}_{nm}(\sin \phi) \cos m\lambda dE$$

and (5)

$$\bar{S}_{nm} = \bar{S}_{nmu} + \frac{1}{(2n+1)kMa^n} \sum_{i=1}^{p} x_i \iint_{\Delta E_i} r^n \, \bar{P}_{nm}(\sin \phi) \sin m\lambda dE ,$$

where $\bar{C}_{nmu}$ and $\bar{S}_{nmu}$ are the harmonic coefficients which define U in (2). The integral over ΔE_i is solved numerically by subdividing ΔE_i into nine elements. Since the coordinate system in which r, ϕ, and λ are defined is geocentric, the density values x_i have to be determined under the constraint that the harmonics of first degree equal zero.

Comparisons of the simple layer model of the geopotential with other representations of the gravity field have been made by Morrison [1971] and Pollack [1973].

2.2 Analysis of Doppler Observations

Satellite orbits should be computed in an inertial reference frame. But since the orbit computations in this analysis do not exceed a time period of one week, one obtains a good approximation to an inertial system by a geocentric coordinate system whose 3-axis is identical with the instantaneous axis of the Earth and whose 1-axis points at an angle east of the true vernal equinox which takes into account the precession and nutation in right ascension. This angle is chosen to be the precession and nutation in right ascension since 1950.0 .

The Doppler data are frequencies f with

$$f = f_b - \frac{f_b}{c} \frac{d}{dt} \left(|\underline{r} - \underline{r}_s| \right) + \delta f, \tag{6}$$

where f_b is the base frequency, c the velocity of light, t the time, $\underline{r}$ and $\underline{r}_s$ the position vectors of the satellite and tracking station in the coordinate system defined above, and δf the tropospheric refraction correction. The coordinates of the tracking stations are also defined in an Earth-fixed geocentric system with the 3-axis pointing toward the mean pole of 1900-1905 and the 1-axis lying in the mean meridional plane of Greenwich. To rotate the Earth-fixed system into the inertial system, the polar motion as determined by the Bureau de l'Heure and the sidereal angle as defined by the Smithsonian Institution [Lundquist and Veis 1966] are used. This rotation is corrected for the difference between UTC and UT1 published by the Bureau de l'Heure because the time of the Doppler observations is given in UTC.

To evaluate the Doppler observations a least squares adjustment based on a differential correction process is applied [Koch and Morrison 1970]. Hence, by means of approximate values the Doppler frequency (6) is expanded as a function of the unknown parameters into a Taylor series which is restricted to the linear terms. This linearization leads to the observation equations of the least squares adjustment for the unknown parameters which are the density values, the coordinates of the tracking stations, a base frequency offset per pass of a satellite over a station, and, for each arc, the six orbital elements. Hence,

$$[\underline{A}, \underline{B}, \underline{C}, \underline{D}] \begin{bmatrix} \underline{\Delta b} \\ \underline{\Delta e} \\ \underline{\Delta x} \\ \underline{\Delta x} \end{bmatrix} = \underline{\ell} + \underline{v} \, . \tag{7}$$

$\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ are matrices of coefficients, and $\underline{\Delta b}$, $\underline{\Delta e}$, $\underline{\Delta x}$, $\underline{\Delta x}$ vectors of corrections to preliminary values of the base frequencies, orbital elements, station coordinates, and density values. $\underline{\ell}$ and $\underline{v}$ are vectors of the observations and residuals.

The orbits of the observed satellites are computed numerically with a 48-second time step using a 12th order Cowell-Störmer integration for positions, a 10th order Adams-Bashforth predictor and a 10th order Adams-Moulton corrector for the velocities, and an 8th order Adams-Moulton integration for the variational equations [Gulick 1971] by which the derivatives of the observations with respect to the unknown parameters are determined. Lagrange's interpolation is applied between time steps. To avoid errors in the orbit computation resulting from the numerical integration in (4), the preliminary density values are set equal to zero so that the geopotential for the orbit computation is given by (2). In addition, the attraction of the sun and moon, and the influence of air drag and radiation pressure, are considered in the orbit computations [Witte 1971].

If $\underline{\Sigma}_{\ell}$ denotes the covariance matrix of the Doppler observations, the normal equations of the least-squares adjustment are given by

$$\begin{vmatrix} \underline{A}^T\underline{\Sigma}_{\ell}^{-1}\underline{A} & \cdots & \underline{A}^T\underline{\Sigma}_{\ell}^{-1}\underline{D} \\ \vdots & & \vdots \\ \underline{D}^T\underline{\Sigma}_{\ell}^{-1}\underline{A} & \cdots & \underline{D}^T\underline{\Sigma}_{\ell}^{-1}\underline{D} \end{vmatrix} \begin{vmatrix} \underline{\Delta b} \\ \vdots \\ \underline{\Delta x} \end{vmatrix} = \begin{vmatrix} \underline{A}^T\underline{\Sigma}_{\ell}^{-1}\underline{\ell} \\ \vdots \\ \underline{D}^T\underline{\Sigma}_{\ell}^{-1}\underline{\ell} \end{vmatrix} . \tag{8}$$

To reduce the number of unknowns, the corrections $\underline{\Delta b}$ to the base frequencies are eliminated whenever the contribution of one pass to the normal equations has been computed. The corrections $\underline{\Delta e}$ to the orbital elements are eliminated after obtaining the contribution of one arc to the normal equations, so we get

$$\begin{vmatrix} \underline{N}_{aa} & \underline{N}_{ab} \\ \underline{N}^T_{ab} & \underline{N}_{bb} \end{vmatrix} \begin{vmatrix} \underline{\Delta x} \\ \underline{\Delta x} \end{vmatrix} = \begin{vmatrix} \underline{f}_a \\ \underline{f}_b \end{vmatrix} , \tag{9}$$

where $\underline{N}$ denotes the matrix of the reduced normals and $\underline{f}$ the reduced absolute column. To avoid a singularity in the normal equations, the longitude of one station must be held fixed. This constraint as well as the constraints for $\overline{C}_{1,0}$ $\overline{C}_{1,1}$ and $\overline{S}_{1,1}$ are introduced as observations with small variances [Koch and Pope 1969]. In addition, the coefficients $\overline{C}_{2,1}$ and $S_{2,1}$ must be small in comparison to the other harmonic coefficients, since for the orbit computation the 3-axis coincides with the rotational axis of the Earth.

2.3 Density Values from Gravity Anomalies

The density values x may be computed also from gravity anomalies Δg [Koch and Morrison 1970], by

$$\chi = \frac{\Delta g - G}{2\pi} + \frac{3}{(4\pi)^2} \iint (\Delta g - G)\ S(\psi)\ \cos\phi\ d\phi d\lambda \tag{10}$$

with

$$G = \frac{1}{4\pi} \iint \Delta g\ \cos\phi\ d\phi d\lambda\ ,$$

where $S(\psi)$ denotes Stokes' function and ψ the spherical distance between the fixed point and the variable point. Equation (10) is the solution of the geodetic boundary value problem for a spherical earth without topography. However, this approximation is sufficient for the present analysis.

