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On the Weight Estimation in Leveling

Rockville, Md.
May 1980

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Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1974 reprinted annually, 12 pp (PB265442). National specifications and tables show the closures required and tolerances permitted for first-, second-, and third-order geodetic control surveys. (A single free copy can be obtained, upon request, from the National Geodetic Survey, OA/C18x2, NOS/NOAA, Rockville, MD 20852.)

Specifications To Support Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1975, reprinted annually, 30 pp (PB261037). This publication provides the rationale behind the original publication, "Classification, Standards of Accuracy, ..." cited above. (A single free copy can be obtained, upon request, from the National Geodetic Survey, OA/C18x2, NOS/NOAA, Rockville, MD 20852.)

Proceedings of the Second International Symposium on Problems Related to the Redefinition of North American Geodetic Networks. Sponsored by U.S. Department of Commerce; Department of Energy, Mines and Resources (Canada); and Danish Geodetic Institute; Arlington, Va., 1978, 658 pp. (GPO #003-017-0426-1). Fifty-four papers present the progress of the new adjustment of the North American Datum at midpoint, including reports by participating nations, software descriptions, and theoretical considerations.

NOAA Technical Memorandums, NOS/NGS subseries

NOS NGS-1 Use of climatological and meteorological data in the planning and execution of National Geodetic Survey field operations. Robert J. Leffler, December 1975, 30 pp (PB249677). Availability, pertinence, uses, and procedures for using climatological and meteorological data are discussed as applicable to NGS field operations.

NOS NGS-2 Final report on responses to geodetic data questionnaire. John F. Spencer, Jr., March 1976, 39 pp (PB254641). Responses (20%) to a geodetic data questionnaire, mailed to 36,000 U.S. land surveyors, are analyzed for projecting future geodetic data needs.

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U.S. DEPARTMENT OF COMMERCE
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ON THE WEIGHT ESTIMATION IN LEVELING[1]

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ABSTRACT. This report addresses the problem of linear propagation of statistically dependent errors. It is shown that the laws of propagation of statistically independent and totally statistically dependent (systematic) errors are in the lower and upper bounds of all the possible linear laws. The particular form of propagation laws covering the region between the bounds depends on the covariance function governing the statistical dependence: different covariance functions give rise to different families of law. Working with continuous models of discrete cases and with one parametric nonnegative function, it is possible to arrive at some general conclusions. The concepts are demonstrated for the case of leveling. Two examples of possible covariance models and the corresponding propagation laws are given.

1. INTRODUCTION

When a leveling network is adjusted, the height differences, Δh, pertaining to individual leveling lines (connections between junction points) are given a weight that is inversely proportional to the length L of these lines. This weighting scheme is justified when the height differences δh of the end points of individual segments (connections between bench marks) within each line are statistically independent. This condition has always been taken for granted within geodetic agencies while research groups have expressed doubts, e.g., Lucht (1972), Müller and Schneider (1968), and Remmer (1975). Under this

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condition the variance $\sigma^2_{\Delta h}$ (of Δh) may be evaluated simply as the summation of the "segment" variances, $\sigma_{\delta h_i}$, (belonging to δh_i), e.g.,

$$\sigma^2_{\Delta h} = \sum_i \sigma^2_{\delta h_i} . \tag{1}$$

If all the height differences, δh_i, within a line are leveled to the same accuracy, which we shall further assume to always be the case, it is expedient to standardize all the variances, $\sigma^2_{\Delta h}$, by expressing them in terms of the "variance of a leveled height difference along a unit distance," ($\ell = 1$). Usually this variance is denoted by σ^2_o and is evaluated for a distance of 1 km. Under the assumption of statistical independence we thus have

$$\sigma^2_{\delta h_i} = \sigma^2_o \, \ell_i \tag{2}$$

and eq. (1) becomes

$$\sigma^2_{\Delta h} = \sigma^2_o \sum_i \ell_i = \sigma^2_o \, L. \tag{3}$$

The following formula, which can be found in any surveying textbook, e.g., (Bomford 1971), is sometimes referred to as the "square-root law" because it ascertains that standard deviations propagate according to the <u>square root of the distance</u>:

$$\sigma_{\Delta h} = \sigma_o \sqrt{L} . \tag{4}$$

Let us now put several leveling lines together to make a closed circuit. (See fig. 1.) The sum of the height differences, Δh_i, properly corrected for the effect of actual gravity and other known systematic effects, is expected to be zero. More precisely, denoting the actual circuit misclosure by m, we have

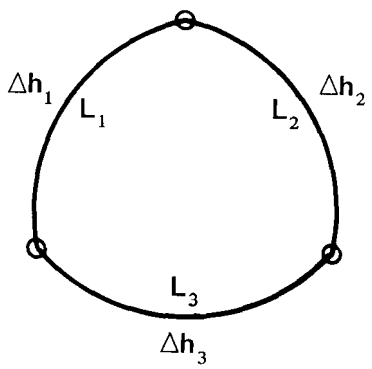

Figure 1.--Leveling circuit.

$$m = \sum_i \Delta h_i \tag{5}$$

and

$$E(m) = 0 \tag{6}$$

where E is the expectation operator if the height differences, Δh_i, are uncorrelated. The variance of m is given by

$$\sigma_m^2 = \sum_i \sigma_{\Delta h_i}^2 = \sigma_o^2 \sum_i L_i \cdot \tag{7}$$

Thus a sample of standardized (actual) circuit misclosures, m*, where

$$m^* = \sum_i \Delta h_i / \sum_i L_i \ , \tag{8}$$

is expected to display a histogram of zero mean and standard deviation σ_o.

For most of the existing national leveling networks the actual standard deviation (of the sample of standardized circuit misclosures) is significantly larger than σ_o, e.g., Lucht (1972: table P-1). This is a well-known fact acknowledged even in specifications for surveys of different orders, e.g., Federal Geodetic Control Committee (1976). There are several possible explanations for this phenomenon.

a. <u>The actually achieved standard deviation for unit length (σ_o) is larger than expected</u>. This explanation may be ruled out because the actually achieved standard deviation is always monitored by the field parties. The basis for the monitoring is repeatability and the achieved repeatability is required to conform to the accuracy level prescribed for the type of survey being carried out.

b. <u>There are unmodeled, or poorly modeled, systematic effects in the leveled height differences</u>. This problem area is being vigorously investigated by scores of researchers. Systematic errors constitute the "upper limit" of statistically dependent errors, as we will see later.

c. <u>The basic assumption of statistical independence of individual δh_i's is not satisfied</u>. This is the area on which we want to focus our attention. The idea is to develop a mathematical apparatus for modeling and handling statistical dependence among leveled height differences.

dependence in a general (as opposed to only a linear) sense. Total statistical dependence here means a deterministic functional relation

$$\delta h_j = f(\delta h_i) \tag{12}$$

between individual quantities δh. Thus, throughout this report we will limit ourselves to <u>positive statistical dependence</u> without a further reminder to the reader. Under these conditions the covariance matrix of totally dependent leveled height differences is

$$\underline{C}_{\delta h} = \sigma_o^2 \begin{bmatrix} 1,1,\ldots,1 \\ 1,1,\ldots,1 \\ \cdot \\ \cdot \\ \cdot \\ 1,1,\ldots,1 \end{bmatrix} . \tag{13}$$

Substituting eq. (13) into (10) gives

$$\sigma_{\Delta h}^2 = \sigma_o^2 \sum_{i=1}^{n} \sum_{j=1}^{n} 1 = \sigma_o^2 \, n^2 . \tag{14}$$

At this stage we must make a subtle, though important point. We note that a similar formulation for the totally independent case yields

$$\sigma_{\Delta h}^2 = \sigma_o^2 \, n . \tag{15}$$

To reconcile this equation with eq. (3) we have to write $\sigma_o^2 \, \ell_o$ instead of just σ_o^2, even though the value of ℓ_o equals 1, to keep track of physical units. When we do this, we get $n \, \ell_o$ instead of n in eq. (15), and because $n \, \ell_o = L$ we get the same result as in eq. (3). Ordinarily, σ_o^2 is thought of as being given in units of height squared per unit of distance (typically in mm^2/km) because of the validity of the square-root law. In the case of total statistical dependence the square root law is evidently no longer valid. It is replaced by the <u>linear law</u> which states: standard deviations propagate linearly with distance. Thus even the physical units of σ_o^2 must change accordingly and, therefore, in eq. (14) we must write $\sigma_o^2 \, \ell^2$ instead of σ_o^2. Equation (14) then changes to

$$\sigma_{\Delta h}^2 = \sigma_o^2 \, \ell^2 \, n^2 = \sigma_o^2 \, (\ell_o n)(\ell_o n) = \sigma_o^2 \, L^2 \tag{16}$$

d. A combination of any of the above is also possible. Separation of purely systematic and statistically dependent effects would require much more effort.

