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Relativistic Geodesy

Arne Bjerhammar

Rockville, MD
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RELATIVISTIC GEODESY

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ABSTRACT. Theoretical formulas for estimation of relativistic geopotential differences are given. Clock frequencies are found to be inversely proportional to the geopotential differences. Rigorously squared clock frequencies are inversely proportional to the g_{00}-elements of the metric tensor. The missing nondiagonal elements of the metric tensor in the Newtonian geopotential introduce a bias (from angular momentum) which is avoided in the relativistic approach.

A technique for measuring potential differences with high-precision clocks (masers or equivalent) is described. The method can operate over arbitrary terrestrial distances using two clocks. The drift between the clocks is estimated by using closed loops. The clocks continuously operated during the entire measuring interval. No satellite links are necessary, but Very Long Baseline Interferometry (VLBI) and the Global Positioning System (GPS) can be combined with this method.

Methods like VLBI and GPS give excellent geometric coordinates but no dynamic information. However, it is possible to obtain geopotential differences by a relativistic approach. Bjerhammar (1975) and Vermeer (1983) suggested a technique that made use of signals transmitted through the atmosphere (eventually also through the ionosphere). This meant that the method was extremely sensitive to corrections for non-relativistic error sources, since the total time delay in the ionosphere is about 30 m in the daytime for measuring a frequency of 400 MHz. A relativistic approach will of course be very sensitive to any transmission of signals. The time delay decreases with the second power of the frequency, and carrier frequencies above 50 GHz are needed to avoid most of the difficulties with the ionosphere.

No such frequencies are available today for geodetic operations. Furthermore, a system that requires satellite support of its own will probably take a long time to develop. Therefore it is desirable to develop a technique which needs only two clocks and is otherwise self-consistent.

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This study was performed during a 6-month stay in 1984 when the author was a Senior Visiting Scientist at the National Geodetic Survey, under the auspices of the Committee on Geodesy, National Research Council, National Academy of Sciences, Washington, DC.

Tape recorders need not be directly available for the kind of operation in mind. Instead we are going to use the clocks themselves as relativistic time traps in the phase mode. All phase comparisons are made over coaxial (short) links when the clocks are available at the same site. Thus the accuracy is limited only by the relative stability of the two clocks. Consequently, Doppler effects of the Newtonian type have no impact on the measuring procedure. Finally, it is also possible to compensate for drifts between the two clocks.

1. ELEMENTARY RELATIVISTIC APPROACH TO GEODESY

According to the relativistic approach, we obtain the following simple relation between the geopotentials W_P and W_Q and the frequencies f_P and f_Q for two points P and Q:

$$f_P^2/f_Q^2 = (1 - 2W_Q/c^2)/(1 - 2W_P/c^2)$$

where c is the velocity of light. We have a weak gravity field which means the following approximation may be used:

$$W_P - W_Q = (f_P - f_Q)c^2 f^{-1}$$

where $f = (f_P + f_Q)/2$. With this approach we can determine geopotential differences after measuring the frequency differences at the two points in question.

For a more rigorous approach see section 2.

Somewhat conflicting statements concerning the absolute stability of atomic clocks can be found in the literature. Reinhardt et al. (1983) found a relative frequency drift of 10^{-15}/days for two hydrogen masers, NP-2 and NX-3, in a study covering 3 days. These clocks are probably among the best on the market today, and have been applied in the National Aeronautics and Space Administration (NASA) crustal motion VLBI program. We consider these accuracies sufficient for interesting geodetic application when using a measuring technique that properly takes care of drift.

The difference in "proper time" between two stations on the ground is extremely difficult to observe over large distances. It is therefore important to develop a measuring technique that takes care of all systematic errors to the highest possible degree. We consider the time needed for a measurement as being of minor importance in this context. Instead we require that the resulting accuracy should always improve when increasing the number of complete measuring steps. It is understood that the measuring procedure can be used over a time span up to a year. This should be acceptable for measuring potential differences over entire continents for accuracies corresponding to the decimeter level or less.

We compose the following definitions:

<u>Universal time</u> is <u>invariant</u> with respect to potential and velocity (coordinate time).

<u>Proper time</u> is a function of potential and velocity. See Landau and Lifshitz (1974), Kilmeister (1973).

2. THEORETICAL CONSIDERATIONS

A description of a process that takes place in nature requires a system of reference. It is necessary to have a coordinate system for indicating a particle in space as well as a clock. In an inertial reference system, a freely moving body (not exposed to external forces) moves with constant velocity. In classical mechanics, the properties of time are independent of the system of reference. There is one and only one time for all reference frames. Consequently, if two phenomena are simultaneous for two observers, then they are simultaneous for all observers. Futhermore, the time interval between two events must be equal for all observers and all reference systems.

The idea of absolute time is rejected in Einstein's (1916) theory of relativity.

In inertial reference systems, the relativistic approach makes use of an infinitesimal invariant interval, which is defined in a four-dimensional coordinate system

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

where c is the velocity of light, t the time coordinate, and x, y, and z position coordinates. If t_0 is proper time, then

$$ds^2 = c^2dt_0{}^2 .$$

In general, one can express this interval ds in tensor notation

$$ds^2 = g_{ij}dx_idx_j. \tag{2}$$

Here g_{ij} is a matrix with special properties that justify the name "tensor". In our application, it is sufficient to know that this matrix is symmetric with the following elements:

$$g_{ij} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} . \tag{3}$$

If

$$g_{00} = 1$$

$$g_{ii} = -1 \text{ for } i \neq 0$$

$$g_{ij} = 0 \text{ for } i \neq j$$

then this four-dimensional coordinate system is called Galilean.

The tensor g_{ij} is called the metric tensor. The inverse of the metric tensor is denoted by g^{ij}.

Schwarzschild (1916) computed the metric tensor for a centrally symmetric gravitational field with the coordinates $x_0 = ct$, $x_1 = r$, $x_2 = \phi$, and $x_3 = \Lambda$

$$g_{00} = 1 - 2\,V/c^2$$
$$g_{11} = -1/(1 - 2V/c^2)$$
$$g_{22} = -r^2$$
$$g_{33} = -r^2\cos^2\phi \qquad \text{and} \qquad g_{ij} = 0,\ i \neq j$$

where ϕ is latitude, Λ longitude, r geocentric distance, and V gravitational potential.

Kerr (1963) included the rotation of the actual body and obtained for a stationary time-independent centrally symmetric gravity field

$$g_{00} = 1 - 2\,V/c^2$$
$$g_{11} = -1/(1 - 2V/c^2)$$
$$g_{22} = -r^2$$
$$g_{33} = -r^2\cos^2\phi$$
$$g_{03} = c^{-3}G\,Mr^{-1}\cos^2\phi \qquad \text{and remaining} \quad g_{ij} = 0$$

where G is the Newtonian constant and M angular momentum with the approximation

$$M = (2\,mR^2\omega)/5 \quad \text{(for the Equator)}$$

with m mass, R radius, and ω angular velocity for the body in question.

This solution holds for an external observer (outside the Earth). We note that our new metric tensor includes a nonsymmetric term. The Earth rotates with a rather low velocity; hence the correction will be small.

The formulas can be transformed for an observer rotating with the body. We introduce new coordinates and ignore the relativistic length correction of g_{11}

$$\lambda = \Lambda - \omega t,\ d\Lambda = d\lambda + \omega dt,\ d\Lambda^2 = d\lambda^2 + \omega^2 dt^2 + 2\,\omega d\lambda dt$$

where λ is the geodetic longitude. Our study will be related first to two stationary clocks at different positions on the Earth.