Gravity anomalies are usually referred to the International Ellipsoid. To define them with respect to the reference surface used here, we have

$$\Delta g = \Delta g_I + |\text{grad } U_I| - |\text{grad } U|\ , \tag{11}$$

where Δg_I are the anomalies referred to the International Ellipsoid and U_I is its potential. According to the theory of the geodetic boundary value problem the gradients of U_I and U must be computed at a height above the ellipsoid and the reference surface that approximately equals the topographic height at which the value of gravity has been measured. To compute grad U_I, the potential U_I has been expanded into spherical harmonics up to degree eight [Heiskanen and Moritz 1967].

Gravity anomalies are usually given for surface elements of a certain size so that density values may be computed from (10) for the same surface elements by replacing the integral by a sum. To obtain the density values for the surface elements ΔE_i used in the analysis of the satellite observations, one has to interpolate with weights proportional to the area of the surface elements. Hence,

$$\underline{x}_0 = \underline{H}\ \underline{F}\ \underline{\Delta g}\ , \tag{12}$$

where $\underline{H}$ represents the matrix for the interpolation and $\underline{F}$ the matrix of summation according to (10).

The density values obtained from (12) may be added as additional observations to the observation equations (7) of the satellite solution; that is,

$$\underline{\Delta x} = \underline{x}_0 + \underline{u}\ , \tag{13}$$

where $\underline{u}$ are the residuals. The covariance matrix $\underline{\Sigma}_x$ of these observations is obtained from the covariance matrix $\underline{\Sigma}_{\Delta g}$ of the gravity anomalies by

$$\underline{\Sigma}_x = \underline{H}\ \underline{F}\ \underline{\Sigma}_{\Delta g}\ \underline{F}^T\ \underline{H}^T\ . \tag{14}$$

Hence, the normal equations of a combination of Doppler data and gravity anomalies are given by

$$\begin{vmatrix} \underline{N}_{aa} & \underline{N}_{ab} \\ \underline{N}_{ab}^T & \underline{N}_{bb} + \underline{\Sigma}_x^{-1} \end{vmatrix} \begin{vmatrix} \underline{\Delta x} \\ \underline{\Delta x} \end{vmatrix} = \begin{vmatrix} \underline{f}_a \\ \underline{f}_b + \underline{\Sigma}_x^{-1}\ \underline{x}_0 \end{vmatrix} . \tag{15}$$

The combination described here uses the same size surface elements for the computation of density values from satellite observations and gravity anomalies. If one wishes to derive more information from the gravity anomalies, density values may be computed for smaller surface elements than the ones used in the analysis of satellite observations. To combine both results the approach in Koch [1970] has to be taken.

2.4 Combination with Results of Satellite Triangulation

Certain stations of the Doppler tracking network lie close to stations of the worldwide satellite triangulation, so results from the geometric satellite triangulation can be combined with the dynamic solution. The results of the satellite triangulation are coordinate differences with respect to one point of the satellite triangulation net. In the least-squares adjustment for the satellite triangulation, however, the coordinates themselves, not coordinate differences, are determined. The coordinates of one station must be held fixed when solving the normal equations to avoid a singular system [Schmid 1974a]. The axes of the coordinate system for the satellite triangulation lie parallel to the axes of the coordinate system for the Doppler stations.

Of interest are only those points of the satellite triangulation that lie close to Doppler tracking sites and that are connected by survey ties. To combine the results for these stations, the normal equations (15) are augmented by the normal equations of the satellite triangulation which have been reduced by eliminating the points without connections [Koch and Pope 1972]. The equation to be used is

$$\begin{vmatrix} \underline{N}_{ee} & \underline{N}_{ef} & \underline{0} & \underline{N}_{eb} \\ \underline{N}_{ef}^T & \underline{N}_{ff} & \underline{0} & \underline{N}_{fb} \\ \underline{0} & \underline{0} & \underline{N}_g & \underline{0} \\ \underline{N}_{eb}^T & \underline{N}_{fb}^T & \underline{0} & \underline{N}_{bb} + \underline{\Sigma}_x^{-1} \end{vmatrix} \begin{vmatrix} \underline{\Delta x}_e \\ \underline{\Delta x}_f \\ \underline{\Delta x}_g \\ \underline{\Delta x} \end{vmatrix} = \begin{vmatrix} \underline{f}_e \\ \underline{f}_f \\ \underline{f}_g \\ \underline{f}_b + \underline{\Sigma}_x^{-1} \underline{x}_0 \end{vmatrix} . \tag{16}$$

Here, $\underline{\Delta x}_e$ are the corrections to the preliminary coordinates of the Doppler tracking stations not connected to points of the satellite triangulation, and $\underline{\Delta x}_f$ and $\underline{\Delta x}_g$ the corrections to the coordinates of Doppler stations and stations of the satellite triangulation tied by survey. $\underline{N}_g$ is the matrix of the reduced normals of the satellite triangulation unconstrained by the fixed point of the geometric solution. $\underline{N}_{ee}$, $\underline{N}_{ef}$, and $\underline{N}_{ff}$ are submatrices of $\underline{N}_{aa}$ in (15) and $\underline{N}_{eb}$ and $\underline{N}_{fb}$ of $\underline{N}_{ab}$. To (16) the contribution of the survey ties have to be added. The observation equations for the survey ties are given by (denoting the corrections to the coordinates of the combined solution by primes)

$$|\underline{I}, - \underline{I}| \begin{vmatrix} \underline{\Delta x}_f' \\ \underline{\Delta x}_g' \end{vmatrix} = \underline{c} + \underline{x}_{go} - \underline{x}_{fo} + \underline{w} = \underline{d} + \underline{w} , \tag{17}$$

where $\underline{I}$ is the unit matrix, $\underline{c}$ the coordinate differences computed from the survey ties in the coordinate system of the Doppler stations, $\underline{x}_{fo}$ and $\underline{x}_{go}$ the preliminary coordinates for the Doppler stations and the stations of the satellite triangulation, and $\underline{w}$ the residuals. By means of the covariance matrix $\underline{\Sigma}_c$ of the differences $\underline{c}$, we obtain the normal equations for the combination of the Doppler data, the gravity anomalies, and the results of the geometric satellite triangulation

$$\begin{vmatrix} \underline{N}_{ee} & \underline{N}_{ef} & \underline{0} & \underline{N}_{eb} \\ \underline{N}_{ef}^T & \underline{N}_{ff} + \underline{\Sigma}_c^{-1} & -\underline{\Sigma}_c^{-1} & \underline{N}_{fb} \\ \underline{0} & -\underline{\Sigma}_c^{-1} & \underline{N}_g + \underline{\Sigma}_c^{-1} & \underline{0} \\ \underline{N}_{eb}^T & \underline{N}_{fb}^T & \underline{0} & \underline{N}_{bb} + \underline{\Sigma}_x^{-1} \end{vmatrix} \begin{vmatrix} \underline{\Delta x}_e' \\ \underline{\Delta x}_f' \\ \underline{\Delta x}_g' \\ \underline{\Delta x} \end{vmatrix} = \begin{vmatrix} \underline{f}_e \\ \underline{f}_f + \underline{\Sigma}_c^{-1} \underline{d} \\ \underline{f}_g - \underline{\Sigma}_c^{-1} \underline{d} \\ \underline{f}_b + \underline{\Sigma}_x^{-1} \underline{x}_0 \end{vmatrix} . \tag{18}$$