2. LIMITING CASES

In this section we propose to show that all possible cases of positively statistically dependent observations ("positively partially statistically dependent" is the expression preferred by some authors) have a lower and an upper bound. The lower bound is given by the totally dependent case. To discuss these cases let us begin by spelling out the relation between Δh and δh's. We have the obvious equation

$$\Delta h = \sum_i \delta h_i = \underline{u}^T \underline{\delta h} \ . \tag{9}$$

Application of the covariance law yields

$$\sigma^2_{\Delta h} = \underline{u}^T \underline{C}_{\delta h} \underline{u} \tag{10}$$

where $\underline{C}_{\delta h}$ is the covariance matrix of the leveled height differences δh. It is clear that statistically independent δh's will have a diagonal covariance matrix because all the covariances, i.e., the off-diagonal terms, will equal zero. Thus eq. (1) will result.

On the other hand, the totally statistically dependent case is slightly more involved. To keep the concepts as clear as possible let us assume, without any detriment to generality, that all the n sections in the leveling line have the same unit length, $\ell = \ell_o = 1$, and therefore the same variance, $\sigma^2_{\delta h} = \sigma^2_o$. For totally dependent δh's the covariances are then given by

$$\sigma_{ij} = \begin{cases} +\sigma^2_o \\ -\sigma^2_o \end{cases} \tag{11}$$

where the + sign stands for positive dependence and - for negative. Only positive dependence makes sense here; we are talking about total statistical

dependence in a general (as opposed to only a linear) sense. Total statistical dependence here means a deterministic functional relation

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between individual quantities δh. Thus, throughout this report we will limit ourselves to <u>positive statistical dependence</u> without a further reminder to the reader. Under these conditions the covariance matrix of totally dependent leveled height differences is

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$$\sigma_{\Delta h}^2 = \sigma_o^2 \, \ell^2 \, n^2 = \sigma_o^2 \, (\ell_o n)(\ell_o n) = \sigma_o^2 \, L^2 \tag{16}$$

or

$$\sigma_{\Delta h} = \sigma_o L. \tag{17}$$

This is a well-known result which we anticipated all along; it is the formula for the total differential of the (deterministic) summation function describing the propagation of deterministic errors and, it is sometimes used in surveying to portray the behavior of systematic errors (Vaníček 1974). It can also be shown that the same equation holds even for segments of different lengths by realizing that

$$\sigma_{ij} = \sigma_i \sigma_j = \sigma_o \ell_i \sigma_o \ell_j = \sigma_o^2 \ell_i \ell_j \tag{18}$$

and substituting this result into eq. (10). Thus the linear law is completely general for totally statistically dependent cases.

We can now proceed to demonstrate that none of the other cases, i.e., those of partial statistical dependence, can escape from these two bounds (square-root law and linear law). To show this, let us again use the leveled line with segments of unit length. Denoting the normalized (positive) covariance by ρ, we get

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_o^2} \varepsilon \langle 0,1 \rangle . \tag{19}$$

We deliberately call this quantity the <u>normalized covariance</u> to distinguish it from the a posteriori estimate of the same, known as correlation coefficient r. We feel that these two quantities should be carefully distinguished because evaluating the latter requires a postulation of the existing deterministic relation. (See eq. (12).) In practice, this postulation is limited to the linear case, which in fact obscures the true nature of the relationship between ρ and r. We simplify the notation by denoting

$$\forall\, k = |i-j| : \rho_k = \rho_{ij} \tag{20}$$

which we may do because of our assumption of uniform length. We then write the covariance matrix $\underline{C}_{\delta h}$ as

$$\underline{C}_{\delta h} = \sigma_o^2 \begin{bmatrix} 1, \rho_1, \rho_2, \ldots, \rho_{n-1} \\ \rho_1, 1, \rho_1, \ldots, \rho_{n-2} \\ \vdots \\ \rho_{n-1}, \rho_{n-2}, \ldots, \rho_1, 1 \end{bmatrix}. \tag{21}$$

(See Lucht 1972.)

Evaluation of $\sigma^2_{\Delta h}$ from eq. (10) using the above covariance matrix yields

$$\sigma^2_{\Delta h} = \sigma^2_o \left[n + 2 \sum_{i=1}^{n-1} (n-i)\rho_i \right]. \tag{22}$$

(See appendix A, eq. A -3.)

[We note that, indeed, if $\forall$ i: $\rho_i = 1$, then eq. (22) becomes identical with (14), and if $\forall$ i: $\rho_i = 0$, eq. (22) coincides with (15). Note also that in the case investigated by Remmer (1975), $\forall$ i>1: $\rho_i = 0$ yields for large n

$$\sigma^2_{\Delta h} \doteq \sigma^2_o \; n(1+2\rho_1) \; . \tag{23}$$

(See eq. A-8.)]

It is not difficult to see from eq. (22) that

$$\sigma^2_o \; n \leq \sigma^2_{\Delta h} \leq \sigma^2_o \; n^2 \tag{24}$$

because all the ρ's are between 0 and 1. More accurately, we should write

$$\forall \; L \geq 1 \; : \; \sigma^2_o \; L \leq \sigma^2_{\Delta h} \leq \sigma^2_o \; L^2. \tag{25}$$

Thus we have proved that any case of (partial) statistical dependence must lie within the limits of total dependence and total independence. This again is an intuitively pleasing, if not unexpected, result. It shows that larger circuit misclosures than those indicated by the value of σ_o can be regarded as being caused by statistical dependence of the leveled height differences. This statement also covers the presence of systematic errors.

In general, any propagation law for (partially) statistically dependent observations must be depictable by a purely monotonous curve that must belong to the stippled space in figure 2. Space delimited by $\sqrt{L}$ and L must then contain all the families of propagation laws. Within each family the individual members differ only by their degree of statistical dependence.

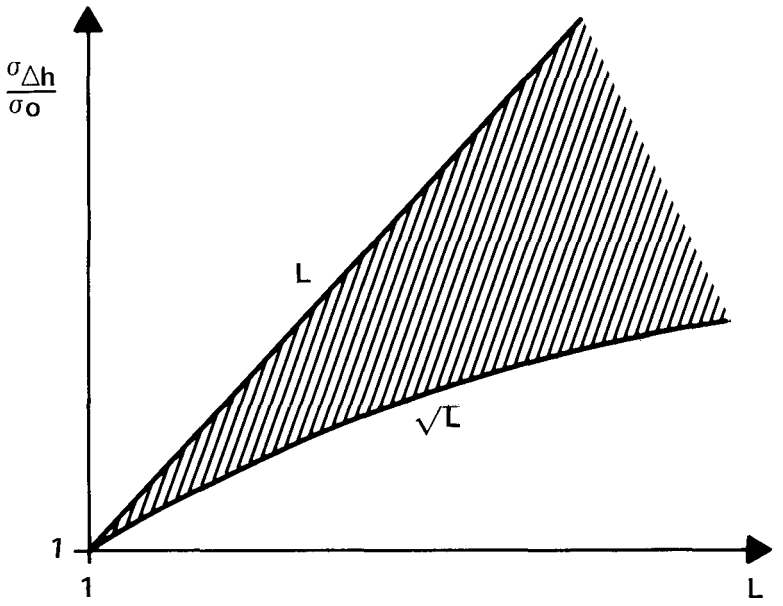

Figure 2.--Family of "laws of propagation of errors."

3. POSTULATION OF COVARIANCE MODEL

To deal with a general case of partially statistically dependent observations, it is more convenient to introduce continuous covariance models. The continuous model of our covariance matrix $\underline{C}_{\delta h}$ is the surface shown on figure 3, defined on the square <0,L> x <0,L>. For any such surface to represent the covariance matrix, it must be symmetrical with respect to $\ell = \ell'$ and linear in any direction parallel to $\ell = \ell'$. The surface is uniquely defined by the equation of a cross section for ℓ' = const. (See the shaded section in fig. 3.) This is the equation of the <u>covariance function</u> $\mathrm{Cov}(\lambda; |\ell-\ell'|)$ for δh.