The "Einstein interval" ds is now defined by new g-values and we obtain

$$\begin{aligned} ds^2 = {} & (1 - 2V/c^2 - r^2\omega^2\cos^2\phi/c^2 + 2\,GM\omega\cos^2\phi/rc^4)c^2dt^2 - \\ & -dr^2 - \\ & -r^2d\phi^2 - \\ & -r^2\cos^2\phi d\lambda^2 - \\ & -(2r^2\omega\cos^2\phi - 2\,GM\cos^2\phi/rc^2)dtd\lambda\ . \end{aligned} \tag{4}$$

Proper time is apparently a function of the Newtonian potential as well as the kinetic potential. Furthermore, there is also a small correction for the angular momentum. The magnitude of these quantities will be estimated for the Earth.

$$2Vc^{-2} = 2\times 6.67\times 10^{-11}\times 5.98\times 10^{24}/(6.4\times 10^{6}\times 9\times 10^{16}) = 1.38\times 10^{-9}$$

$$r^2\omega^2\cos^2\phi/c^2 \quad \text{(at Equator)} = 2.40\times 10^{-12}$$

$$2\ GM\omega\ \cos^2\phi/rc^4 \quad \text{(at Equator)} = 1.3\times 10^{-21}$$

The ratio between the contribution from the Newtonian potential and the angular momentum is about 10^{12}. This means that the latter can be disregarded for most studies. For further details, see Landau and Lifshitz (1974: p. 323).

Two clocks at different positions 1 and 2 will generate the two frequencies f_1 and f_2. We put $W^* = V + (1/2)r^2\omega^2\cos^2\phi - GM\omega\ \cos^2\phi/rc^2$ and obtain

$$f_1^2/f_2^2 = (1 - 2W_2^*/c^2)/(1 - 2W_1^*/c^2) \tag{5}$$

where W_1 and W_2 are the geopotentials at 1 and 2.

With all mass concentrated inside a sphere of radius $2\ Gmc^{-2}$ (0.9 cm for the Earth) we obtain a "black hole." For zero angular momentum, the Kerr metric is identical to the Schwarzschild metric. For zero mass, the Schwarzschild metric becomes the Galilean metric

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

For the case with moving clocks we introduce the two vectors X and Z

$$X = r\begin{vmatrix} \cos\phi & \cos\lambda \\ \cos\phi & \sin\lambda \\ \sin\phi & \end{vmatrix} \qquad Z = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix},$$

where X is a position vector for a point on the Earth and Z the unit vector for the polar axis. Furthermore, r represents the geocentric distance of the actual point. The Einstein interval ds is now interpreted as a vector and the kinetic contribution is separated as

$$ds = dX + \omega(Z\times X)dt\ , \tag{6}$$

and

$$ds^2 = c^2dt_0^2 = dX^2 + \omega^2(Z\times X)^2dt^2 + 2\ \omega(Z\times X)dX\ dt\ . \tag{7}$$

Vector multiplication is denoted by (×). Proper time at an arbitrary point on the Earth is represented by t_0, and t represents universal time. Neglecting c^{-4} terms, we rewrite eq. (4)

$$dt_0^2 = (1 - 2W^*/c^2)dt^2 - 2\ \omega Z(X\times dX)dt/c^2 - (dX)^2/c^2\ . \tag{8}$$

We have a weak gravity field and it will generally be justified to make use of the approximation (neglecting c^{-4} -terms and still smaller terms)

$$dt_0 = (1 - W^*/c^2 - (dX/dt)^2/2c^2)dt - \omega Z(X \times dX)/c^2 \tag{9}$$

where dX/dt represents the velocity of the moving clock relative to the Earth. This formula allows us to correct for the difference in movements between two clocks used for a determination of potential differences. For stationary clocks, dX/dt is zero and we can use our original formula. For small velocities of a clock, we can write

$$dt = (1 + W^*/c^2 + (dX/dt)^2/2c^2)dt_0 + \omega Z(X \times dX)/c^2. \tag{10}$$

Any movement of a clock will have a velocity that is small compared to the velocity of light. The difference in universal time is therefore obtained by the integration

$$t_2 - t_1 = \int_{t_{01}}^{t_{02}} (1 + W^*/c^2 + (dX/dt)^2/2c^2)dt_0 + \omega c^{-2}\int_{X_1}^{X_2} Z(X \times dX) \ . \tag{11}$$

If the clock is moved along sea level at very low speed then the first integral is of no interest. The geopotential here should be a constant. The second term will be evaluated for a constant latitude. Then

$$dX = (Z \times X)d\lambda \tag{12}$$

and

$$Z(X \times dX) = Z[X \times (Z \times X)]d\lambda \tag{13}$$

$$Z \times X = r \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \times \begin{vmatrix} \cos\phi \ \cos\lambda \\ \cos\phi \ \sin\lambda \\ \sin\phi \end{vmatrix} = r \begin{vmatrix} -\cos\phi \ \sin\lambda \\ \cos\phi \ \cos\lambda \\ 0 \end{vmatrix} . \tag{14}$$

Furthermore,

$$Z\ [X \times (Z \times X)] = r^2 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} . \begin{vmatrix} \cos\phi \ \cos\lambda \\ \cos\phi \ \sin\lambda \\ \sin\phi \end{vmatrix} \times \begin{vmatrix} -\cos\phi \ \sin\lambda \\ \cos\phi \ \cos\lambda \\ 0 \end{vmatrix} = r^2\cos^2\phi. \tag{15}$$

The final formula is then (with longitude counted positive east)

$$t_2 - t_1 = \omega r^2 c^{-2} \cos\phi^2 \int_0^{\lambda} d\lambda \ . \tag{16}$$

This formula gives +207 ns (nanoseconds) for one circumnavigation at the Equator. However, the error is eliminated if two clocks are compared after making identical trips. This procedure is used in the relativistic time trap described above.

Sometimes a transmission of signals is included for a comparison of two clocks. This case corresponds to eq. (10) with $dt_0 = 0$

$$dt = |dX|/c + \omega Z(X \times dX)/c^2. \tag{17}$$

The total time delay is computed for a case with transmission along a straight line $X = X_1 + y(X_2 - X_1)$ with $dX = dy(X_2 - X_1)$ and $0 \leq y \leq 1$

$$t_2 - t_1 = c^{-1}\int_{X_1}^{X_2} |dX| + \omega Z(X_1 \times X_2)/c^2. \tag{18}$$

The first term of the right-hand side represents the time for transmission between the two points, including delay in atmosphere and ionosphere. The second term is the relativistic correction which should be positive for X_2 east of X_1.

This correction is evaluated for two points with $r_1 = r_2 = r$

$$\omega Z\ (X_1 \times X_2)/c^2 = \omega r^2 c^{-2}(\cos\phi_1\cos\lambda_1\cos\phi_2\sin\lambda_2 - \cos\phi_1\sin\lambda_1\cos\phi_2\cos\lambda_2)\ . \tag{19}$$

We transcribe this expression

$$\omega Z(X_1 \times X_2)/c^2 = \omega r^2 c^{-2}(\cos\phi_1\cos\phi_2\sin(\lambda_2-\lambda_1))\ . \tag{20}$$

This delay is about 32.8 ns for a separation of 90° on the Equator.

In atomic clocks, there is a relativistic frequency change for an atom with the frequency f_0. The velocity $dX/dt = v$ gives the distorted frequency f_v

$$f_v^2/f_0^2 = 1 - v^2/c^2\ . \tag{21}$$

The kinetic energy of an atom must correspond to its absolute temperature T according to the relation (with 3 degrees of freedom for the atom)

$$mv^2/2 = 3\ kT/2 \tag{22}$$

where k is the Boltzmann constant and m the mass of the atom. The frequency change increases with T. This secondary Doppler effect can, for example, be reduced by laser cooling down to less than 1°K. (See sec. 5.) For further details, see Misner et al. (1973) and Murray (1983).

We also note that the secondary Doppler effect can be reduced by selecting atoms with large mass.

2.1 The Relativistic Geoid

The (covariant) elements g_{ij} of the metric tensor are defined by the Einstein equations (Misner et al. 1973). In vacuo (outside a body), these equations can be written

$$R_{ij} = 0 \tag{23}$$

where

$$R_{ij} = \delta\Gamma^h_{ij}/\delta x^h - \delta\Gamma^h_{ih}/\delta x^j + \Gamma^h_{ij}\Gamma^m_{hm} - \Gamma^m_{ih}\Gamma^h_{jm}\ . \quad \text{(Ricci tensor)} \tag{24}$$

We have used the Chirstoffel symbols

$$\Gamma^h_{ij} = g^{hk}(\delta g_{kj}/\delta x^i + \delta g_{ki}/\delta x^j - \delta g_{ij}/\delta x^k)/2 \quad . \tag{25}$$

The metric tensor is a nonsingular symmetric matrix in our application. The elements of the inverse metric tensor are denoted by superscripts. The inverse relation is given by

$$g_{ik}g^{kh} = \delta^h_i \ (\delta = 1 \text{ for } i = j; \text{ otherwise } \delta = 0) \ .$$

See also Hotine (1969) for the tensor definition.