The results of the geometric satellite triangulation do not contain any information about the mass center of the Earth, while the coordinates of the Doppler sites are given with respect to the mass center. Hence, by means of the combined solution (18) the coordinates $\underline{s}$ of the origin of the coordinate system of the satellite triangulation may be given with respect to the mass center of the Earth. It can be shown from a limiting case of the combined solution that the mass center determination (leaving the internal geometry of the satellite triangulation undisturbed) is given by

$$\underline{s} = \left(\underline{E}^T \underline{\Sigma}_k^{-1} \underline{E}\right)^{-1} \underline{E}^T \underline{\Sigma}_k^{-1} \left(\underline{x}_k - \underline{x}_g\right) \tag{19}$$

with

$$\underline{E}^T = \begin{vmatrix} 100 & 100 & \\ 010 & 010 & \dots \\ 001 & 001 & \end{vmatrix}$$

and

$$\underline{x}_k = \underline{x}_{go} + \underline{\Delta x}'_g, \quad \underline{x}_g = \underline{x}_{go} + \underline{\Delta x}_g,$$

where $\underline{x}_k$ are the coordinates of the stations of the satellite triangulation from the combined solution (18), and $\underline{\Sigma}_k$ their covariance matrix obtained by the inversion of the left side of (18).

$\underline{s}$ may also be derived using the adjustment of observation equations with constraints. In (15) we separate $\underline{\Delta x}$ into the corrections $\underline{\Delta x}_e$ of the stations without survey ties to stations of the satellite triangulation net and the corrections $\underline{\Delta x}_f$ of stations tied by survey. After eliminating $\underline{\Delta x}_e$ and $\underline{\Delta x}$ we obtain

$$\underline{\bar{N}}_{ff} \underline{\Delta x}_f = \underline{\bar{f}}_f \quad , \tag{20}$$

where $\underline{\bar{N}}_{ff}$ denotes the matrix of the reduced normal equations and $\underline{\bar{f}}_f$ the reduced absolute column. On combining (20) with the reduced normal equations of the satellite triangulation, we obtain, with (17),

$$\begin{vmatrix} \underline{\bar{N}}_{ff} + \underline{\Sigma}_c^{-1} & -\underline{\Sigma}_c^{-1} \\ -\underline{\Sigma}_c^{-1} & \underline{N}_g + \underline{\Sigma}_c^{-1} \end{vmatrix} \begin{vmatrix} \underline{\Delta x}'_f \\ \underline{\Delta x}'_g \end{vmatrix} = \begin{vmatrix} \underline{\bar{f}}_f + \underline{\Sigma}_c^{-1} \underline{d} \\ \underline{f}_g - \underline{\Sigma}_c^{-1} \underline{d} \end{vmatrix}, \tag{21}$$

which is equivalent to (18). After eliminating $\Delta x_f'$ we find

$$\left(\underline{N}_g + \underline{\Sigma}_c^{-1} - \underline{\Sigma}_c^{-1}\left(\overline{\underline{N}}_{ff} + \underline{\Sigma}_c^{-1}\right)^{-1} \underline{\Sigma}_c^{-1}\right)\underline{\Delta x}_g' = \underline{f}_g - \underline{\Sigma}_c^{-1}\underline{d} + \underline{\Sigma}_c^{-1}\left(\overline{\underline{N}}_{ff} + \underline{\Sigma}_c^{-1}\right)^{-1}\left(\overline{\underline{f}}_f + \underline{\Sigma}_c^{-1}\underline{d}\right). \quad (22)$$

Since

$$\left(\overline{\underline{N}}_{ff} + \underline{\Sigma}_c^{-1}\right)\left(\overline{\underline{N}}_{ff} + \Sigma_c^{-1}\right)^{-1} = \underline{I} ,$$

it follows that

$$\underline{I} - \underline{\Sigma}_c^{-1}\left(\overline{\underline{N}}_{ff} + \underline{\Sigma}_c^{-1}\right)^{-1} = \overline{\underline{N}}_{ff}\left(\underline{\Sigma}_c \underline{N}_{ff} + \underline{I}\right)^{-1}\underline{\Sigma}_c ,$$

so that we obtain instead of (22)

$$\left(\underline{N}_g + \overline{\underline{N}}_{ff}\left(\underline{\Sigma}_c\overline{\underline{N}}_{ff} + \underline{I}\right)^{-1}\right)\underline{\Delta x}_g' = \underline{f}_g + \left(\overline{\underline{N}}_{ff}\underline{\Sigma}_c + \underline{I}\right)^{-1}\overline{\underline{f}}_f - \overline{\underline{N}}_{ff}\left(\underline{\Sigma}_c\overline{\underline{N}}_{ff} + \underline{I}\right)^{-1}\underline{d} . \quad (23)$$

By setting $\underline{\Sigma}_c = \underline{0}$ in (23) we recognize that the observation equations (17) serve to obtain identical points for the combined solution. Putting

$$\underline{N}_k = \underline{N}_g + \underline{N}_d ,$$

where

$$\underline{N}_d = \overline{\underline{N}}_{ff}\left(\underline{\Sigma}_c\overline{\underline{N}}_{ff} + I\right)^{-1} .$$

Then using (16) and (20), (23) becomes

$$\underline{N}_k\underline{\Delta x}_g' = \underline{N}_g\underline{\Delta x}_g - \underline{N}_d\underline{d} + \underline{\Sigma}_c^{-1}\left(\overline{\underline{N}}_{ff} + \underline{\Sigma}_c^{-1}\right)^{-1}\overline{\underline{N}}_{ff}\underline{\Delta x}_f ;$$

and with

$$\underline{\Sigma}_c^{-1}\left(\overline{\underline{N}}_{ff} + \underline{\Sigma}_c^{-1}\right)^{-1}\overline{\underline{N}}_{ff} = \underline{\Sigma}_c^{-1} - \Sigma_c^{-1}\left(\overline{\underline{N}}_{ff} + \underline{\Sigma}_c^{-1}\right)^{-1}\underline{\Sigma}_c^{-1} = \underline{N}_D$$

we obtain

$$\underline{N}_k\underline{\Delta x}_g' = \underline{N}_g\underline{\Delta x}_g + \underline{N}_d\left(\underline{\Delta x}_f - \underline{d}\right) . \quad (24)$$

From (17) we have

$$\underline{\Delta x}_f - \underline{d} = \underline{\Delta x}_f - \underline{c} - \underline{x}_{go} + \underline{x}_{fo} = \underline{x}_d - \underline{x}_{go} ,$$

where $\underline{x}_d$ denotes the Doppler coordinates plus survey ties, and

$$\underline{x}_d = \underline{x}_{fo} + \underline{\Delta x}_f - \underline{c} \quad .$$

By adding $\underline{N}_k \underline{x}_{go}$ to both sides of (24) we find

$$\underline{N}_k\left(\underline{x}_{go} + \underline{\Delta x}'_g\right) = \underline{N}_g \underline{\Delta x}_g + \underline{N}_d\left(\underline{x}_d - \underline{x}_{go}\right) + \left(\underline{N}_g + \underline{N}_d\right) \underline{x}_{go} \quad .$$

With the coordinates $\underline{x}_k$ of the combined solution and the coordinates $\underline{x}_g$ of the geometric solution

$$\underline{x}_k = \underline{x}_{go} + \underline{\Delta x}'_g , \quad \underline{x}_g = \underline{x}_{go} + \underline{\Delta x}_g$$

we finally obtain

$$\underline{N}_k \underline{x}_k = \underline{N}_g \underline{x}_g + \underline{N}_d \underline{x}_d \quad . \tag{25}$$