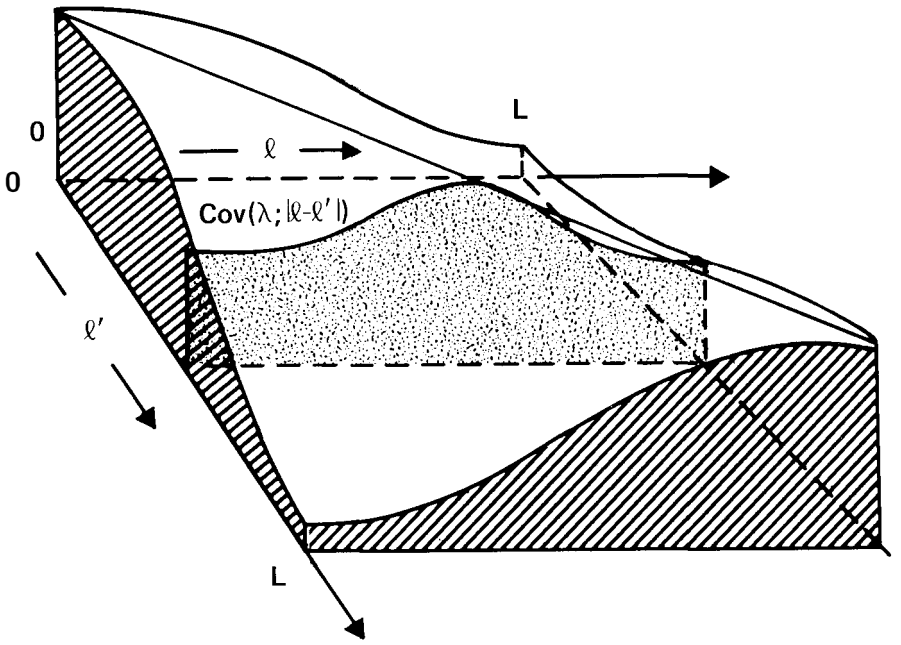

Figure 3.--Continuous model of covariance matrix.

In our investigations let us limit ourselves to only one parametric (nonnegative) covariance functions and let us denote this single parameter by λ. To satisfy the above conditions, any admissible covariance function must be a function of only $|\ell-\ell'|$. Further, the following equation must be satisfied:

$$\mathrm{Cov}(\lambda;0) = 1 \tag{26}$$

and cov $(\eta, \ell-\ell')$ must never increase with increasing $|\ell-\ell'|$ and must be nonnegative. Then, the continuous analog of the general propagation law (eq. 10), is given by the following integral

$$\sigma_{\Delta h}^{2} = \sigma_{o}^{2}(\lambda) \int_{o}^{L} \int_{o}^{L} \mathrm{Cov}(\lambda;|\ell-\ell'|)\,d\ell\, d\ell' \tag{27}$$

where σ_o^2 (λ) is the continuous equivalent of σ_o^2. σ_o^2 (λ) also plays the role of the density distribution for the covariance function. Its introduction is necessary to compensate for the undetermined physical units of σ_o^2. (See sec. 1 .)

Any covariance model must have the following five properties: denoting by λ_{min} the value of λ, for which there is no covariance (statistical independence), and by λ_{max} the value of λ, for which there is total statistical dependence (fig. 4) these are written as

$$\text{(i)} \quad \lim_{\lambda\to\lambda_{min}} \mathrm{Cov}(\lambda;|\ell-\ell'|) = \begin{cases} 1 & \forall\ \ell=\ell' \\ 0 & \forall\ \ell\neq\ell'; \end{cases} \tag{28}$$

$$\text{(ii)} \quad \lim_{\lambda\to\lambda_{max}} \mathrm{Cov}(\lambda;|\ell-\ell'|) = 1 \ \forall\ \ell,\ell' \in \langle 0,L\rangle; \tag{29}$$

$$\text{(iii)} \quad \lim_{\lambda\to\lambda_{min}} \sigma_o^2(\lambda) \int_o^L \int_o^L \mathrm{Cov}(\lambda;|\ell-\ell'|)\,d\ell\,d\ell' = \sigma_o^2 L \quad \forall\ L\geq 1; \tag{30}$$

$$\text{(iv)} \quad \lim_{\lambda\to\lambda_{max}} \sigma_o^2(\lambda) \int_o^L \int_o^L \mathrm{Cov}(\lambda;|\ell-\ell'|)\,d\ell\,d\ell' = \sigma_o^2 L^2 \quad \forall\ L\geq 1; \tag{31}$$

$$\text{(v)} \quad \sigma_o^2(\lambda) \int_o^1 \int_o^1 \mathrm{Cov}(\lambda;|\ell-\ell'|)\,d\ell\,d\ell' = \sigma_o^2 \quad \forall\ \lambda_{min}\leq\lambda\leq\lambda_{max}. \tag{32}$$

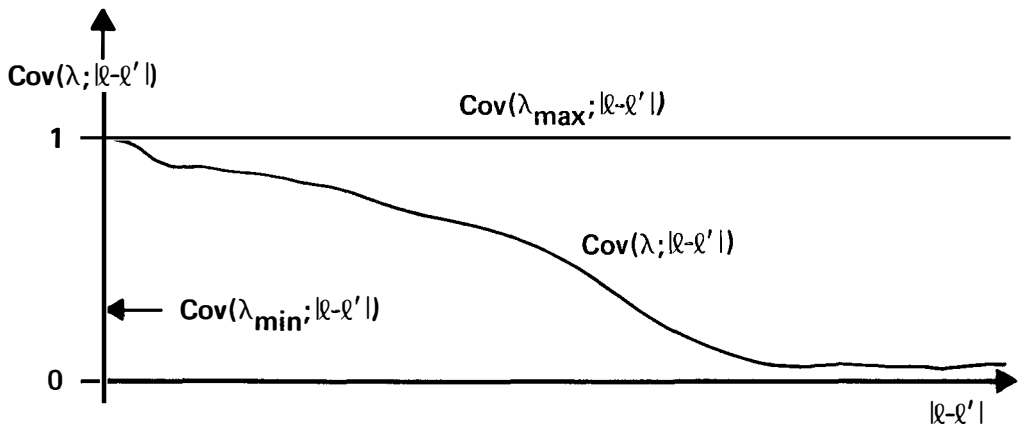

Figure 4.--Extreme cases of covariance functions.

The first four conditions are straightforward and natural. The fifth condition ensures that the covariance density is properly normalized.

There are infinitely many families of one-parametric covariance functions that satisfy the above five conditions. Every one of them contains infinitely many curves (functions with a specific value of λ). It is required that the curves $\sqrt{L}$ and L represent the limits for each family. It is worth noting that the shape of the mathematical formulas for the bounds ($\sigma_{\Delta h}/\sigma_o = \sqrt{L}$ and $\sigma_{\Delta h}/\sigma_o = L$, respectively) has led geodesists to suspect that the <u>power law</u>

$$\forall\ 0.5 \leq \alpha \leq 1 \ :\ \sigma_{\Delta h}/\sigma_o = L^{\alpha} \tag{33}$$

governs the propagation of statistically dependent errors in general. (See for example, Müller and Schneider (1968).) Indeed such a law can be forced on any of the families but there is little merit in doing so. The fact is that generally α is a function of both λ and L. The shape of the function $\sigma_{\Delta h}/\sigma_o$ depends on λ and the selected covariance function.

As examples we have selected two different one-parametric covariance models (families of functions). The first is

$$\mathrm{Cov}(\lambda; |\ell-\ell'|) = \lambda^{|\ell-\ell'|} \quad \forall\ 0 \leq \lambda \leq 1 \tag{34}$$

chosen in accordance with Lucht's (1972) preference. It is shown graphically in figure 5. We have shown (appendix B) that the appropriate density for this family is

$$\sigma_o^2\ (\lambda) = \frac{\sigma_o^2 \quad \ln^2\lambda}{2(\lambda-1-\ln\lambda)} \ . \tag{35}$$

The other covariance model we have selected is

$$\mathrm{Cov}(\lambda; |\ell-\ell'|) = \exp(-(\ell-\ell')^2/\lambda^2) \ \forall\ \lambda \geq 0 , \tag{36}$$

shown in figure 6. Its density is (appendix C)