An equipotential surface has to satisfy the condition

$$g_{oo} = \text{Constant}$$

or

$$g_{oo} = (1 - 2V/c^2 - r^2\omega^2\cos^2\phi/c^2 + 2\ GM\omega\cos^2\phi/rc^4) = \text{constant} \tag{26}$$

where V is the mass potential (including extraterrestrial bodies). The Newtonian geopotential is obtained by adding the centrifugal potential ($r^2\ \omega^2\ \cos^2\phi/2$). The relativistic geopotential is finally obtained after correcting for angular momentum ($GM\omega\ \cos^2\phi/rc^2$). This means that there is a minor bias in the Newtonian geopotential. It will be harmless for normal geodetic operations. (See eq. 4.)

The classical definition of the geoid was given as the equipotential surface nearest to mean sea level (Hotine 1969: p. 200).

Molodensky (1948) found that there was no way to determine the geoid under the continents. He introduced the concept "quasigeoid," which made use of the disturbance potentials from the physical surface instead of the true values inside the Earth. However, no proof was given for the existence and the uniqueness of the solution of the actual free boundary value problem.

Hörmander (1975) presented proof of the existence and uniqueness of a solution of the free boundary value problem. The solution was valid for the nonlinear case of smooth surfaces.

For the existence and uniqueness in the nonlinear case with given surface, see Bjerhammar and Svensson (1984). Modern satellite methods such as GPS and VLBI make these new solutions fully realistic. However, only quasigeoids can be determined with these methods.

Relativistic geodesy lends itself to the following new definition of the geoid:

<u>The relativistic geoid is the surface nearest to mean sea level on which precise clocks run with the same speed.</u>

This definition of the geoid is valid over the oceans as well as the continents. The geoid is directly observable anywhere it is accessible. Mine shafts can be used for observations on the continents. The observed geoid represents the "true equipotential surface" with respect to the energy aspects. (Newtonian physics

gives equipotential surfaces which ignore the relativistic correction for angular momentum.)

The Dirichlet problem is directly applicable to geodesy given a known surface and known potential differences from our relativistic measuring technique. All oblique derivative problems are avoided in this kind of approach.

3. A TIME TRAP FOR RELATIVISTIC PROPER TIME DIFFERENCES

We postulate using two clocks, (1) and (2), continually in an operational mode while determining the geopotential difference between the two stations P and Q. The following measuring procedure is suggested.

Clock positions:	Counts: Clock (1)	Counts: Clock (2)	Comments
1. Both clocks at P. Calibration phase 1.	N_{11}	N_{21}	$N_{11} \neq N_{21}$ Frequency check f_1/f_2
2. Clock (1) at P. Clock (2) moves to Q. Transportation phase 1.	N_{12}	N_{22}	$N_{12} \neq N_{22}$
3. Clocks rest at P and Q. Measuring phase 1.	N_{13}	N_{23}	$N_{13} \neq N_{23}$
4. Clock (1) moves to Q. Clock (2) at Q. Transportation phase 2.	N_{14}	N_{24}	$N_{14} \neq N_{24}$
5. Both clocks at (Q) Calibration phase 2.	N_{15}	N_{25}	$N_{15} - N_{25} = (W_Q - W_p)c^{-2}f + C$ Frequency check f_1/f_2
6. Clock (2) moves to P Clock (1) at Q Transportation phase 3.	N_{16}	N_{26}	$N_{16} \neq N_{26}$
7. Clock (1) at Q. Clock (2) at P. Measuring phase 2.	N_{17}	N_{27}	$N_{17} \neq N_{27}$
8. Clock (2) at P Clock (1) moves to P. Transportation phase 4.	N_{18}	N_{28}	$N_{18} \neq N_{28}$
9. Both clocks at P. Calibration phase 3.	N_{19}	N_{29}	$N_{19} - N_{29} = D$ Frequency check f_1/f_2

There is an unknown quantity C representing the drift between the clocks in position 5. This unknown is determined when closing the loop in step 9. All these steps can be repeated several times. If the drift is constant, then the correction C can be "exactly" determined by this approach by putting C = D/2.

The transportation phases should be identical in an ideal time trap. It is, of course, impossible to fulfill this condition in a practical application. The relativistic errors from the transportation phase can be estimated if we know the differences which can be expected.

There is a general relativistic time difference when the two clocks are exposed to different geopotentials during transportation. This time difference can be computed by:

$$\Delta t = \int_0^T (g_1 h_1 - g_2 h_2) c^{-2} dt$$

where g_1 and h_1 represent gravity and height for clock 1, and g_2 and h_2 represent gravity and height for clock 2. We put $g_1 = g_2 = g$ and $h_1 - h_2 = \Delta h$. T represents the actual measurement interval during transportation. A constant height error of Δh will give a relative error

$$\Delta t/T = g\Delta h/c^2; \qquad \Delta t/T = 10^{-16} \quad \text{for} \quad \Delta h = 1 \text{ m} .$$

It seems advisable to make use of a number of checkpoints along the transportation road. These points should be passed at exactly the same time intervals for both clocks. There are also more sophisticated techniques for a definition of the transportation phase. However, in this section we restrict our study to the errors that are directly linked with relativistic effects.

If the two clocks are transported at different speeds, then the integration time will be different for the two clocks. Proper records of the movements of the clocks will allow for the necessary corrections.

There is also a special relativistic time difference caused by the difference in the velocities of the two moving clocks. The corresponding time difference here is:

$$\Delta t = \int_0^T (1/2)(v_1^2 - v_2^2) c^{-2} dt$$

where v_1 and v_2 are the velocities relative to the surface of the Earth for the clocks.

With $v_1 = 30$ km/hr and $v_2 = 31$ km/hr, the relative error $\Delta t/T$ will be on the order of 10^{-17}. For a more rigorous approach see eq.(10).

If the transportation phase has a duration of 10,000s, then the corresponding offset in time Δt will be

$$\Delta t \approx 1 \text{ ps (picosecond)} \times \Delta h \quad \text{(general relativity)}$$

where Δh is the systematic error in height (in units of meters).

If the transportation phase has a duration of 10,000s and the moving clock has a constant velocity of 30 km/hr, then the corresponding offset in time Δt will be:

$$\Delta t \approx 3.7 \text{ ps} \quad \text{(special relativity).}$$

The most suitable clock for a relativistic determination of geopotential differences is probably the hydrogen maser. The transition frequency of hydrogen is strongly affected by the magnetic field. The entire transportation van should have additional magnetic shielding in this application. (See sec. 4.) This means that the moving clock approach seems very attractive, if phase measurements can be utilized for the determination of differences of proper time.

If good corrections are applied for differences in velocities as well as differences in altitude during the transportation, then the corresponding offset between the two clocks that made the same loop will be very small when considering errors caused by relativistic effects (probably less that 0.01 ps).

An unknown factor in this comparison is the eventual detuning caused by the transportation. No records are available for the behavior of hydrogen clocks during transportation. Such a detuning can be detected if a third clock is available at the primary point P in the time trap. The detuning of each moving clock can then be estimated after closing the loops for clocks (1) and (2). If the third clock generates the frequency f_0 we will have two additional beat frequencies at point P. The two new beat frequencies $(f_1 - f_0)$ and $(f_2 - f_0)$ should both be close to zero if there has been no detuning. Even with $(f_1 - f_2) = 0$, we could very well have an unacceptable detuning which remains unknown. Consequently, we can control the measuring procedure in a better way if we utilize a third clock. If excessive detuning has been recorded, then the observations have to be corrected or disregarded. However, the third clock is not compulsory.

4. THE HYDROGEN MASER

The hydrogen maser is probably the most accurate clock now available. The physical unit includes a number of subdevices:

1. An atomic dissociator is filled with hydrogen. Dissociated hydrogen leaves through a collimator.

2. A variable field quadruple magnet selects the atoms of appropriate energy level in a vacuum chamber. Expelled atoms leave the unit and enter the storage bulb which is made of quartz and internally coated with Teflon™ (polyfluorine polymer).