Now, for $\underline{E}^T \underline{\Sigma}_k^{-1} \underline{E}$ in (19) we have

$$\underline{E}^T \underline{\Sigma}_k^{-1} \underline{E} = \underline{E}^T\left(\underline{N}_g + \underline{N}_d\right) \underline{E} = \underline{E}^T \underline{N}_d \underline{E} \tag{26}$$

with

$$\underline{N}_g \underline{E} = \underline{0}$$

because solutions of the normal equations of the geometric solution may differ by translations. In addition, we find with (25)

$$\underline{\Sigma}_k^{-1}\left(\underline{x}_k - \underline{x}_g\right) = \underline{N}_g\underline{x}_g + \underline{N}_d\underline{x}_d - \left(\underline{N}_g\underline{x}_g + \underline{N}_d\underline{x}_g\right)$$

$$= \underline{N}_d\left(\underline{x}_d - \underline{x}_g\right).$$

Hence, instead of (19) we finally obtain

$$\underline{s} = \left(\underline{E}^T\underline{N}_d\underline{E}\right)^{-1}\ \underline{E}^T\underline{N}_d\left(\underline{x}_d - \underline{x}_g\right)\ . \tag{27}$$

On the other hand (27) can be derived if, to the reduced normal equations of the dynamic solution with the contribution

$$\underline{N}_d\ \underline{x}_d = \underline{g}_d \tag{28}$$

of the survey ties included, the contribution of the constraints

$$\underline{x}_d - \underline{E}\ \underline{s} = \underline{x}_g$$

is added. The solution is given by

$$\begin{vmatrix} \underline{N}_d & \underline{0} & \underline{I} \\ \underline{0} & \underline{0} & -\underline{E}^T \\ \underline{I} & -\underline{E} & \underline{0} \end{vmatrix} \begin{vmatrix} \underline{x}_d \\ \underline{s} \\ \underline{k} \end{vmatrix} = \begin{vmatrix} \underline{g}_d \\ \underline{0} \\ \underline{x}_g \end{vmatrix}$$

where $\underline{k}$ is the vector of correlates. Rearrangement and elimination leads to

$$\begin{vmatrix} -\underline{N}_d^{-1} & -\underline{E} \\ -\underline{E}^T & \underline{0} \end{vmatrix} \begin{vmatrix} \underline{k} \\ \underline{s} \end{vmatrix} = \begin{vmatrix} \underline{x}_g - \underline{N}_d^{-1}\underline{g}_d \\ \underline{0} \end{vmatrix}$$

so that, with (28) and the elimination of $\underline{k}$, equation (27) is obtained.

3. RESULTS

3.1 Dynamic Solution

The previous solution by Koch and Witte [1971] for the Earth's gravity field represented by a potential of a simple layer was based on Doppler tracking of five satellites. For each satellite, two arcs of data were processed, each arc corresponding to a duration of one week. The inclinations of the five satellites ranged between 40° and 80° so that data from a low-inclination satellite and a polar satellite were missing in the solution.

Subsequently Doppler data from a polar satellite and a low-inclination satellite became available, so a new solution could be prepared. In the old solution [Koch and Witte 1971], 104 density values were determined; this is equivalent to an expansion into spherical harmonics to the 10th degree and order. Judging from the results of the old solution, it appears doubtful that, when only satellite observations are used, a solution can be obtained for the gravity field which is complete in its harmonic coefficients much beyond the 10th degree. This is confirmed by Gaposchkin and Lambeck [1971], Lerch *et al.* [1972], and Yionoulis *et al.* [1972], whose satellite solutions are complete to the 12th order. To avoid ill-conditioned normal equations in his satellite solution, Anderle [1971] introduces for each harmonic coefficient, except $\overline{C}_{2,0}$, an observation of zero with a standard deviation according to the rule-of-thumb of $10^{-5}/n^2$, where n is the degree of the harmonic coefficient. Based on these results, it was decided not to increase the number of unknown density values in order to avoid ill-conditioned normal equations.

Instead of two arcs, three arcs of observations, also of one week duration each, are processed for the new solution. A summary of the observed data is given in table 1. Tracking data from 41 stations have been used. Although all tracking stations are treated as unknown stations, 11 stations lie close to one of the remaining 30 stations so that observation equations such as (17) are introduced by means of the survey ties. The 30 stations with their connected stations and the satellites observed by them are given in table 2.

Table 1.--Observational data

No.	Satellite	Inclin. (deg)	Perigee ht. (km)	Apogee ht. (km)	No. of arcs	No. of passes	No. of Observations
1	1961 αnl TRANSIT 4B	32.4	956	1104	3	389	6,691
2	1965 32A BE-C	41.2	940	1320	3	526	9,680
3	1962 βμl ANNA 1B	50.1	1080	1180	3	823	19,607
4	1965 89A GEOS 1	59.4	1120	2270	3	979	33,193
5	1964 64A BE-B	79.7	880	1080	3	288	5,161
6	1967 48A	89.6	1074	1105	3	926	22,573
7	1968 02A GEOS II	105.8	1080	1570	3	384	11,586
Total					21	4,315	108,491

Table 2.--Tracking stations

Station no.	Latitude (deg)	Longitude (deg)	Location	Observed satellites of table 1
1*	39	283	Maryland	1,2,3,4,6
2*	30	262	Texas	1,3,4
3*	32	253	New Mexico	1,2,3,4,6
6*	51	359	England	3,4
8	-23	314	Brazil	1,3,4
10	19	204	Hawaii	1
11*	15	120	Philippines	1,3,4
12*	-35	139	Australia	1,3,4
13	41	141	Japan	1,3,4
14	61	210	Alaska	3,4,6
15*	-26	28	South Africa	1,3,4
17*	-14	189	Samoa	2,3,4,5,6,7
18	76	291	Greenland	4,5,6,7
19	-78	167	Antarctica	5
100*	22	202	Hawaii	2,3,4,6
200*	34	241	California	3,6
203	38	284	Virginia	7
310	44	292	Maine	6
321	45	267	Minnesota	6
717	- 5	55	Seychelles	4,6
722	- 8	346	Ascension	2,5
723	-12	97	Cocos Islands	5
729	33	343	Madeira	4
738	47	241	Washington	2
809	-46	168	New Zealand	4
811	21	204	Hawaii	4
817	36	60	Iran	7
820	-32	295	Argentina	6,7
822	12	15	Chad	7
837	- 6	324	Brazil	6,7

*Station 1 is connected by survey ties with station 111, 2 with 92, 3 with 103, 6 with 106, 11 with 121, 12 with 112, 15 with 115, 17 with 117, 100 with 340, 200 with 330 and 334.

The harmonic coefficients up to (11,11) of the old solution [Koch and Witte 1971] have been taken to define the known potential U in (2). Furthermore, the coefficients (12,11), (12, 12), (13, 11), (13, 12), (13, 13), (14, 11), (14, 12), (14, 13), (14, 14), (15, 11), (15, 12), (15, 13), (15, 14) from Gaposchkin and Lambeck [1971] are added to the coefficients for the potential U to account for the resonant terms of the satellites being observed. With this force field variances of unit weight considerably smaller than in the old solution were obtained.