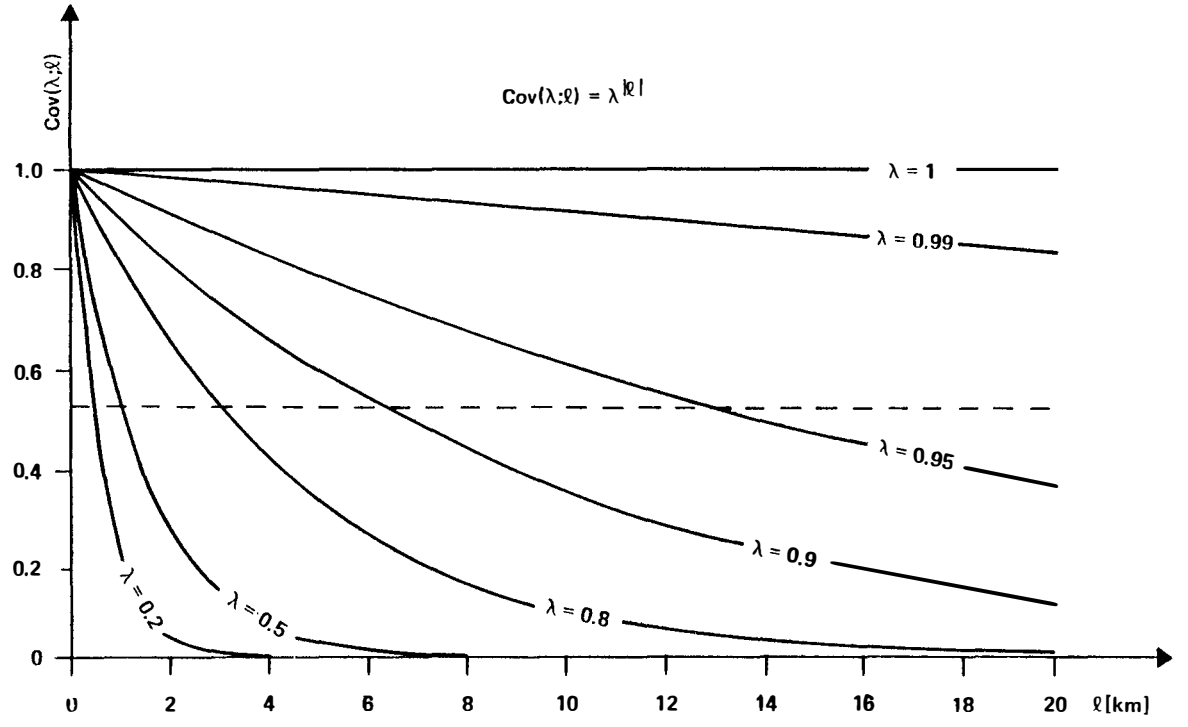

Figure 5.-- First family of covariance functions.

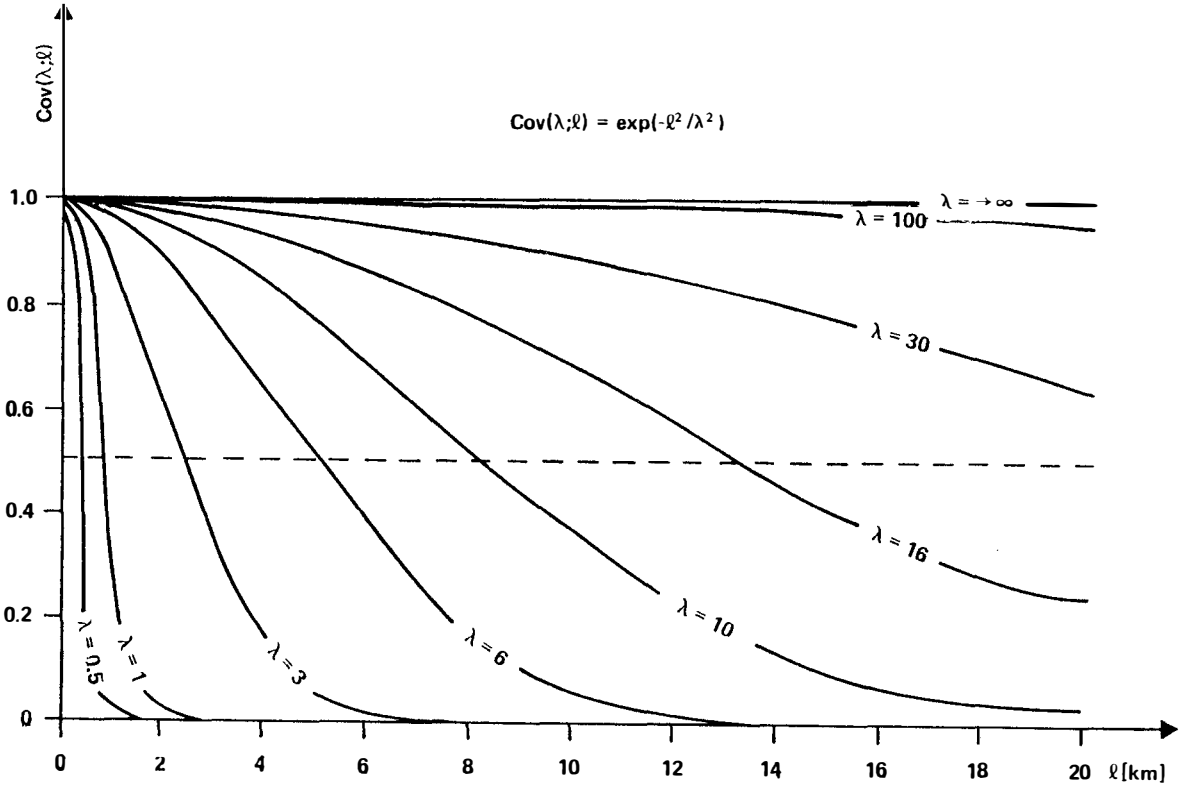

Figure 6.--Second family of covariance functions.

$$\sigma_o^2(\lambda) = \frac{\sigma_o^2}{\lambda^2 \sqrt{\pi} \int_o^{1/\lambda} \operatorname{erf} t \, dt} \tag{37}$$

where (Abramowitz and Stegun 1964)

$$\operatorname{erf} t = \frac{2}{\sqrt{\pi}} \int_o^t \exp(-x^2) dx. \tag{38}$$

In section 4 we will see how these are used to obtain weight matrices for the adjustment of the leveling network.

Ideally, the covariance model should not be chosen arbitrarily, but should be constructed to represent the situation as it exists in leveling. The construction of a model reflecting the actual process of leveling is beyond our scope here. The first steps in this direction were taken by Lucht (1972) and they should be pursued further.

All we can do here is offer a crude, common sense check on the admissibility of an arbitrary covariance model. For any covariance function there exists the <u>radius of statistical semi-dependence</u> ℓ_s. This is the distance at which the normalized covariance drops to 0.5. This distance is related to the parameter λ through the covariance function. In the first case we have

$$\lambda^{|\ell_s|} = 0.5 \tag{39}$$

yielding

$$\ell_s = \ell n \; 0.5/\ell n \; \lambda \doteq -0.693/\ell n \; \lambda \; . \tag{40}$$

In the second case we get

$$\exp(-\ell_s^2/\lambda^2) = 0.5 \tag{41}$$

and thus

$$\ell_s = \lambda \sqrt{\ell n \; 2} \doteq 0.833 \; \lambda \; . \tag{42}$$

When we get a certain value of λ from the given $\sigma_{\Delta h}/\sigma_o$ and L, corresponding to the chosen covariance model, we can convert it easily to ℓ_s. The appropriateness of the value for ℓ_s is more readily assessed using common sense than is the appropriateness of λ.

4. EVALUATION OF WEIGHTS

Assume now that we get an estimate for the average value of $\sigma_{\Delta h}/\sigma_o$ in a certain region which is characterized by some common characteristic (climate, morphology, etc.). Then, after selecting a plausible covariance model we can get both the average parameter of statistical dependence λ and the average radius of statistical semidependence ℓ_s. Once the estimate of the parameter λ is known we can assemble the fully populated covariance matrices for segments within the lines as well as for the lines to be adjusted and invert them to get more realistic weight matrices.

Given the ratio $\sigma_{\Delta h}/\sigma_o$ and its L, we can find the corresponding value of λ for the selected covariance model from either nomograms or tables. Table 1 gives the numerical values of $\ell n(\sigma_{\Delta h}/\sigma_o)$, associated with the first covariance model (eq. 34) constructed from the following formula (appendix B):

$$L^{\alpha} = \sigma_{\Delta h}/\sigma_o = \sqrt{\frac{\lambda^L - 1 - L\ \ell n\lambda}{\lambda - 1 - \ell n\lambda}}\ . \tag{43}$$

The corresponding nomogram is shown in figure 7. The use of either the table or the nomogram should be evident.