3. A resonant cavity encloses the storage tube. Radio-frequency signals are taken out by a coaxial unit.

4. A vacuum chamber encloses the resonant chamber.

5. External magnetic shielding and multistage temperature control are supplied outside the active units.

A hydrogen maser has a resonant frequency of about 1.420 GHz. For an offset of 0.3 Hz there will be an attenuation of the signal to about half the peak-to-peak value. This gives a Q-value of approximately 3×10^9. The hydrogen maser, which has been available for more than 20 years, is manufactured by the Applied Physics Laboratory, Johns Hopkins University; National Bureau of Standards; Smithsonian Astrophysical Observatory; and others. The short-term accuracy (or stability) is closely correlated with the Q-value of the system. Estimates of short-term stability are primarily obtained by observing the beat frequency between the signals from two masers. If it can be postulated that the two signals are obtained without any drift, then the so-called Allan variance gives a direct measure of the short-term instability. Rueger (1981) presented results that were obtained from a hydrogen maser NR-1 (APL).

(Allan variance)$^{1/2}$ ($10^{-16} \times \Delta f/f$)	Time interval (s)
± 263	10
152	20
93.3	40
39.8	100
23.5	200
16.3	400
12.9	1,000
8.71	2,000
9.67	4,000
9.86	10,000

These figures clearly show that a strictly stochastic behavior occurs up to 2,000 s. When increasing the time interval above 2,000 s, there is no further improvement. The explanation is obvious. The Allan variance postulates that no systematic trend is hidden in the data.

Separating the stochastic and nonstochastic parts of the "Allan variance" leads to the following estimate of the standard deviations for an interval of 1,000 s:

$$s_{1000} = \pm 1.3 \times 10^{-15}.$$

An alternative estimate is found in Reinhardt et al. (1983)

$$s_{1000} = \pm 2.1 \times 10^{-15}.$$

These estimates of the standard deviation refer only to the stochastic part of the Allan variance. However, they give the lower limit for successful application when correcting for systematic effects.

If we disregard Earth tides and transmission errors, then we can estimate the minimum observation times for obtaining a desired accuracy when using the first clock (with unchanged performance in the field)

Limiting standard deviation in height (m)	Time interval (day)
±1.4	1
0.4	10
0.1	100

Such "accuracies" can only be expected in the relativistic approach if there is a technique for eliminating the systematic errors. This means that the needed observation time will be at least twice as long in any present application with the actual instrument.

Rueger (1981) stated that this laser can be set over a range of $\pm 5\times 10^{-8}$ to a resolution of 7 parts in 10^{17}. This means that the laser has a device for tuning the cavity to the resonant frequency.

Bollinger et al. (1983) give the following data for a passive NBS H-maser: Frequency drift = 3×10^{-16}/day, and square root of Allan variance

$$s(t) = 1.5 \times 10^{-12} t^{-1/2} \quad \text{(t in seconds).}$$

The long time stability of hydrogen masers is not well known. For the SAO maser clock Mattison and Vessot (1982) state:

"Over intervals longer than 1000 seconds, maser output frequency is influenced chiefly by systematic effects that include changes in the resonance frequency of the microwave cavity and variations in the external magnetic field.

"Cavity frequency shifts, which result in pulling of the atomic line shape and consequently of the output frequency, can be caused by temperature-induced changes in the dielectric coefficient of the quartz storage bulb; by changes in cavity dimensions due to variations in ambient temperature, barometric pressure, or cavity mechanical properties; and by other mechanisms. Work...has shown that some masers display a regular long-term frequency drift on the order of parts of 10^{14} per day that is accompanied by a corresponding increase in the tuned cavity resonance frequency. The temporal behavior of these drifts is typically $1-e^{-t/T}$, with T on the order of 40 days. We believe that this effect is due to "bedding" of the ground surfaces of the low-expansion Cervit cavity at the joints between the cylinder.... We have taken steps to reduce this drift by optically polishing the

cavity endplates and cylinder end surfaces, and assembling the cavity under clean room conditions....

"Other possible sources of cavity frequency drift are thermal expansion of the cavity material and relaxation of surface stress in the cavity's conductive coating.

"Over an ambient temperature range of 21 to 28°C it yields an estimated temperature sensitivity (1/f)(df/dT) of approximately 8×10^{-16} °C....

"The measured shielding factor and magnetic sensitivity of a maser with the new magnetic shields are:

$$\frac{dH_{ext}}{dH_{int}} = 18x16^3; \frac{1}{f}\frac{df}{dH_{ext}} = 2 \times 10^{-13} G^{-1} \quad \text{for} \quad dH_{ext} = 0.44 \text{ G} \ldots."$$

The NR-maser is available directly mounted in a van, which means it can be operated during transport. However, no information is available concerning frequency stability or phase control during transportation. The present tuning system will probably not allow transportation without serious restrictions. A refined tuning technique is being developed for this kind of hydrogen maser.

The relativistic approach will require an advanced tuning technique. Mattison and Vessot (1982) state: "To remove any long term cavity drifts that remain, we are working on the development of an electronic cavity frequency stabilization system that will lock the cavity resonance to the atomic transition frequency with minimum perturbation of the atomic lines."

It seems reasonable that such a tuning technique can be developed with accuracies close to 10^{-16}, but at present nobody seems to have a working solution. The tuning consistency will be of utmost importance if the clock does not have such stability, so that it can be transported by conventional means without significantly influencing the setting of the tuning.

Manual tuning, which is normally done on the hydrogen maser, allows the maser to be set over a wide range. Once the tuning is completed, then the maser "stays" on the selected frequency with high stability. The relativistic approach has to satisfy at least one of the following two conditions:

1. After tuning is completed, the instrument must be able to withstand whatever type of transportation is used to a new observation site, without losing the preset tuning. If this condition is satisfied, then there is a possibility of using a time trap of some kind.

2. The tuning can be reset without the help of any external signal or information.

These conditions were probably not very important in the earlier design of hydrogen masers. If the hydrogen maser is to be considered a geodetic instrument for relativistic operations, then some design modifications must be contemplated. The most promising strategy would probably be the application of electronic cavity

tuning which would lock the cavity resonance strictly to the atomic frequency transitions.

4.1 Accuracy

The accuracy of a frequency standard is measured with respect to "absolute values." There is a frequency shift when the hydrogen atoms are reflected by the walls of the container. The storage bulb is coated mostly with Teflon^{tm} to reduce reflections. Petit et al. (1980) found that the "wall shift" was reduced to zero at a temperature of 112°C. This temperature seems too high for practical applications; furthermore, the wall shift has to be determined individually for each maser. The lack of reproducibility of the coating limits the accuracy of the hydrogen maser to the 10^{-12} level which is less than the accuracy of the best cesium standards. Petit et al. used a refined technique for measuring the wall shift in the range 27°C - 87°C. They obtained the following value for the unperturbed hydrogen transition frequency (f_H):

$$f_H = 1{,}420{,}405{,}751.773 \pm 0.001 \text{ Hz}.$$

The claimed corresponding accuracy was 6×10^{-13}. There was also a study of the frequency stability.

t	1	10	10^2	10^3	10^4
s(t)	4×10^{-13}	4×10^{-14}	8×10^{-15}	4×10^{-15}	2×10^{-14}

Here t is the time interval in seconds and s(t) the relative standard deviation (square root of the Allan variance).

This indicates that the accuracy of the hydrogen maser is only at the 10^{-12} level, and therefore we cannot expect to use hydrogen masers without a previous calibration or synchronization. The stability of the actual hydrogen maser reaches its optimal value for a time interval of 1,000 s.

Present clocks cannot be used for relativistic determination of potential differences without careful synchronization and calibration. The accuracy of 10^{-14} corresponds to only ±100 m. However, the absolute accuracy of each of the two clocks need not be high in a relativistic application. It is required only that the two selected clocks have almost identical drifts over a long time interval. For this purpose we suggest using a pair of clocks that have been chosen from a set of clocks, with all having, as close as possible, the same physical parameters. Aging of the clocks for several months is recommended before selection of the "best pair."

The following steps should be considered:

1. Two matched clocks are selected.
2. The beat frequency is observed for a time interval of n days.
3. The phase difference (time difference) is measured simultaneously for the same time interval.

If n is 180 days, then the elapsed time is about 1.5×10^{19} ps. The time difference is expected to be determined within ±1 ps.