The least-squares adjustment (8) of the new solution contains as unknown parameters 104 density values, 123 station coordinates, 126 orbital elements, and 4,315 frequency offsets. The longitude of the station Maryland was held fixed. The variance of unit weight for the adjustment equals 4.06. A value of 1.02 is obtained if the standard deviations of the Doppler data are increased by a factor of 2. Judging from the residuals of the orbit fits, this seems to be justified. By multiplication with the square root of 4.06, the maximum and minimum standard deviations for one coordinate of a tracking station are found to be ± 40 m and ± 1 m respectively. The maximum value belongs to a tracking station that tracks only one satellite (table 2). The determination of the coordinates of these stations can be improved by combination with the results of the worldwide satellite triangulation. The results for station coordinates are, therefore, given in the following section.

Because of the resonant terms in the potential U, the computed density values are transformed by (5) into spherical harmonics to the 15th degree and order. The results are given in table 3. Choosing $kM = 3.986\,013 \times 10^{14}\ m^3\ sec^{-2}$ and a = 6,378,145 m for this analysis, the value of $\overline{C}_{2,0} = -484.1703$ (table 3) corresponds to a flattening of 1/298.256 for the best fitting ellipsoid. The geoid heights of figure 1 refer to this ellipsoid.

If the computed rectangular station coordinates are transformed into ellipsoidal coordinates and the leveling heights as well as the geoid undulations obtained from the coefficients of table 3 are subtracted from the ellipsoidal heights, a new reference ellipsoid may be computed. Applying a least-squares fit, an equatorial radius of a = 6,378,140 m is obtained. A recent determination for kM by Mariner 9 data gives $kM = 3.986\,008 \times 10^{14}\ m^3 sec^{-2}$ [Esposito and Wong 1972]. With this new value a correction da/a = dkM/3kM obtained from Kepler's Third Law can be computed; this yields an equatorial radius of a = 6,378,137 m.

The comparison of the new solution with the combined solution of Gaposchkin and Lambeck [1971], for example, shows good agreement. The rms discrepancy between geoid heights computed at 10° intervals is ± 10.3 m and between common coefficients $\pm 0.11 \times 10^{-6}$. Table 4 compares the zonal harmonics $\overline{C}_{n0}$ of table 3 with the coefficients of Cazenave *et al.* [1972], which are derived by means of the results of Kozai [1969]. Table 5 gives the positive square roots of the mean degree variances σ_n of the coefficients of table 3, which indicate the decay of the harmonic coefficients. $\sigma_d = 10^{-5}/n^2$ gives the values for the decay from the rule-of-thumb, while the last column contains the anomaly degree variances $\sigma^2_{\Delta g,n}$. The values for σ_n and $\sigma^2_{\Delta g,n}$ beyond n = 6 are smaller than the corresponding values of the old solution, thus indicating the improvement over the latter.

3.2 Combined solution

Gravity anomalies given as mean values for 2,592 surface elements, 5° × 5° in size [Koch 1970], were used for a combination with the Doppler observations according to (15). The harmonic coefficients and the geoid undulations coming out of the combination are only

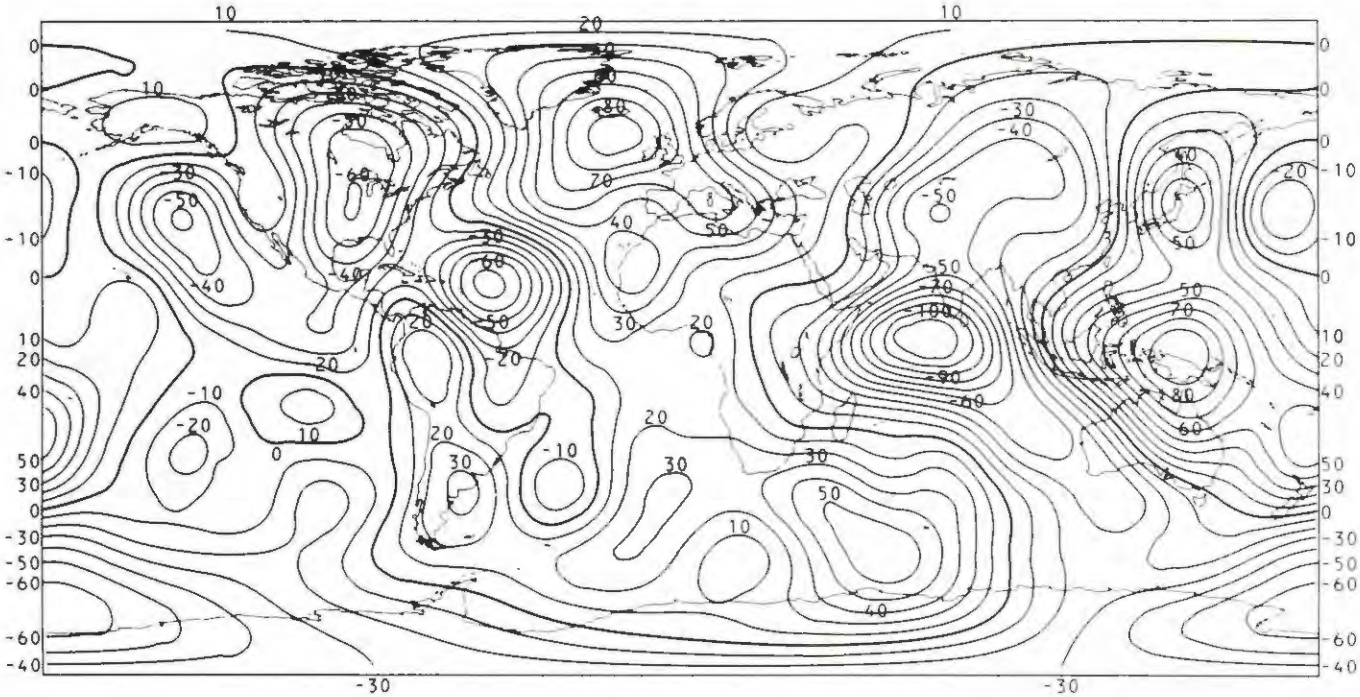

Figure 1.--Geoid heights in meters from the dynamic solution referred to the best fitting ellipsoid of flattening, 1/298.256 .