Similarly, table 2 lists the numerical values of $\ell n(\sigma_{\Delta h}/\sigma_o)$ for the second covariance model (eq. 36). It is based on the following formula (appendix C):

$$L^{\alpha} = \sigma_{\Delta h}/\sigma_o = \sqrt{\frac{\int_o^{L/\lambda} \mathrm{erf}\ t\ dt}{\int_o^{1/\lambda} \mathrm{erf}\ t\ dt}}\ . \tag{44}$$

Because integration is an awkward operation to perform numerically, we have transformed eq. (44) to a more convenient form (appendix D)

$$L^{\alpha} = \sigma_{\Delta h}/\sigma_o = \sqrt{\frac{\lambda(\exp(-L^2/\lambda^2)-1)+L\ \sqrt{\pi}\ \mathrm{erf}(L/\lambda)}{\lambda(\exp(-1/\lambda^2)-1)+\sqrt{\pi}\ \mathrm{erf}(1/\lambda)}}\ . \tag{45}$$

Table 1.--Relation between L and λ for the first covariance model

L	lambda											
	.030	.100	.300	.500	.600	.700	.800	.900	.950	.970	.990	.995
1.0	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
3.0	.66	.72	.83	.92	.96	.99	1.03	1.06	1.08	1.09	1.10	1.10
10.0	1.30	1.38	1.54	1.71	1.81	1.91	2.03	2.16	2.23	2.26	2.29	2.30
30.0	1.86	1.94	2.12	2.31	2.43	2.57	2.75	3.01	3.18	3.27	3.35	3.38
100.0	2.46	2.55	2.73	2.93	3.06	3.21	3.41	3.74	4.04	4.21	4.45	4.53
300.0	3.01	3.10	3.29	3.49	3.61	3.77	3.98	4.32	4.66	4.89	5.31	5.48
1000.0	3.62	3.70	3.89	4.09	4.22	4.37	4.58	4.94	5.28	5.53	6.05	6.34
3000.0	4.07	4.25	4.44	4.64	4.77	4.92	5.14	5.49	5.84	6.10	6.63	6.96
10000.0	4.77	4.85	5.04	5.24	5.37	5.52	5.74	6.09	6.44	6.70	7.25	7.59

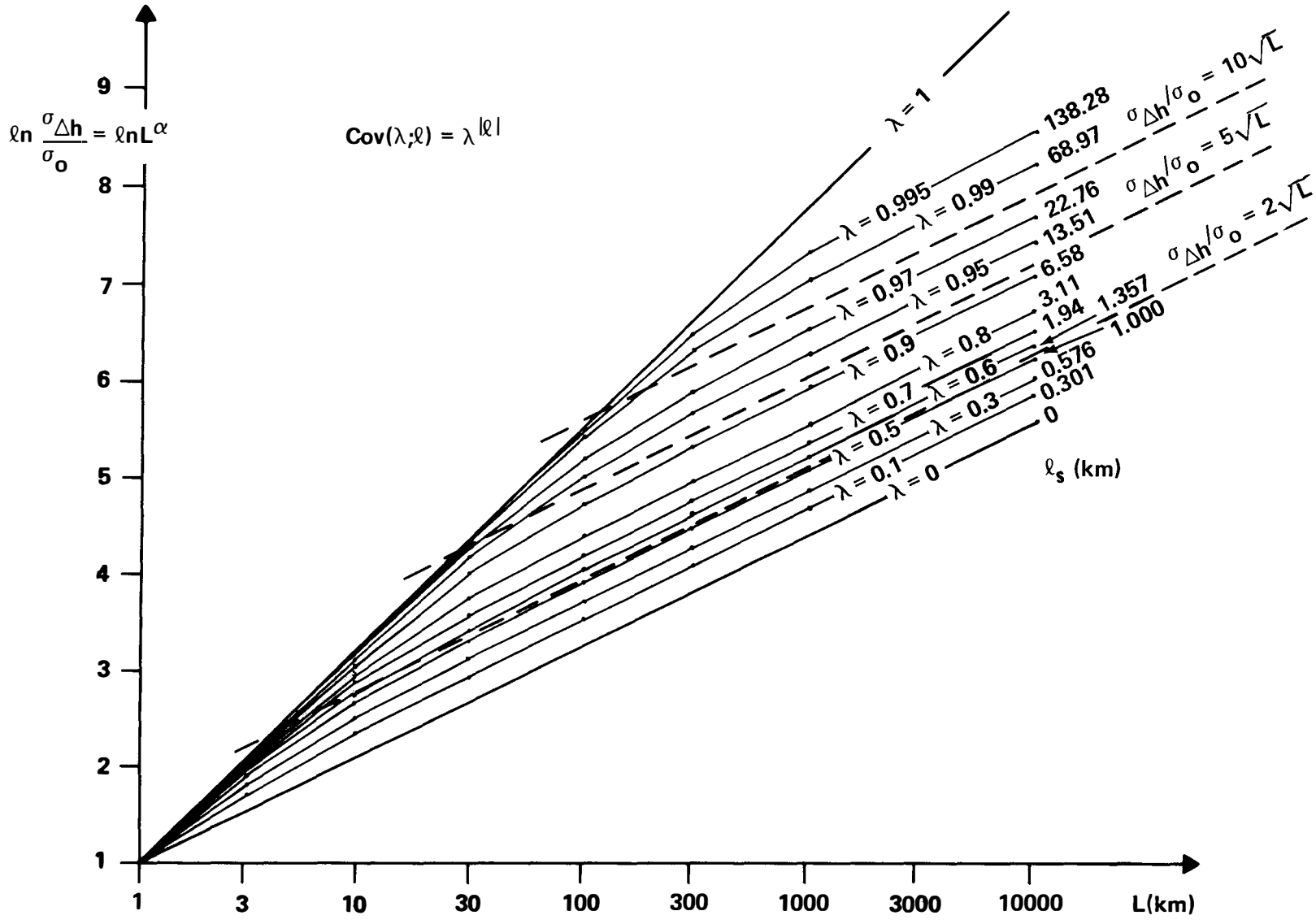

Figure 7.--Family of error propagation laws for the first covariance model.

Table 2.--Relation between L and λ for the second covariance model

L	lambda										
	.000	.100	.500	1.000	2.000	5.000	10.000	30.000	100.000	300.000	1000.00
1.0	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
3.0	.55	.57	.67	.81	.97	1.07	1.09	1.10	1.10	1.10	1.10
10.0	1.15	1.18	1.30	1.48	1.74	2.08	2.23	2.29	2.30	2.30	2.30
30.0	1.70	1.73	1.86	2.05	2.33	2.75	3.03	3.33	3.39	3.40	3.40
100.0	2.30	2.33	2.47	2.66	2.95	3.38	3.71	4.20	4.53	4.60	4.60
300.0	2.85	2.88	3.02	3.21	3.50	3.94	4.28	4.81	5.34	5.63	5.70
1000.0	3.45	3.48	3.62	3.81	4.11	4.55	4.89	5.43	6.01	6.50	6.83
3000.0	4.00	4.03	4.17	4.36	4.66	5.10	5.44	5.99	6.58	7.11	7.64
10000.0	4.61	4.63	4.77	4.97	5.26	5.70	6.04	6.59	7.19	7.73	8.32

Figure 8 shows the corresponding nomogram.

Although the two covariance models exemplified here are of very different character, it is interesting to observe that the shapes of the two nomograms are not all that different. The only visible difference is at the beginning (for low L); after a certain point, the slope of the curves levels off, and from there on the increase of $\sigma_{\Delta h}/\sigma_o$ is the same as that of the statistically independent case. Intuitively, this behavior makes sense.

However, at a closer look one discovers that there is an appreciable difference between the corresponding radii of statistical semidependence. For example, taking L=10000 km and $\sigma_{\Delta h}/\sigma_o$ equal to $2\sqrt{L}$ we get $\ell_s \sim 1$ km versus 2 km for the two covariance models. When $\sigma_{\Delta h}/\sigma_o$ equals $5\sqrt{L}$ we have $\ell_s \sim 8$ km versus 14 km and for $10\sqrt{L}$ we have $\ell_s \sim 30$ versus 55 km. It appears that the ratio $\sigma_{\Delta h}/\sigma_o$ depends more strongly on the degree of statistical dependence than on the selected covariance model.

Once the regional value of the parameter of statistical dependence λ is obtained (for the selected covariance model), it is easy to construct the proper covariance matrix $\underline{C}_{\delta h}$. Denoting

$$\underline{C}_{\delta h} = \begin{bmatrix} \sigma_1^2, \sigma_{12}, \ldots, \sigma_{1n} \\ \sigma_{21}, \sigma_2^2, \ldots, \sigma_{2n} \\ \vdots \\ \sigma_{n1}, \sigma_{n2}, \ldots, \sigma_{nn}^2 \end{bmatrix} \tag{46}$$

we can evaluate the variances from the following formula

$$\sigma_i = (\sigma_o L^{\alpha})^2 \qquad i=1,2,\ldots,n \tag{47}$$

where L^{α} is obtained from the nomogram or table for the appropriate L and λ. The covariances are then calculated as

$$\sigma_{ij} = \sigma_i \sigma_j \ \mathrm{Cov}(\lambda; |\ell_i - \ell_j|) \quad i,j=1,2,\ldots,n. \tag{48}$$

If the lengths of individual leveling sections are approximately uniform, it is advantageous to treat them as equal. This gives us a covariance matrix