If the utmost accuracy is wanted, then the geopotentials should be measured for two points that are at almost the same height above sea level. The magnetic field should be as similar as possible (Zeeman effects). Electrical alternating current fields should be avoided. One of the clocks is now moved to the new position in accordance with the principles of the time trap discussed in section 3.

We conclude that it is sufficient to have two clocks with rather modest accuracies if their drifts are almost identical. A constant drift between the two clocks is also acceptable, because such a drift can be completely eliminated. This means that we ask only for good relative stability (a matched pair of clocks).

Improved performance can be expected if a third clock is used exclusively for frequency calibrations. The beat frequency is then measured between the third clock and alternatively clocks 1 and 2 during all the measuring phases.

Rueger (1981) found for the APL maser a jitter of 0.3 ps. This is a stochastic error source of minor importance for a relativistic approach, where the shortest observation intervals will be on the order of 1 day (8.64×10^{16} ps).

4.2 Hydrogen Masers With Automatic Tuning

Mattison and Vessot (1982) mentioned work on a new clock with automatic cavity frequency stabilization locked to the atomic frequency transition.

Peters (1984) gave a detailed description of a new hydrogen clock that successfully utilized a servosystem for automatic cavity tuning to the hydrogen frequency transition. A technique had been used earlier for "autotuning" which could benefit from simultaneous utilization of two masers. There seems to be no way of using this autotuning for a single oscillator. (Two oscillators can be phase-locked anyhow).

The new clocks, according to Peters, have a number of new features that deserve to be mentioned:

Size: 80 × 40 × 50 cm (alternatively with 20% increased values)
Weight: 125 kg (150 kg)
"Settability": ± 2.5 parts in 10^{17} (phase coherent)

The new design is based on a cavity frequency switching technique that should eliminate cavity related drifts. The system has been designed in such a way that the "cavity is continuously maintained at the spin exchange offset frequency without the use of a secondary reference oscillator." Each maser is therefore serving as a self-consistent frequency standard which is formally free from cavity pulling. The temperature of the cavity is used for controlling the cavity frequency. The servosystem operates between the two equilibrium levels that are reached within approximately 10 μs. In this way the cavity resonance frequency is

switched rapidly between two different frequencies. The frequency switching introduces a phase modulation of the maser output signal. This phase modulation is eliminated in the maser synthesizer circuit. Therefore, the short term stability is more or less unaffected by the servo-operations. Peters (1984) derived the following theoretical value for the long-time stability of the new hydrogen maser:

$$\Delta f/f = 1.55 \times 10^{-14} t^{-1/2} \quad \text{(t is time interval in seconds)}$$

or

$$\Delta f/f = \pm 0.5 \times 10^{-16} \quad \text{(1-day interval)}$$

$$\Delta f/f = \pm 1.7 \times 10^{-17} \quad \text{(10-day interval)}$$

Test results from two different designs of the servoswitching system were presented (graphically displayed):

Time interval (s)	(Allan variance)$^{1/2}$ ($\Delta f/f$)
10	$\pm 3\times10^{-14}$
10^2	7×10^{-15}
10^3	6×10^{-15}
10^4	4×10^{-15}
10^5	2×10^{-15}
6×10^5	3×10^{-15}

Peters concludes: "No linear drift between the tuned masers has yet been observed within the statistical uncertainty of measurement. (The calculated drift for this 13-day period was 6×10^{-16}/day.)"

Several tests were made with the new masers:

1. A frequency offset of 3×10^{-12} was introduced. The servosystem responded with a linear correction rate of 1.5×10^{-12}/hr. The correction continued until the original frequency was restored.

2. A similar offset in the opposite direction gave the same result.

3. The magnetic field (external) was changed from 1.5 to -1.6 Gauss. The shielding factor was found to be about 10^5 for the best shielded instrument and 3.7×10^4 for about three times the smaller changes.

4. No statistically significant temperature coefficient was found for the best behaving instrument.

It should be noted that the new instruments incorporated several new techniques not applied in the previously mentioned servosystem. For example, new techniques for magnetic shielding and temperature control have been used.

Peters (1984) concluded:

"There are no apparent barriers to achieving the cavity related stability level implied by Equation 6." (Equation 6 refers to the section containing this quotation.)

"The present results reported in this paper give a basis for a very optimistic projection of the performance level which may be ultimately achieved."

There seems to be little doubt that the new technology for hydrogen maser design represents a very promising development. It seems to have the power of making the relativistic determination of geopotential difference quite attractive.

4.3 Hydrogen Masers Especially Designed for Relativistic Operations

Present hydrogen masers have been designed for laboratory operation. The APL masers are mounted in special portable vans. Masers from NBS as well as SAO can all be transported, but there is no indication that this can be done without the obvious risk of excessively detuning the clocks. Considering the kind of tuning done in these instruments, we conclude that the transportation phase would be so critical that a meaningful geodetic application of relativistic measurements becomes very intricate.

The hydrogen maser of Peters, thus far, is made only in two copies. However, it has been shown that the servoswitching technique responds to a detuning of magnitude 3×10^{-12} with complete retuning after about 2 hours. These results are sensational and open the possibility for geodetic applications in a relativistic mode. (This is an improvement by a factor of 100 above the classical technique.)

The first two clocks were built according to two different concepts. Their relative performance could probably have been considerably improved if maximum relative stability had been requested.

Transportation of the clocks seems unavoidable in a geodetic relativistic operation and, therefore, some instrumental comments are justified:

1. The geodetic clocks should be mounted in vans similar to the ones used in the crustal dynamic program (VLBI).

2. The van itself should be used for additional magnetic shielding. The clocks have magnetic shielding consisting of four different shields with the first

two inside the vacuum enclosure. A shielding factor of 20,000 has been obtained in this way. The transportation factor is expected to introduce

Zeeman effects of the second order. It would be necessary to use a large external shield, for example, the entire transportation van itself. (The shielding factor is on the order of $\sim 10^6$.)

3. A zero temperature coefficient was found for one of the two new clocks. This state was related to a higher beam intensity. The clocks should still benefit from additional temperature control in the entire shield volume.

4. Corrections for gravity need to be included or added afterwards.

5. Two matched twin clocks need to be used.

6. A third clock will be convenient for calibration and for allowing stationary operations with the two measuring clocks. (Satellite links can also be considered.)

5. NEW PRECISE CLOCKS

The hydrogen maser apparently already reached an impressive short term stability many years ago. Vessot (1979) reported clocks with a short term stability of 6×10^{-16}. In a later publication Vessot (1981) wrote, "...atomic clocks, whose stability is now well below the 10^{-15} level for intervals between 20 min. and 1 hour." Very long term stability is not well documented. The most promising development seems to be linked with methods for cavity tuning directly to the transition frequency of the hydrogen atom. In a still longer perspective we have to consider the changing of the wall shift for the hydrogen maser.

Reports have been given concerning a number of new designs which make use of radio frequency ion traps eventually in combination with laser cooling. It is known that several laboratories are developing new frequency standards. A radio frequency ion trap is favorable for clock application because the ions can be trapped in confinement for a long period (Bollinger et al. 1983). Stored mercury ions are attractive because of the high frequency (f=40.5 GHz) and the small line width (Δf = 8.8 Hz). The quality factor Q is

$$Q = 2f/\Delta f = 1 \times 10^{10}.$$

The ions will be pumped from the ground state (F = 1) up to the excited state and then will decay back to the ground state. Microwave radiation at the resonance frequency between the first and second excited state of Mercury 199 will produce fluorescence which can be used for detecting the hyperfine resonance. The decay passes over to Mercury 202 which has no hyperfine structure (no nuclear spin). The excited state and the ground state of Mercury 199 match well with the resonance frequency from the two excited states of Mercury 199.

The most important source of frequency shift in a system of this kind is the second-order Doppler shift caused by the special relativity effect on emitted atoms. The technique commonly used for cooling in such devices is sophisticated. Cutler et al. (1981) outlined a method of cooling with the aid of a light inert gas

like helium. This system is under development by Hewlett Packard laboratories. (See also Jardino et al. (1981) for a similar approach.) Equations (21) and (22) show that the second-order Doppler effects vanish at T = 0.