Table 3.--Potential coefficients to (15, 15) from the dynamic solution

n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$	n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$
2	0	-484.1703		8	0	0.0099	
2	2	2.3885	-1.4587	8	1	.0346	-0.1259
3	0	.9779		8	2	- .0543	.2985
3	1	2.1175	.3323	8	3	.0169	- .2492
3	2	.8831	- .6689	8	4	- .5327	.0058
3	3	.7815	1.2478	8	5	- .1114	.0502
4	0	.5207		8	6	.2020	.2772
4	1	- .5288	- .5458	8	7	.0755	- .1375
4	2	.3362	.6336	8	8	- .2112	.1996
4	3	1.0208	- .2501	9	0	.0036	
4	4	- .2161	.2100	9	1	.1796	- .1346
5	0	.0500		9	2	- .0044	- .0075
5	1	- .0992	- .2133	9	3	- .1110	- .2187
5	2	.7642	- .3978	9	4	.0694	- .0193
5	3	- .6413	- .2601	9	5	.1840	.0880
5	4	- .2857	.1510	9	6	.0442	.4828
5	5	.3307	- .3873	9	7	- .0321	- .4485
6	0	- .1749		9	8	- .1517	.2206
6	1	- .1342	.1126	9	9	.0061	.2248
6	2	.0934	- .4475	10	0	.0045	
6	3	.1730	.0540	10	1	.0343	- .1009
6	4	- .1939	- .5837	10	2	.0062	- .1897
6	5	- .1346	- .3271	10	3	.0029	- .1173
6	6	- .0462	- .1856	10	4	- .0088	- .4659
7	0	.1244		10	5	- .0504	- .0163
7	1	.3142	.1693	10	6	.0636	- .0596
7	2	.3345	.1697	10	7	- .0212	- .1445
7	3	.3074	- .2703	10	8	- .0009	.0617
7	4	- .4039	- .2067	10	9	.1147	.0913
7	5	.1352	.1367	10	10	.1651	- .0072
7	6	- .3580	- .1519				
7	7	- .1106	- .4457				

Table 3.--Continued

n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$	n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$
11	0	-0.0048		13	10	0.0155	0.0314
11	1	.0381	0.0201	13	11	- .0917	.0001
11	2	.0052	- .2002	13	12	- .0038	.0739
11	3	.0234	- .0293	13	13	- .0802	.0385
11	4	.0472	- .0239	14	0	.0022	
11	5	.1415	- .3528	14	1	- .0095	- .0049
11	6	- .0121	.0560	14	2	.0013	- .0250
11	7	.0398	- .2301	14	3	- .0082	.0256
11	8	- .0257	.2560	14	4	.0045	.0008
11	9	- .0035	.0243	14	5	.0234	.0248
11	10	- .0503	- .0143	14	6	- .0015	- .0711
11	11	- .0334	.0015	14	7	.0002	.0025
12	0	- .0038		14	8	- .0220	- .0007
12	1	.0087	.0439	14	9	- .0021	.0569
12	2	- .0591	.1337	14	10	- .0009	.0341
12	3	- .0023	- .0086	14	11	.0274	- .0896
12	4	- .0688	.0011	14	12	.0036	- .0522
12	5	.0167	- .0080	14	13	.0262	.0304
12	6	- .0199	- .1020	14	14	- .0315	.0156
12	7	- .0142	.0020	15	0	- .0023	
12	8	.0055	.1575	15	1	- .0011	.0047
12	9	- .0211	.0511	15	2	.0091	.0023
12	10	.0242	- .0541	15	3	.0062	.0066
12	11	- .0417	- .0337	15	4	.0123	.0031
12	12	- .0581	- .0633	15	5	- .0077	- .0017
13	0	.0047		15	6	.0046	- .0462
13	1	.0269	- .0363	15	7	- .0117	.0166
13	2	- .0183	.0100	15	8	- .0130	- .0719
13	3	- .0141	.0047	15	9	.0191	.0185
13	4	- .0460	.0436	15	10	.0184	.0202
13	5	.0801	- .0775	15	11	.0005	.1002
13	6	.0672	- .1235	15	12	.0093	.0095
13	7	.0586	.0112	15	13	- .0476	- .0337
13	8	.0578	.1198	15	14	.0120	- .0236
13	9	.0419	.0012	15	15	.0083	.0105

Table 4.--Zonal harmonics $\overline{C}_{n0}$

$\overline{C}_{n0}$	Cazenave *et al.* [1972]	King-Hele *et al.* [1969]	Table 3	Table 8
$10^6\overline{C}_{2,0}$	-484.1670	--	-484.1703	-484.1775
$10^6\overline{C}_{3,0}$	.9612	0.9615	.9779	.9770
$10^6\overline{C}_{4,0}$	.5397	--	.5207	.5358
$10^6\overline{C}_{5,0}$	.0681	.0648	.0500	.0505
$10^6\overline{C}_{6,0}$	- .1548	--	- .1749	- .1918
$10^6\overline{C}_{7,0}$	.0942	.1030	.1244	.1244
$10^6\overline{C}_{8,0}$	.0507	--	.0099	.0286
$10^6\overline{C}_{9,0}$	.0271	.0000	.0036	.0036
$10^6\overline{C}_{10,0}$	.0508	--	.0045	- .0059
$10^6\overline{C}_{11,0}$	- .0492	.0000	- .0048	- .0047
$10^6\overline{C}_{12,0}$	.0376	--	- .0038	- .0002
$10^6\overline{C}_{13,0}$	.0389	.0000	.0047	.0047
$10^6\overline{C}_{14,0}$	- .0158	--	.0022	.0013
$10^6\overline{C}_{15,0}$	.0145	.0365	- .0023	- .0023

Table 5.--Degree variances of the dynamic solution

n	$10^6\sigma_n$	$10^6\sigma_d$	$\sigma^2_{\Delta g,n}$ in mgal^2
2	216.53	2.50	7.5
3	1.13	1.11	34.4
4	0.53	0.62	20.4
5	0.39	0.40	25.2
6	0.25	0.28	20.2
7	0.27	0.20	36.5
8	0.20	0.16	32.8
9	0.19	0.12	44.0
10	0.13	0.10	28.2
11	0.12	0.08	30.6
12	0.06	0.07	9.4
13	0.06	0.06	11.7
14	0.03	0.05	4.4
15	0.03	0.04	4.6

slightly changed in comparison to the adjustment (8) of the satellite observations alone. This means that the Doppler observations overwhelm the information coming out of the gravity anomalies. Hence, the normal equations of the Doppler solution are divided by a factor of 4, which can be justified by the results of the orbit fits as explained in the preceding section. The covariance matrix Σ_x of the density values computed from (14) remains unchanged. This combination gives a variance of unit weight close to one.

For the combination with the worldwide satellite triangulation, 17 stations were selectected. Table 6 gives the stations of the satellite triangulation together with the Doppler tracking sites to which they are connected by survey ties. In the least-squares adjustment according to (18), with the station coordinates from Schmid [1974a] and the corresponding reduced normals, the longitude of the station Maryland of the satellite triangulation was held fixed. Hence, the definition of the zero meridian in the combined solution (18) agrees with the definition in the geometric satellite triangulation. If the reduced normal equations of the satellite triangulation are introduced without scaling, the corrections to the coordinates of the satellite triangulation in some cases exceed three standard deviations. Hence, for the combined solution the normal equations from the satellite triangulation are multiplied by a factor of four. This scaling affects all geoid undulations by less than 2 m.

Table 6.--Stations of the satellite triangulation

Station no.	Location	Connected Doppler station no.
G 1	Greenland	18
G 2	Maryland	1 and 203
G 3	Washington	738
G 11	Hawaii	811
G 15	Iran	817
G 19	Argentina	820
G 22	Samoa	17
G 31	New Zealand	809
G 40	Cocos Islands	723
G 53	Antarctica	19
G 55	Ascension	722
G 60	Australia	12
G 64	Chad	822
G 67	Brazil	837
G 68	South Africa	15
G 75	Seychelles	717
G 111	California	200

The variance of unit weight of the combined solution (18) equals 2.06 after introducing the scale factors for the Doppler solution and the satellite triangulation. Using the square root of this variance of unit weight to compute the standard deviations of the station coordinates, values between ± 1 m and ± 5 m for all three coordinates are obtained with the exception of the standard deviations at the following stations:

station 10 : $m_x = \pm 9$ m, $m_y = \pm 11$ m, $m_z = \pm 19$ m

station 31D: $m_x = \pm 43$ m, $m_y = \pm 35$ m, $m_z = \pm 21$ m,

station 321: $m_x = \pm 49$ m, $m_y = \pm 32$ m, $m_z = \pm 24$ m,

station 729: $m_x = \pm 9$ m, $m_y = \pm 7$ m, $m_z = \pm 9$ m.