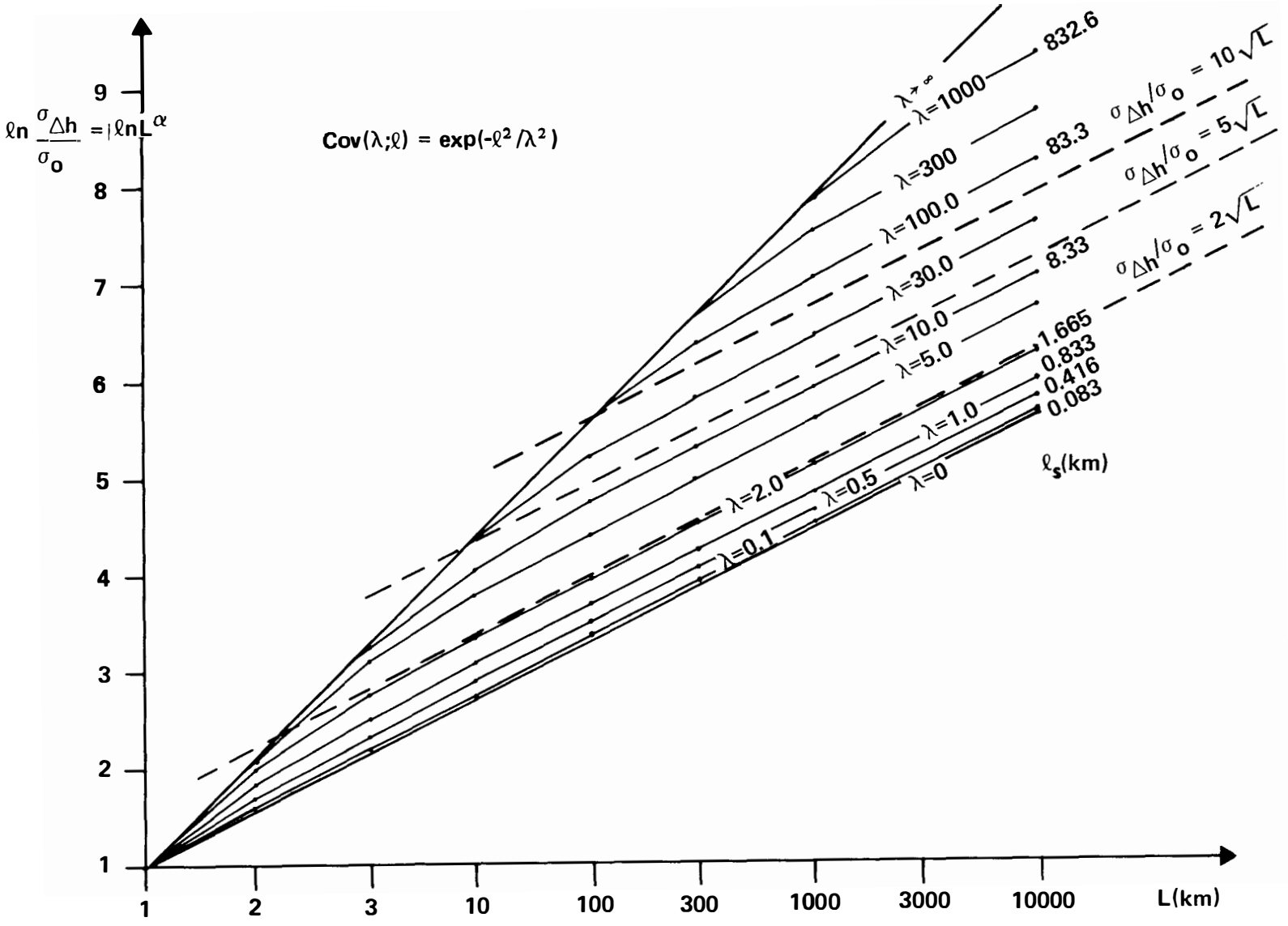

Figure 8.--Family of error propagation laws for the second covariance model.

with Toeplitz's structure (appendix A), which is much more readily invertible than a general covariance matrix. Ways exist to evaluate the elements of the inverse matrix directly.

The final point we wish to make is that the present study should be considered as only a building block in a conglomerate of techniques needed to deal adequately with leveling. One generalization of the technique presented here comes to mind: we started by assuming (in accordance with the custom in geodesy) that the δh's depend on only one parameter, ℓ. Thus the whole technique is geared to quantify the statistical dependence of δh on ℓ. One should be able to look into statistical dependence with respect to various other parameters, e.g., temperature, time, and height itself. The problem with these parameters is that they are not the "natural parameters" along the leveling line, but we cannot see any reason why this difficulty should not be overcome.

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APPENDIX A.--MATRICES WITH TOEPLITZ'S STRUCTURE

A matrix of the following form

$$M = \begin{bmatrix} m_{11}, & m_{12}, \ldots, m_{1n} \\ m_{21}, & m_{22}, \ldots, m_{2n} \\ \vdots & \\ m_{n1}, & m_{n2}, \ldots, m_{nn} \end{bmatrix} = \begin{bmatrix} C_0, & C_1, & C_2, \ldots, C_{n-1} \\ C_1, & C_0, & C_1, \ldots, C_{n-2} \\ \vdots & & \\ C_{n-1}, \ldots, C_2, & C_1, & C_0 \end{bmatrix} \tag{A-1}$$

is said to have Toeplitz's structure. It can be shown that for matrices with this structure, the sum of all its elements can be written as

$$\begin{aligned} S = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} &= nC_o + \sum_{k=2}^{n} [2n - 2(k-1)]\, C_{k-1} \\ &= nC_o + 2 \sum_{k=1}^{n-1} (n-k)\, C_k \,. \end{aligned} \tag{A-2}$$

Our correlation matrix, eq. (21), is a special case of a Toeplitz's matrix so for the summation of its elements we write:

$$S = n + 2 \sum_{k=1}^{n-1} (n-k)\, \rho_k . \tag{A-3}$$

The limiting case of

$$\forall\, i\ \rho_i = 0$$

yields the expected result, i.e.,

$$S = n \,. \tag{A-4}$$

In the other limiting case, i.e.,

$$\forall\, i\ \rho_i = 1$$

eq. (A-3) gives

$$S = n + 2 \sum_{k=1}^{n-1} (n-k) = n + 2n\ (n-1) - 2 \sum_{k=1}^{n-1} k$$

$$= n + 2n(n-1) - 2\ \frac{n(n-1)}{2} = n + n(n-1) = n^2 \ . \tag{A-5}$$

More directly, substitution of 1's for m_{ij}'s in eq. (A-2) yields

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} 1 = \sum_{i=1}^{n} n = n^2 \ . \tag{A-6}$$

If the correlation matrix is tridiagonal, i.e.,

$$\rho_1 = \rho \neq 0, \ \forall\ i \neq 1 \quad \rho_i = 0$$

we get in particular

$$S = n + 2 \sum_{k=1}^{1} (n-k)\ \rho_1 = n + 2\ (n-1)\ \rho \ . \tag{A-7}$$

It is clear that for larger n, eq. (A-7) reduces to

$$S \doteq n\ (1+2\rho) \ . \tag{A-8}$$

APPENDIX B.--FIRST COVARIANCE MODEL

For all $\ell,\ell' \in \langle 0,L\rangle$, let

$$\mathrm{Cov}\,(\lambda;\ |\ell-\ell'|) = \lambda^{|\ell-\ell'|},\quad 0\leq\lambda\leq 1\ . \tag{B-1}$$

Obviously, we have

$$\lim_{\lambda=0} \lambda^{|\ell-\ell'|} = \begin{cases} 0\ , & \forall\ \ell\neq\ell' \\ 1\ , & \forall\ \ell=\ell' \end{cases} \tag{B-2}$$

and

$$\lim_{\lambda=1} \lambda^{|\ell-\ell'|} = 1,\quad \forall\ \ell,\ell' \in \langle 0,L\rangle \tag{B-3}$$

as stipulated by eqs. (28) and (29).

Let us now evaluate the following integral which will be needed in the forthcoming argument:

$$I(L,\lambda) = \int_0^L\int_0^L \lambda^{|\ell-\ell'|}\ d\ell\ d\ell'\ . \tag{B-4}$$

We get

$$I(L,\lambda) = 2\int_0^L\int_{\ell'}^L \lambda^{\ell-\ell'}\ d\ell\ d\ell' = 2\int_0^L \left(\lambda^{-\ell'}\int_{\ell'}^L \lambda^{\ell}\ d\ell\right) d\ell'$$

$$= \frac{2}{\ln\lambda}\int_0^L \lambda^{-\ell'}\left(\lambda^L - \lambda^{\ell'}\right) d\ell' = \frac{2}{\ln\lambda}\left(\lambda^L\int_0^L \lambda^{-\ell'}\ d\ell' - \int_0^L d\ell'\right),$$

This yields

$$I(L,\lambda) = \frac{2}{\ln\lambda}\left[\frac{-\lambda^L}{\ln\lambda}\left(\lambda^{-L}-1\right) - L\right] = \frac{2}{\ln\lambda}\left(\frac{\lambda^L-1}{\ln\lambda} - L\right)\ . \tag{B-5}$$

The last equation suggests that we may want to choose the following covariance density function:

$$\sigma_o^2(\lambda) = \sigma_o^2/I(1,\lambda) = \frac{\sigma_o^2(\ell n\lambda)^2}{2(\lambda-1-\ell n\lambda)} \ . \tag{B-6}$$

Next we must ask if eqs. (30), (31), and (32) are satisfied. The rest of this appendix is devoted to showing that all three equations are indeed satisfied.