Wineland et al. (1981) described the principles of a microwave frequency standard utilizing Mercury 201 ions stored in a static trap charged with a given quadruple electrical potential in a uniform magnetic field. Since only static fields are used in the trap, cooling the ions to reduce the second-order Doppler shift is greatly simplified.

The technique chosen for cooling appears promising. The trapped ions are exposed to a laser beam tuned slightly below the resonant transition frequency. If the laser frequency is above the resonant frequency, the the trap will be heated. The physical explanation is complex but mainly founded on the fact that if the stream of ions is directed against the laser beam, then the light frequency is shifted closer to resonance and this requires energy. Laser cooling has successfully been applied to magnesium and beryllium ions. (See, for example, Itano et al. (1983) and Wineland et al. (1981).) It is expected that it will be possible to make an equivalent application for trapped Mercury 201 ions. This will require a tunable continuous-wave, narrow-band laser. Such a source has recently been developed by NBS. The required frequency is obtained by mixing independent laser signals (Wayne et al. 1983). The proposed system is predicted to have a stability corresponding to (Allan 1966):

$$s(t) = \pm 2 \times 10^{-15} t^{-1/2} \qquad t > 100 \text{ s}$$

where s(t) is the square root of Allan variance and t is the selected time interval. For one day of operation the limiting value will be

$$s(86400) = \pm 6 \times 10^{-18} .$$

The expected Q-value has been computed to be

$$Q = 2.6 \times 10^{12} .$$

This is about 1,000 times better than for the hydrogen maser.

Still more exciting results are expected from "laser clocks" based on Mercury 201 ions. The corresponding values were presented as

$$s(t) = 2 \times 10^{-18} t^{-1/2} \qquad t > 2 \text{ s}$$

$$Q = 7.4 \times 10^{14}$$

(See Wineland et al. (1981) and Wayne et al. (1983) for further details.)

Some competitive ions are perhaps found in barium with a Q-value of 1.6×10^{16} and indium with a second-order Doppler shift of only 10^{-19}. Such clocks might have up to 10^6 times better Q-values than present hydrogen masers. (For superconducting cavity resonators, see Turneaure et al. (1983).)

6. SYSTEMATIC FREQUENCY DRIFTS

At present, relativistic determinations of geopotentials or dynamic heights are of interest only for distances larger than 1,000 km.

A relativistic determination of dynamic heights has not yet been made for geodetic applications. In a moving clock experiment all errors would be generated by the clocks (and transportation). The clock errors in an experiment will be estimated.

1. Clocks operated without sophisticated calibration (H^2 masers): Drift rate/day: 10^{-15}.

   Corresponding height drift rate: ±10 m/day.

2. Clocks operated in closed loops with special calibration for the daily drift: Repeated calibrations are expected to furnish mathematical corrections to the calibration which should reduce the final frequency errors to about 2×10^{-16} or ±2 m.

Geodesists have had some earlier experience in working with instruments that need calibrations in closed loops to give satisfactory results. For comparison we select here a gravity meter of the older type. These instruments could have a drift of about 10 mgal a day. If the instrument had been operated in closed loop, then the remaining errors could normally be reduced by a factor of 10. If there had been a linear drift only, then better results would have been obtained. Still better results can perhaps be expected from a relativistic determination of heights, which can benefit from much longer observation.

If the observations are made during very long observation intervals, then submeter accuracies should be completely realistic. Observations by the Mossbauer technique more than 20 years ago proved the validity of these considerations. See Pound and Rebka (1960).

A comparison with precise leveling seems somewhat less favorable but also less meaningful. The systematic errors of leveling remain mostly hidden.

Somewhat conflicting information can be obtained concerning the stability of various "precise" clocks. Different procedures can be considered for obtaining useful estimates of proper time at two points a large distance apart on the Earth. We will restrict our study to the case where the observations have been made in such a way that both clocks are exposed to longitudinal displacements in the same manner. Otherwise, additional corrections have to be applied. See the appendix for further details.

We postulate that our problem will be to determine the difference in proper time for two selected points P and Q. There is no need to know "absolute proper time" for the study we have in mind.

Some general rules for selecting adequate clocks can be contemplated. If a large number of clocks are available, then it should be possible to make a preliminary study which results in a pair of clocks with closest match. It is understood that present hydrogen masers mostly have frequency drifts too large for an immediate

meaningful geodetic application in a relativistic approach. Different modes of operations can be considered:

1. Clocks are used with a primary calibration and then without further corrections.

   Expected relative frequency drift: $3 \times 10^{14} - 10^{15}$.
   (See Reinhardt et al. (1983) and Knowles et al. (1982).)
   Relativistic height drift: 30 - 10 m.
   Comments: The stability indicated above is not useful if corrections for the drift cannot be determined by reliable methods.

   If the relative frequency drift is constant, then it should be possible to make the necessary correction without a more sophisticated technique. Probably the drift will change with time and this will make accurate height determinations difficult.

2. The relative drift between two clocks can be determined with high accuracy from the beat frequency obtained in a multiplicative mixing.

   $$2 \sin(c_1 + \omega_1 t) \times \sin(c_2 + \omega_2 t) =$$

   $$\sin(c_3 + (\omega_1 - \omega_2)t) + \sin(c_4 + (\omega_1 + \omega_2)t) \quad (\omega = 2\pi f) .$$

   The high frequency signal is filtered out and we simply obtain the beat frequency $(\omega_1 - \omega_2)$. This approach is most conveniently applied to direct observations of frequency differences.

3. The phase difference between the signals from two clocks is most easily interpreted if the clock frequencies are available as two low frequencies, which are selected in such a way that drift is taken care of inside one full wavelength. A phase meter will then automatically give the integrated value of the frequency drift during the measuring interval. The advantage of a phase measuring system is that it gives us an opportunity to make use of extremely long measuring times.

A relativistic application of differences in proper time makes it necessary to separate instrumental clock drift from true drift caused by difference in proper time.

Instrumental drift and true drift cannot be separated in a straightforward study. Such is the case in the following situation.

Two clocks (1) and (2) are calibrated and give identical frequencies at a point. Clock (2) is then moved to a new position (Q). The beat frequency is now $(f_1 - f_2)$. What is the instrumental drift and what is the true drift? This is a singular problem without any meaningful solution. If either the true drift or the instrumental drift is known, then the remaining one can be determined.

The previous experiment is followed up by transporting clock (1) to point Q. Now the beat frequency $(f_1 - f_2)$ can be measured again. If $(f_1 - f_2) \neq 0$, then there is an instrumental drift. We can repeat this measuring operation in the reverse direction. Each time we obtain a new measure of the different drifts. Each time the clocks are at P and Q we obtain a direct measure of the total drift. Each time two clocks are in the same position, we obtain a direct determination of the actual instrumental drift.

This measuring procedure requires a link between the two stations P and Q by satellite or direct transmission. Transportation between the two stations can be accomplished by air, and the time-dependent instrumental drift can probably be greatly reduced. However, any change in the instrumental drift during transportation would remain unknown.

We now consider a case where the frequency difference ($f_1 - f_2$) is recorded over a longer time interval. This frequency difference is presented in the following way:

$$f_1 - f_2 = c^{-2} f(W_P - W_Q) + x_0 + x_1 t + x_2 t^2 + \ldots$$

where t is time and x_0, x_1, and x_2 are instrumental drift parameters. Here x_0 represents the offset and this constant can be determined with high accuracy when P and Q are coincident. Furthermore, we can consider using a large number of observations for a determination of the remaining unknowns. However, x_0 and the geopotential difference will be harder to separate when P and Q are noncoincident. One approach could be to determine the frequency difference after interchanging the clocks. The geopotential and x_0 will then enter with opposite signs and the mean of the observed frequency differences should now be free of bias.

Observations with two clocks in the same position

$$f_1 - f_2 = x_0 + x_1 t + x_2 t^2 + \ldots \qquad \text{(1) + (2) in P}$$

If there is no satellite link, we cannot directly observe the beat frequency over long distances when we operate in the measuring phase of the time trap. The beat frequency over a selected time interval t is expressed by three unknown parameters x_0, x_1 and x_2.