These Doppler stations (tables 2 and 6) track only one satellite and are not connected to stations of the satellite triangulation. The station coordinates resulting from the combined solution are given in table 7.

The results of the transformation into spherical harmonics of the density values from the combined solution are given in table 8. With the values for $kM = 3.986\,013 \times 10^{14} m^3 sec^{-2}$ and a = 6,378,145 m, the value of $\overline{C}_{20}$ in table 8 corresponds to a flattening of 1/298.254 for the best fitting ellipsoid. The geoid heights of figure 2 are computed with respect to this ellipsoid. The least-squares fit for the equatorial radius described in section 3.1 gives a = 6,378,137 m; and with the new value for kM we finally obtain a = 6,378,134 m.

The comparison of the combined solution with the combined solution of Gaposchkin and Lambeck [1971], as before, yields an rms discrepancy between geoid heights, computed at 10° intervals, of ± 8.7 m and between common coefficients, of $\pm 0.09 \times 10^{-6}$. Table 4 gives a comparison of the zonal harmonics and table 9 contains the degree variances equivalent to the values given in table 5 for the dynamic solution.

By means of (19) the origin of the coordinate system of the satellite triangulation with respect to the mass center of the Earth has been determined as

$$s_x = 0.7 \pm 0.6 \text{ m}, \; s_y = -2.6 \pm 2.6 \text{ m}, \; s_z = -17.0 \pm 4.4 \text{ m} . \tag{29}$$

The variance of unit weight of the adjustment (19) on which the given standard deviations are based equals 10.0. Corresponding coordinate shifts of the origin of the system for the satellite triangulation have also been determined by Schmid [1974b] who obtained

$$s_x = 19.6 \text{ m}, \; s_y = 17.7 \text{ m}, \; s_z = -14.3 \text{ m} . \tag{30}$$

Table 7.--Station Coordinates from the combined solution

Station No.	x (meters)	y (meters)	z (meters)
1	1,122,533	-4,823,078	4,006,460
2	-741,622	-5,462,205	3,198,112
3	-1,555,988	-5,169,357	3,387,512
6	4,005,474	-71,743	4,946,669
8	4,083,879	-4,209,824	-2,499,144
10	-5,499,587	-2,485,280	2,056,918
11	-3,088,094	5,333,115	1,638,807
12	-3,942,247	3,468,827	-3,608,192
13	-3,779,659	3,024,725	4,138,963
14	-2,656,165	-1,544,370	5,570,624
15	5,051,484	2,727,316	-2,773,655
17	-6,100,021	-997,199	-1,568,462
18	539,387	-1,388,383	6,181,056
19	-1,310,709	310,443	-6,213,363
92	-741,631	-5,462,192	3,198,158
100	-5,504,157	-2,224,162	2,325,298
103	-1,556,213	-5,169,445	3,387,254
106	4,005,436	-71,748	4,946,706
111	1,122,634	-4,823,054	4,006,457
112	-3,942,237	3,468,840	-3,608,191
115	5,051,971	2,725,645	-2,774,482
117	-6,100,018	-997,206	-1,568,461
121	-3,088,091	5,333,121	1,638,816
200	-2,572,073	-4,618,422	3,556,655
203	1,261,660	-4,881,254	3,893,544
310	1,708,845	-4,232,150	4,440,298
321	-243,892	-4,532,325	4,466,348
330	-2,568,353	-4,621,095	3,556,620

Table 7.--Continued

Station No.	x (meters)	y (meters)	z (meters)
334	-2,568,383	-4,621,084	3,556,630
340	-5,504,042	-2,224,356	2,325,421
717	3,602,863	5,238,202	-515,924
722	6,118,419	-1,571,567	-878,452
723	-741,986	6,190,781	-1,338,562
729	5,142,508	-1,566,189	3,421,601
738	-2,127,835	-3,785,842	4,656,044
809	-4,313,793	893,023	-4,596,963
811	-5,468,020	-2,381,418	2,253,207
817	2,604,348	4,444,147	3,750,328
820	2,280,562	-4,914,563	-3,355,453
822	6,023,401	1,617,916	1,331,703
837	5,186,364	-3,654,227	-653,047
G 1	546,575	-1,389,990	6,180,230
G 2	1,130,763	-4,830,836	3,994,694
G 3	-2,127,831	-3,785,865	4,656,022
G 11	-5,466,022	-2,404,434	2,242,212
G 15	2,604,356	4,444,151	3,750,312
G 19	2,280,610	-4,914,550	-3,355,434
G 22	-6,099,956	-997,365	-1,568,584
G 31	-4,313,815	891,328	-4,597,260
G 40	-741,966	6,190,783	-1,338,548
G 53	-1,310,839	311,252	-6,213,276
G 55	6,118,341	-1,571,739	-878,613
G 60	-4,751,637	2,792,048	-3,200,161
G 64	6,023,389	1,617,929	1,331,727
G 67	5,186,410	-3,653,937	-654,296
G 68	5,084,827	2,670,328	-2,768,088
G 75	3,602,821	5,238,231	-515,949
G111	-2,448,852	-4,667,992	3,582,746

Table 8.--Potential coefficients to (15, 15) from the combined solution

n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$	n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$
2	0	-484.1775		8	0	0.0286	
2	2	2.3676	-1.4587	8	1	.0439	-0.0530
3	0	0.9770		8	2	.0241	.1705
3	1	2.0565	0.3597	8	3	.0443	- .2739
3	2	.8732	- .6335	8	4	- .4952	.0862
3	3	.8031	1.2784	8	5	- .1602	.0425
4	0	.5358		8	6	.1227	.1788
4	1	- .5432	- .4658	8	7	.0730	- .0769
4	2	.3840	.6289	8	8	- .2353	.1865
4	3	1.0785	- .2652	9	0	.0036	
4	4	- .1983	.2387	9	1	.1747	- .0796
5	0	.0505		9	2	- .0017	.0066
5	1	- .1040	- .1800	9	3	- .0323	- .1695
5	2	.7074	- .3299	9	4	.0958	.0262
5	3	- .5465	- .2671	9	5	.1594	.0783
5	4	- .1919	.1153	9	6	- .0048	.2549
5	5	.3078	- .4167	9	7	- .0428	- .3245
6	0	- .1918		9	8	- .1147	.1034
6	1	- .1411	.0927	9	9	.0293	.1604
6	2	.0161	- .3605	10	0	- .0059	
6	3	.2512	.0751	10	1	.0431	- .0371
6	4	- .1239	- .5546	10	2	- .0297	- .0978
6	5	- .1883	- .3256	10	3	.0133	- .1289
6	6	- .0298	- .1947	10	4	.0012	- .3901
7	0	.1244		10	5	- .0849	.0097
7	1	.2682	.1615	10	6	- .0503	- .1364
7	2	.3871	.1436	10	7	- .0209	- .0467
7	3	.3188	-.2418	10	8	.0449	- .0093
7	4	-.2506	-.1916	10	9	.1014	.0497
7	5	.0953	.0587	10	10	.1938	.0051
7	6	-.3748	-.2191				
7	7	-.1777	-.3537				