1. Equation (32) is the easiest to satisfy. We merely write

$$\begin{aligned} 2\int_o^1 \int_{\ell'}^1 \sigma_o^2(\lambda)\ \lambda^{\ell-\ell'}\ d\ell\ d\ell' &= \sigma_o^2(\lambda)\ I(1,\lambda) \\ &= \sigma_o^2\ I(1,\lambda)/I(1,\lambda) = \sigma_o^2 \end{aligned} \tag{B-7}$$

which is what we have set out to show.

2. Equation (31) is more difficult. First, let us write

$$\lim_{\lambda=1}(\sigma_o^2(\lambda)\ I(L,\lambda)) = \lim_{\lambda=1} \sigma_o^2(\lambda) \ \lim_{\lambda=1} I(L,\lambda) \tag{B-8}$$

and try to evaluate the second term first. Applying the L'Hospital's rule we get

$$\lim_{\lambda=1} I(L,\lambda) = \lim_{\lambda=1} \frac{2(\lambda^L-1-L\ \ell n\lambda)}{(\ell n\lambda)^2} = 2 \lim_{\lambda=1} \frac{L\lambda^{L-1}-L\lambda^{-1}}{2\ \ell n\lambda\cdot\lambda^{-1}} = \lim_{\lambda=1} \frac{L\lambda^L-L}{\ell n\lambda} = L \lim_{\lambda=1} \frac{\lambda^L-1}{\ell n\lambda}.$$

A new application of the L'Hospital's rule yields

$$\lim_{\lambda=1} I(L,\lambda) = L \lim_{\lambda=1} \frac{L\lambda^{L-1}}{\lambda^{-1}} = L^2 \lim_{\lambda=1} \lambda^L = L^2 \ . \tag{B-9}$$

From the above equation we have the following implication:

$$\lim_{\lambda=1} I(L,\lambda) = L^2 \Rightarrow \lim_{\lambda=1} I(1,\lambda) = 1$$

and

$$\lim_{\lambda=1} \sigma_o^2(\lambda) = \sigma_o^2 \ . \tag{B-10}$$

Substituting eqs. (B-9) and (B-10) back into (B-8) we obtain

$$\lim_{\lambda=1} \sigma_o^2(\lambda)\ I(L,\lambda) = \sigma_o^2\ L^2 \tag{B-11}$$

which was to be proven.

3. Finally, we have to prove that even eq. (30) is satisfied, i.e., we have to prove the validity of the following relation:

$$\lim_{\lambda=0} \sigma_o^2(\lambda)\ I(L,\lambda) = \sigma_o^2 L \ . \tag{B-12}$$

We write

$$\begin{aligned} \lim_{\lambda=0} \sigma_o^2(\lambda)\ I(L,\lambda) &= 2 \lim_{\lambda=0} \int_o^L \int_{\ell'}^L \sigma_o^2(\lambda)\ I(L,\lambda)\ d\ell\ d\ell' \\ &= \sigma_o^2 \int_o^L \int_{\ell'}^L \lim_{\lambda=0} \frac{(\ell n\lambda)^2\ \lambda^{\ell-\ell'}}{\lambda-1-\ell n\lambda}\ d\ell\ d\ell' \ . \end{aligned} \tag{B-13}$$

The subintegral function simplifies to

$$\lim_{\lambda=0} \frac{(\ell n\lambda)^2\ \lambda^{\ell-\ell'}}{\lambda-1-\ell n\lambda} = - \lim_{\lambda=0} \ell n\lambda \cdot \lambda^{\ell-\ell'}$$

which is known to equal 0 for $\ell-\ell' > 0$ (e.g., Rektorys (1969). On the other hand, we have

$$\forall\ \ell=\ell' \quad - \lim_{\lambda=0} \ell n\lambda \cdot \lambda^{\ell-\ell'} = - \lim_{\lambda=0} \ell n\lambda \cdot \lim_{\lambda=0} \lambda^0 = - \lim_{\lambda=0} \ell n\lambda \to \infty \ .$$

Thus

$$\lim_{\lambda=0} \frac{(\ell n\lambda)^2 \lambda^{\ell-\ell'}}{\lambda-1-\ell n\lambda} = \begin{cases} 0, & \forall\ \ell>\ell' \\ \infty, & \forall\ \ell=\ell' \end{cases}, \tag{B-14}$$

and the subintegral function constitutes the Dirac function $\delta(\ell-\ell')$. Hence, we can write

$$\begin{aligned} \lim_{\lambda=0} \sigma_o^2(\lambda)\ I(L,\lambda) &= \sigma_o^2 \int_o^L \int_{\ell'}^L \delta(\ell-\ell')\ d\ell\ d\ell' \\ &= \sigma_o^2 \int_o^L d\ell' = \sigma_o^2\ L\ , \end{aligned} \tag{B-15}$$

which was to be shown.

For all $\ell,\ell' \in \langle 0,L\rangle$, let

$$\mathrm{Cov}\,(\lambda;|\ell-\ell'|) = \exp\left(-\frac{(\ell-\ell')^2}{\lambda^2}\right), \quad \lambda \geq 0. \tag{C-1}$$

We have

$$\lim_{\lambda=0} \exp\left(-\frac{(\ell-\ell')^2}{\lambda^2}\right) = \begin{cases} 0, & \forall\ \ell\neq\ell' \\ 1, & \forall\ \ell=\ell' \end{cases}, \tag{C-2}$$

and

$$\lim_{\lambda\to\infty} \exp\left(-\frac{(\ell-\ell')^2}{\lambda^2}\right) = 1 \quad \forall\ \ell,\ell' \in \langle 0,L\rangle \tag{C-3}$$

as required by eqs. (24) and (26).

Let us now denote the double integral over the covariance function by $I(L,\lambda)$ and write

$$I(L,\lambda) = \int_0^L \int_0^L \exp\left(\frac{-(\ell-\ell')^2}{\lambda^2}\right) d\ell\, d\ell' . \tag{C-4}$$

It can be evaluated by using substitution $(\ell-\ell')/\lambda=t$

$$\begin{aligned} I(L,\lambda) &= \int_0^L \left(\lambda \int_{-\frac{\ell'}{\lambda}}^{\frac{L-\ell'}{\lambda}} \exp(-t^2)\, dt\right) d\ell' \\ &= \lambda \int_0^L \left[\int_0^{\ell'/\lambda} \exp(-t^2)\, dt + \int_0^{\frac{L-\ell'}{\lambda}} \exp(-t^2)\, dt\right] . \end{aligned} \tag{C-5}$$

By introducing the "error function" (eq. 28), we can rewrite the previous equation as

$$I(L,\lambda) = \lambda \int_0^L \left(\frac{\sqrt{\pi}}{2} \operatorname{erf} \frac{\ell'}{\lambda} + \frac{\sqrt{\pi}}{2} \operatorname{erf} \frac{L-\ell'}{\lambda} \right) d\ell'$$

$$= \frac{\lambda\sqrt{\pi}}{2} \left(\int_0^L \operatorname{erf} \frac{\ell'}{\lambda} \, d\ell' + \int_0^L \operatorname{erf} \frac{L-\ell'}{\lambda} \, d\ell' \right) .$$

Substitutions $t=\ell'/\lambda$ and $t=(L-\ell')/\lambda$ respectively yield

$$I(L,\lambda) = \frac{\lambda\sqrt{\pi}}{2} \left(\lambda \int_0^{L/\lambda} \operatorname{erf} t \, dt - \lambda \int_{L/\lambda}^0 \operatorname{erf} t \, dt \right)$$

$$= \lambda^2 \sqrt{\pi} \int_0^{L/\lambda} \operatorname{erf} t \, dt . \tag{C-6}$$

Referring to appendix B, let us select the covariance density function as

$$\sigma_o^2(\lambda) = \sigma_o^2/I(1,\lambda) = \frac{\sigma_o^2}{\lambda^2 \sqrt{\pi} \int_0^{1/\lambda} \operatorname{erf} t \, dt} , \tag{C-7}$$

Again we must prove that our covariance function, eq. (C-1), with density given in eq. (C-7), satisfies eqs. (30) to (32).

1. Equation (32) obviously gives

$$\int_0^1 \int_0^1 \sigma_o^2(\lambda) \exp\left(- \frac{(\ell-\ell')^2}{\lambda^2} \right) d\ell \, d\ell' = \sigma_o^2(\lambda) \, I(1,\lambda)$$

$$= \sigma_o^2 \, I(1,\lambda)/I(1,\lambda) = \sigma_o^2 , \tag{C-8}$$

which is the correct answer.