Observations with the two clocks in different positions

$$f_1 - f_2 = c^{-2} f(W_P - W_Q) + x_0 + x_1 t + x_2 t^2 + \ldots \qquad \text{(1) in P and (2) in Q}$$

After interchanging the two clocks we write

$$f_1 - f_2 = -c^{-2} f(W_P - W_Q) + x_0 + x_1 t + x_2 t^2 + \ldots \qquad \text{(2) in P and (1) in Q}$$

Observations with the two clocks in the same positions

$$f_1 - f_2 = x_0 + x_1 t + x_2 t^2 + \ldots \qquad \text{(1) and (2) in P}$$

Records for the determination of the instrumental drift are obtained with both clocks in the same position at three different steps of the previous time trap.

(1) and (2) at P
(1) and (2) at Q
(1) and (2) at P

These records can be used for a determination of the instrumental drift parameters x_0, x_1 and x_2. After subtracting the instrumental drift we obtain the geopotential difference from the beat frequency after proper integration with respect to observation time or phase difference.

The three observations of beat frequencies at P-Q-P give a model of the instrumental drift, and the misclosure in time can be computed. The difference between observed and computed misclosure is an estimate of the transportation error.

No records are available for a study of the behavior of precise clocks during transportation. However, some indications can perhaps be found from the experiments concerning the red shift. Vessot (1979) described a test with a hydrogen maser in a spacecraft. A near-vertical trajectory with a maximum altitude of 10,000 km was used. There was a second maser on the ground. Three microwave links transmitted signals between ground and the spacecraft. The first-order Doppler shift was cancelled with the help of a closed loop from ground to spacecraft and back. The expected gravitational red shift was on the order of 4×10^{-10} and the expected stability of the maser was about 10^{-14}. Vessot states: "Our present conclusion is that the red shift agrees with prediction to about 200 ppm." This seems to indicate that there was no major loss of stability during the space flight. Very strong accelerations were involved in this experiment.

The Vessot experiment is not directly applicable to geodetic determinations of geopotential differences, and its seems advisable to use a measuring technique which supervises the transportation to the utmost. Therefore, we postulate that a large number of checkpoints would be used. For each checkpoint, we can make use of the following records:

1. Beat frequency relative external precise clock: GPS.

2. Geometric height above the ellipsoid: GPS or movable VLBI.

3. Gravity: Precise gravity meter.

4. Inertial navigation coordinates: PADS.

5. Temperature.

6. Magnetic flux.

Present hydrogen masers used for movable VLBI would be mounted in vans. The clocks would be in operation during the transportation, but with no records available for the beat frequency from such an experiment. There is a risk of irregular behavior during transportation (mainly because of magnetic disturbances), and additional magnetic shielding should be justified. (A large external shield is considered useful.)

It is now postulated that the observation intervals have the same duration before and after interchanging the clocks. We would then have the following integrals of the beat frequencies:

$$I_1 = \int_0^T (f_1 - f_2)dt = \int_0^T (c^{-2} f(W_P - W_Q) + x_0 + x_1 t + x_2 t^2 + \dots)\, dt$$

$$I_2 = \int_T^{2T} (f_1 - f_2)dt = \int_T^{2T} (-c^{-2} f)(W_P - W_Q) + x_0 + x_1 t + x_2 t^2 + \dots)\, dt.$$

Here I_1 represents the phase difference at the end of the first measuring phase and I_2 the phase difference at the end of the second measuring phase.

$$I_1 - I_2 = 2Tc^{-2}f(W_P - W_Q) - x_1T^2 - 2x_2T^3 + \ldots$$

and

$$I_1 + I_2 = 2Tx_0 + 2T^2x_1 + (8/3)T^3x_1 + \ldots$$

$$W_P - W_Q = (I_1 - I_2)/(2Tc^{-2}f) + (x_1T^2 + 2x_2T^3 + \ldots)/(2Tc^{-2}f)$$

where $(I_1 - I_2)$ is obtained without the linear drift.

There is also a possibility of modeling the instrumental drift from records of the beat frequencies at the three positions where the clock operate in the calibration phase.

$(f_1 - f_2)_0 = x_0$ (Beat frequency: both clocks at P.)

$(f_1 - f_2)_1 = x_0 + Tx_1 + T^2x_2 + \ldots$ (Beat frequency: both clocks at Q.)

$(f_1 - f_2)_2 = x_0 + 2Tx_1 + 4T^2x_2 + \ldots$ (Beat frequency: both clocks at P.)

These records determine the three drift parameters x_0, x_1, and x_2.

Regarding the transportation phases, it is assumed that the errors of the transportation phases cancel in the computation of I_1 and I_2. Modeling of the instrumental drift will require a correction for the transportation phases if there are significant errors. Furthermore, there is a possibility of using a large number of beat frequency observations inside each calibration phase. This would make it possible to gain degrees of freedom or a more detailed model.

If a satellite link is used in combination with the time trap, then the beat frequency can also be observed with the clocks at P and Q.

The interchanging of the two primary clocks is the most fundamental operation for a determination of the proper time difference or the corresponding true proper beat frequency. If transportation of the clocks can be made without influencing the tuning, then the two-clock approach with the time trap appears to be attractive.

If a satellite link is available, then direct observation of time will be unnecessary. However, even in this case we have to consider an interchange of the two clocks. This seems to be the best way to eliminate the errors from an incorrect setting of the frequencies for the two clocks. If the clocks are stable, then it should be sufficient to determine the beat frequency when the two clocks are at the same position, and then apply a correction in any future application.

The use of a third clock is an attractive approach because it leaves the two primary clocks stationary for most of the time. The transportation of the primary clocks might be a risky enterprise, eventually resulting in malfunction of one or both clocks.

The weakest point in this chain of observations is probably the tuning of the two primary clocks.

Formally, we must always consider:

1. The beat frequency between the two primary frequencies should be a constant, disregarding tidal variations. The change of beat frequency is a direct measure of systematic errors like "flickering."

2. The resolution (theoretical) of a precise hydrogen maser is of the order of ±1 ps. (Observations during half a year cover a time span of about 1.5×10^{19} ps. This means that our system will have a height resolution of about ±1 mm after operating half a year. A more realistic resolution is ±10 ps.)

Considerable loss in resolution follows from using signals transmitted through the atmosphere and the ionosphere. Fiber optic links are under rapid development in the commercial telephone system. A transatlantic cable is already projected. The relativistic approach offers exceptional advantages from this kind of technology.

7. GEOSTATIONARY SATELLITE LINK

The time trap outlined above opens the possibility of using two clocks as the only measuring tools needed for obtaining relativistic determinations of geopotential differences. The technique can make use of extremely long integration times which should be helpful in an early experiment. Formally, the whole measurement would have the character of a phase-measuring technique, where the total phase displacement eventually is observed at several frequencies in order to avoid ambiquity. The technique does not require a direct communication line between the two clocks. This means that the errors of a communication line are also avoided. Instead we are exposed to the errors from the transportation phases of the measuring system.

There is another possibility which has some interesting features. This technique uses an almost geostationary satellite as a link between the two clocks.

The satellite generates a signal with frequency f_0 and a suitable beacon for frequency control. The two ground stations generate signals with frequencies f_P and f_Q. Furthermore, there is communication in both directions to the satellite, to determine the movement of the satellite. Finally, the satellite should generate two auxiliary signals at different frequencies, which allow determination of the delay in the ionosphere.

Our main measuring problem is solved in the following way:

Superheterodyning 1: $f_P - f_0$

Superheterodyning 2: $f_Q - f_0$

Superheterodyning 3: $(f_P - f_0) - (f_Q - f_0) = f_P - f_Q$

We note that this procedure benefits from the invariance with respect to the reference frequency f_0. The beat frequency $f_P - f_Q$ is transmitted over the satellite link (or along the ground). Only geostationary satellites would serve well in a strictly continuous mode.

Knowles et al. (1982) reported a successful application of a phase coherent satellite link with the use of the geostationary ANIK-satellite. They concluded, "Our present link should be better than a hydrogen maser over periods of a day or longer; the improved link should be superior for periods of 3 hours or longer." Figures in the report indicate that the "two-way" link was capable of eliminating all first-order effects of the satellite motion. The uplink was operated at 14 GHz and the downlink at 12 GHz. Unfortunately, there was no beacon transmitted from the satellite, which made it difficult to compensate for the drift of the satellite oscillator. This correction was instead computed by using two tones separated by 60 MHz. The experiment involved VLBI stations in British Columbia, Ontario, and Maryland.