Table 8.--Continued

n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$	n	m	$10^6\bar{C}_{nm}$	$10^6\bar{S}_{nm}$
11	0	-0.0047		13	10	-0.0077	0.0245
11	1	.0169	0.0249	13	11	- .1153	- .0029
11	2	- .0194	- .1651	13	12	.0009	.0784
11	3	.0087	- .0319	13	13	- .0801	.0380
11	4	.0455	.0120	14	0	.0013	
11	5	.1273	-.2584	14	1	- .0105	.0048
11	6	- .0319	.0143	14	2	.0038	- .0185
11	7	.0566	- .1135	14	3	- .0176	.0255
11	8	- .0065	.1392	14	4	.0085	- .0016
11	9	.0196	.0149	14	5	.0082	.0354
11	10	- .0594	- .0195	14	6	.0041	- .0362
11	11	- .0424	.0277	14	7	- .0011	- .0248
12	0	- .0002		14	8	- .0189	- .0100
12	1	.0093	.0493	14	9	.0102	.0604
12	2	- .0309	.1248	14	10	- .0187	.0312
12	3	.0024	.0012	14	11	.0233	- .0860
12	4	- .0613	.0390	14	12	.0040	- .0458
12	5	- .0064	.0250	14	13	.0327	.0283
12	6	.0038	- .0597	14	14	- .0342	.0197
12	7	- .0145	.0353	15	0	- .0023	
12	8	.0708	.1063	15	1	- .0044	.0067
12	9	- .0067	.0474	15	2	.0088	.0120
12	10	.0310	- .0563	15	3	- .0034	.0041
12	11	- .0488	.0005	15	4	.0113	.0002
12	12	- .0527	- .0592	15	5	- .0134	- .0020
13	0	.0047		15	6	- .0047	- .0484
13	1	.0234	- .0293	15	7	- .0136	- .0199
13	2	- .0112	.0162	15	8	- .0126	- .0569
13	3	- .0170	- .0067	15	9	.0239	.0176
13	4	- .0175	.0526	15	10	- .0001	.0149
13	5	.0721	- .0249	15	11	.0134	.0963
13	6	.0686	- .0424	15	12	.0131	.0127
13	7	.0539	- .0056	15	13	- .0470	- .0321
13	8	.0652	.0660	15	14	.0133	- .0268
13	9	.0392	- .0467	15	15	.0110	.0050

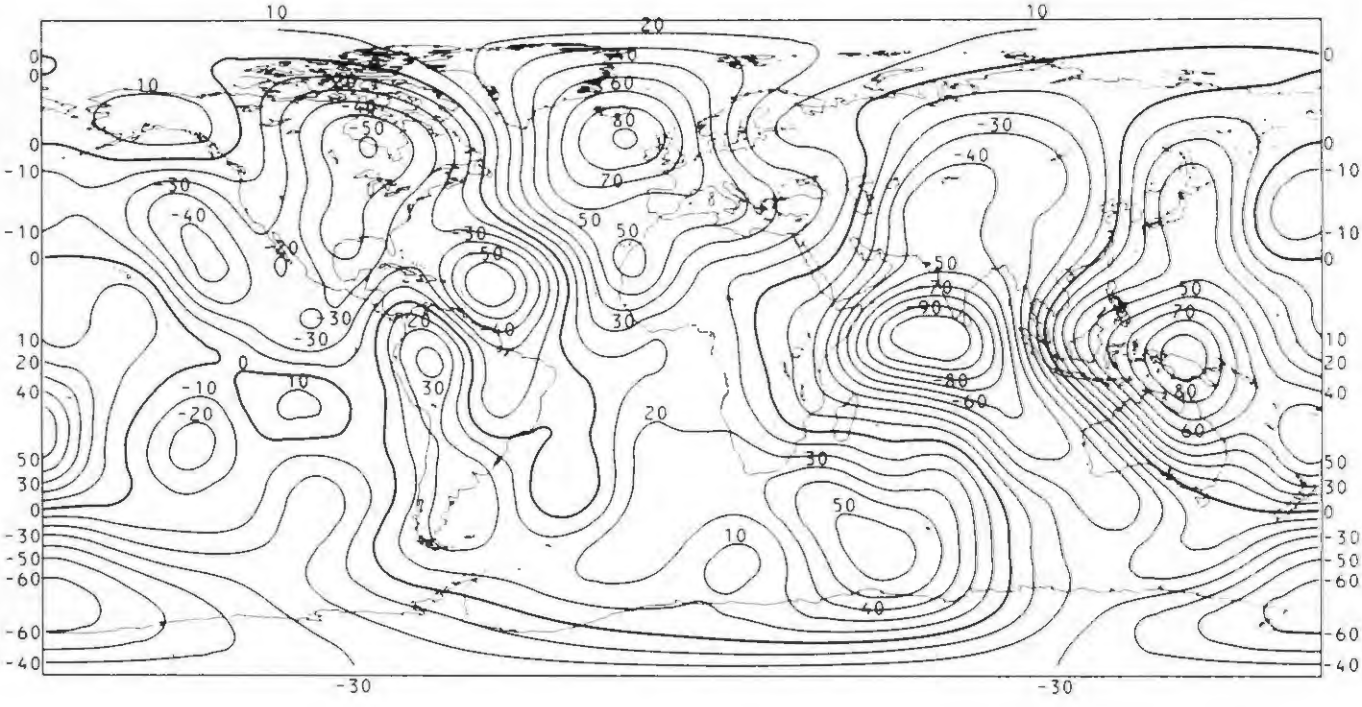

Figure 2.--Geoid heights in meters from the combined solution referred to the best fitting ellipsoid of flattening, 1/298.254.

Table 9.--Degree variances of the combined solution

n	$10^6\sigma_n$	$10^6\sigma_d$	$\sigma^2\Delta g$, n in mgal^2
2	216.53	2.50	7.4
3	1.12	1.11	33.6
4	0.54	0.62	21.2
5	0.35	0.40	20.6
6	0.24	0.28	18.4
7	0.25	0.20	31.1
8	0.18	0.16	25.4
9	0.13	0.12	20.3
10	0.11	0.10	20.7
11	0.08	0.08	15.2
12	0.05	0.07	7.1
13	0.05	0.06	8.4
14	0.03	0.05	3.9
15	0.03	0.04	4.2

The large discrepancy between the two results probably is due to the fact that the latter was based on a comparison with 37 Doppler tracking sites whose coordinates were obtained from a larger amount and a better distribution of Doppler data than the Doppler data of this analysis. However, the discrepancy might be partly explained by the computation according to (19) or (27), where the full covariance matrix of the Doppler solution has been utilized; this solution was not available for the computation of the results of (30). For example, if one takes the arithmetic mean of the differences between the coordinates arising from the geometric solution and the coordinates of table 7 from the combined solution for the 17 stations of the satellite triangulation, one obtains for the origin

$$s_x = 5.9 \text{ m}, \; s_y = 3.3 \text{ m}, \; s_z = -17.0 \text{ m}. \tag{31}$$

These values come closer to the values of (30).

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