2. Equation (31) can be written as

$$\lim_{\lambda\to\infty} \sigma_o^2(\lambda) \, I(L,\lambda) = \sigma_o^2 \lim_{\lambda\to\infty} \frac{\int_0^{L/\lambda} \operatorname{erf} t \, dt}{\int_0^{1/\lambda} \operatorname{erf} t \, dt} . \tag{C-9}$$

From Abramowitz and Stegun (1964)

$$\operatorname{erf} t = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} (-1)^i \frac{t^{2i+1}}{(2i+1)\,i!} \tag{C-10}$$

we get

$$\int_0^{L/\lambda} \operatorname{erf} t \, dt = \frac{2}{\sqrt{\pi}} \int_0^{L/\lambda} \sum_{i=0}^{\infty} (-1)^i \frac{t^{2i+1}}{(2i+1)\,i!} \, dt$$

$$= \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)\,i!} \int_0^{L/\lambda} t^{2i+1} \, dt = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)(2i+2)\,i!} \left(\frac{L}{\lambda}\right)^{2i+2}$$

$$= \left(\frac{L}{\lambda}\right)^2 \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)(i+1)!} \left(\frac{L}{\lambda}\right)^{2i} .$$

Similarly,

$$\int_0^{1/\lambda} \operatorname{erf} t \, dt = \left(\frac{1}{\lambda}\right)^2 \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)(i+1)!} \left(\frac{1}{\lambda}\right)^{2i}$$

so that

$$\lim_{\lambda\to\infty} \sigma_o^2(\lambda)\, I(L,\lambda) = \sigma_o^2 \lim_{\lambda\to\infty} \frac{\left(\frac{L}{\lambda}\right)^2 \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)(i+1)!} \left(\frac{L}{\lambda}\right)^{2i}}{\left(\frac{1}{\lambda}\right)^2 \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)(i+1)!} \left(\frac{1}{\lambda}\right)^{2i}} \tag{C-11}$$

$$= \sigma_o^2 \lim_{\lambda\to\infty} L^2 \frac{1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i+1)(i+1)!} \left(\frac{L}{\lambda}\right)^{2i}}{1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i+1)(i+1)!} \left(\frac{1}{\lambda}\right)^{2i}} .$$

In the limit, both series in eq. (C-11) go to zero and we get the right answer, i.e.,

$$\lim_{\lambda\to\infty} \sigma_o^2(\lambda)\, I(L,\lambda) = \sigma_o^2 L^2 . \tag{C-12}$$

3. To show that eq. (30) also is satisfied, let us write

$$\lim_{\lambda=0} \sigma_o^2(\lambda)\ I(L,\lambda) = \sigma_o^2 \lim_{\lambda=0} \frac{1}{\lambda^2 \sqrt{\pi}\int_o^{1/\lambda} \text{erf } t\ dt} \times$$

$$\times \lim_{\lambda=0} \int_o^L \int_o^L \exp\left(-\left(\frac{\ell-\ell'}{\lambda}\right)^2\right) d\ell\ d\ell' \ . \tag{C-13}$$

The first term can be written as

$$\lim_{\lambda=0} \frac{1}{\lambda^2 \sqrt{\pi}\int_o^{1/\lambda} \text{erf } t\ dt} = \lim_{\lambda=0} \frac{1}{\sqrt{\pi}\ \lambda-\lambda^2}$$

because

$$\lim_{\lambda=0} \int_o^{1/\lambda} \text{erf } t\ dt = \lim_{\lambda=0} \lambda^{-1} - 1/\sqrt{\pi} \ .$$

(This is a consequence, as we can easily see, of the following equation (Abramowitz and Stegun 1964):

$$\int_o^\infty \text{erfc } t\ dt = \int_o^\infty (1 - \text{erf } t)\ dt = 1/\sqrt{\pi}\Big) \ .$$

Substituting this result back into eq. (C-13), we get

$$\lim_{\lambda=0} \sigma_0^2(\lambda)\ I(L,\lambda) = \sigma_o^2 \int_o^L \int_o^L \lim_{\lambda=0} \frac{\exp\left(-\frac{(\ell-\ell')^2}{\lambda^2}\right)}{\sqrt{\pi}\lambda - \lambda^2} d\ell\ d\ell'$$

$$= \sigma_o^2 \int_o^L \int_o^L \lim_{\lambda=0} \frac{\exp\left(-\frac{(\ell-\ell')^2}{\lambda^2}\right)}{\lambda\sqrt{\pi}} d\ell\ d\ell' \ . \tag{C-14}$$

Here, the function within the limit sign is the Gaussian function for which the limit is known to be the Dirac's function. Thus we have

$$\lim_{\lambda=0} \sigma_o^2(\lambda)\ I(L,\lambda) = \sigma_o^2 \int_o^L \int_o^L \delta(\ell-\ell')\ d\ell\ d\ell' = \sigma_o^2 \int_o^L d\ell' = \sigma_o^2\ L \tag{C-15}$$

which was to be shown.

As an alternative, an integral of the error function may be written as (Abramowitz and Stegun 1964):

$$\int_o^x \text{erf}\ t\ dt = \frac{1}{\sqrt{\pi}}[(y(1,x)-1)] + x \tag{C-16}$$

where y(1,x) is a function for which

$$\lim_{x\to\infty} y(1,x) = 0\ .$$

Then

$$\sigma_o^2 \lim_{\lambda=0} \frac{\int_o^{L/\lambda} \text{erf}\ t\ dt}{\int_o^{1/\lambda} \text{erf}\ t\ dt} = \sigma_o^2 \lim_{\lambda=0} \frac{1/\sqrt{\pi} - L/\lambda}{1/\sqrt{\pi} - 1/\lambda} = \sigma_o^2 \lim_{\lambda=0} \frac{L/\lambda}{1/\lambda} = \sigma_o^2\ L\ , \tag{C-17}$$

which is identical to eq. (C-15).

APPENDIX D.--AN ALTERNATIVE EXPRESSION FOR THE INTEGRAL OF ERF FUNCTION

We want to find an alternative expression for

$$F(x) = \int_0^x \operatorname{erf} t \, dt \tag{D-1}$$

that would be more convenient for numerical evaluation. To carry out the conversion, let us first list all the relations we are going to use (Abramowitz and Stegun 1964):

$$\operatorname{erfc} t = 1 - \operatorname{erf} t \, , \tag{D-2}$$

$$i^n \operatorname{erfc} t = \int_t^\infty i^{n-1} \operatorname{erfc} z \, dz \, , \quad i = \sqrt{-1} \, , \tag{D-3}$$

$$i^n \operatorname{erfc} t = -\frac{t}{n} i^{n-1} \operatorname{erfc} t + \frac{1}{2n} i^{n-2} \operatorname{erfc} t \, , \; i = \sqrt{-1} \, , \tag{D-4}$$

$$i^{-1} \operatorname{erfc} t = \frac{2}{\sqrt{\pi}} \exp(-t^2) \, , \; i = \sqrt{-1} \, . \tag{D-5}$$

Now we can write eq. (D-1) as

$$\int_0^x \operatorname{erf} t \, dt = \int_0^x (1 - \operatorname{erfc} t) \, dt = x - \int_0^x \operatorname{erfc} t \, dt \tag{D-6}$$

$$= x - \left(\int_0^\infty \operatorname{erfc} t \, dt - \int_x^\infty \operatorname{erfc} t \, dt \right) .$$

Let us evaluate the second integral first. For n=1 eq. (D-3) gives

$$\int_t^\infty \operatorname{erfc} z \, dz = i \operatorname{erfc} t, \; i = \sqrt{-1} \, . \tag{D-7}$$

The use of the recurrence formula, eq. (D-4), for n=1 yields

$$i \text{ erfc } t = - t \text{ erfc } t + \frac{1}{2i} \text{ erfc } t \ , \ i = \sqrt{-1} \ ,$$

where the last term can be written as (see eq. D-5)

$$\frac{1}{2i} \text{ erfc } t = \frac{1}{\sqrt{\pi}} \exp\ (-t^2) \ , \ i = \sqrt{-1} \ .$$

Thus

$$\int_t^\infty \text{erfc } z \ dz = - t \text{ erfc } t + \frac{1}{\sqrt{\pi}} \exp\ (-t^2) \ . \qquad \text{(D-8)}$$

For t=0, we get especially:

$$\int_0^\infty \text{erfc } z \ dz = \frac{1}{\sqrt{\pi}} \ .$$

Substituting eqs. (D-8) and (D-9) back to (D-6) we obtain

$$\int_0^x \text{erf } t \ dt = x - \frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \exp(-x^2) - x \text{ erfc } x \ .$$

Finally, we can write

$$\int_0^x \text{erf } t \ dt = x(1-\text{erfc } x) + \pi^{-1/2}(\exp\ (-x^2) -1) \qquad \text{(D-10)}$$

$$= x \text{ erf } x + \pi^{-1/2}(\exp(-x^2) -1).$$

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