Some figures from the study might be of special interest:

Short-term noise of the link	±25 ps
1-day stability	±175 ps
10-day "	±250 ps
1-day relative stability	2×10^{-15}
10-day "	3×10^{-16}

The figure for 10-day stability is uncertain because it is based on prediction.

Several modifications were suggested. There seem to be convincing arguments indicating a satellite link of this kind is the most promising system for generating "universal time." All numerical data from the report refer to an experimental design. The results are so excellent that the method deserves further application.

It should be noted that the main advantage of an ANIK-link compared with GPS is that the ANIK-link gives continuous operation over almost unlimited time. This means that such a link, which makes use of a phase coherent system over a very long time, should be capable of defining relative long-time stabilities good to 10^{-17}. However, further studies will be needed before stabilities of this kind can be successfully utilized in a relativistic approach. (See Campbell and Lohmar (1982) for studies over the delay in the ionosphere.)

A satellite link can serve as a suitable tool generating the beat frequency between the two clocks involved in a relativistic measurement of proper time differences. The following measuring procedure can be used:

1. The beat frequency is determined over the satellite link with both clocks at the selected stations (P and Q).

2. The two clocks are interchanged and the beat frequency is determined again.

This measuring procedure can be repeated an unlimited number of times. The advantage of this technique is that the "flickering" from the two instruments will be gradually eliminated in the final mean. For flickering see Vessot (1976).

If the complete system has an operational stability of only 10^{-14}, then the limiting standard deviation after 400 days of observation will be ±0.5 m. Improved performance would require the use of several clocks in simultaneous operation at the observation sites. The technique should be investigated carefully at a test site, where points with known geopotential differences are available. It should be emphasized that the time trap of section 3 should also benefit from using a satellite link.

8. GLOBAL POSITIONING SYSTEM

A geostationary satellite system has obvious advantages for a relativistic approach where long observation intervals can be important. The Global Positioning System (GPS) will make use of 18 satellites when it is fully deployed. At present, much less are in orbit and this means that the observation intervals are restricted to only a few hours.

The satellite oscillators have stabilities of about 10^{-11} to 10^{-13} for short intervals, and are either rubidium or cesium. The geodetic GPS receivers mostly have crystal oscillators with a short-term stability of 10^{-10} to 10^{-12} (Goad and Remondi 1983).

The stabilities of the oscillators are so low that no direct relativistic operations are meaningful. The short observation intervals with the present satellites in orbit further reduce the possibilities with this technique.

Some modifications of the GPS approach might be considered as necessary for relativistic operations:

1. The local oscillators at the ground station can be replaced by a hydrogen maser of highest quality.

2. A dual or triple frequency technique is applied to reduce the errors in the ionosphere.

3. Advanced new techniques are applied to reduce the orbital errors.

Goad and Remondi (1983) report on the successful determination of "the relative variations of the ground clocks to the subnanosecond level." This is impressive accuracy for this kind of system, but it is still not quite satisfactory when using observation intervals of only 5 hours. The relativistic approach will be good only for height differences of about ±1,000 m. More promising results can be expected from a dual frequency approach.

Available records indicate that it should be possible to determine the offset between two distant clocks to about ±10 ps. Instruments of this kind are now being tested. We can consider using such instruments for a determination of potential differences.

One way of using the system is to incorporate two perfect clocks which replace the existing clocks at the two ground units in question. With only 5 hours of operation, the instability of the time interval would be on the order of $\pm 10^{-15}$.

The corresponding height error in a relativistic approach is about ±10 m. Improved accuracies can, of course, be obtained by repeating the observations. (Clock drifts are disregarded.)

An experiment of the kind indicated demonstrates how a relativistic height determination can be made.

1. Two perfect clocks are running freely at the positions P and Q. Proper time has different speed at different geopotentials. The difference in the geopotentials is directly proportional to the frequency difference of the two clocks.

2. The offset and the drift between the two clocks are determined over the GPS link.

3. The height difference is obtained from the frequency difference

$$h = (f_P - f_Q)c^2 f^{-1} g^{-1}$$

   where g is gravity and h dynamic height difference.

When all GPS satellites will be in orbit, then this system could serve well as the timekeeping unit in a relativistic height determination approach.

However, we should note that we get only a time reference system by GPS. The final differences in proper time will be only as good as the clocks allow. In most situations we have to accept a moving clock technique to allow for proper calibration, but the usefulness is limited as long as the operational time is so small. The GPS instruments have been operated at a frequency of 1.575 GHz.

9. VERY LONG BASELINE INTERFEROMETRY (VLBI)

The VLBI technique has so far been used in an untuned mode. This means that the offset between the two clocks operating at the ends of a baseline is computed by cross-correlation between the signals received at the two stations without any direct reference to the difference in proper time.

If we want to use the VLBI technique for a relativistic determination of geopotential differences, then the following procedure can be considered:

1. The two clocks at the end of the baseline are tuned to the utmost for optimal performance. The NR maser of APL is "settable" over a range of $\pm 5 \times 10^{-8}$ to a resolution of 7 parts in 10^{17} (Rueger 1981).

2. The VLBI operation is completed in the usual way and the geometric height difference is computed. The two clocks are synchronized by the cross-correlation technique (mathematically). A common universal time system is established.

3. The VLBI operation is repeated after interchanging the clocks. The beat frequency now corresponds to twice the geopotential difference if there are no instrumental errors. (This step can be considered only when using movable clocks. A practical application has to be restricted to mobile VLBI.)

4. Frequency checks can be included as in the time trap. Both clocks have to be compared at P and Q.

Stationary VLBI instruments have perhaps the most accurate clocks with the most efficient temperature control. There is a temperature drift of about $10^{-14}/1°$ if no additional external temperature stabilization is included. Stationary instruments and VLBI instruments for crustal movement have an external oven which should reduce the temperature variations below 0.1° (or less). Additional temperature corrections can eventually be contemplated.

If movable VLBI instruments are available for an experiment of this kind, then the following procedure can be considered:

1. A test area is selected where two points P and Q have a difference in altitude of at least 1,000 m. The distance between the points is not important, but a short distance will speed up the test.

2. The two clocks from the VLBI stations are used in a measuring sequence according to section 3. This measuring procedure has to be preceded by a careful tuning of the clocks for long-time stability. About 10 intermediate checkpoints between the given points P and Q can be considered. The movements of the two clocks should follow a strict pattern and be properly mapped.

3. The geometric coordinates for points P and Q are determined by using the standard VLBI technique. If precise leveling data are available, then this is an advantage.

The observation time is important for an experiment of this kind. The shortest meaningful observation time seems to be on the order of 10 days. This is sufficient to define height differences at the 10-meter level if the offset between the clocks can be determined within ±1 ns. The drift rate between the two clocks is predicted to be about 10^{-15}/day, which has to be eliminated by calibration.

Our entire measuring procedure for the time trap is based on the technique of phase-measuring. It has the advantage of benefiting from very long integration times. The movable VLBI instruments have masers with so-called "auto-tuning." Technical data for this kind of tuning are rather limited, but the advantage of the tuning technique seems to be limited to a situation where two masers can be operated simultaneously at the same position. This might indicate that two masers have to be used together in the transportation van to preserve optimal stability. Any practical application has to be preceded by field tests of the masers.

Early application of the VLBI technique can be done without previous advanced tuning if two stations of known geopotential difference are available. We can then compute the theoretical beat frequency caused by the difference in proper time. We will have

$$\Delta f_{observed} = \Delta f_{proper} + \Delta f_{instrumental}$$

where $\Delta f_{observed}$ is the beat frequency computed from the VLBI operation, Δf_{proper} is the theoretical beat frequency corresponding to the known geopotential difference, and $\Delta f_{instrumental}$ is the tuning error.

The tuning error is now determined for the system of two instruments. One of the instruments can then be moved to a new position and the geopotential difference is directly obtainable from the observed beat frequency after due correction for the tuning error. All determinations of new geopotential differences can be made in a sequence that is terminated by a new determination of the tuning error at the primary points. This procedure assumes that the atomic clock can be moved without being detuned.

The wavelength in the VLBI system is 3.8 cm. The portable instrument has an antenna of only 4 m diameter. This means that the signal-to-noise ratio is less favorable than for the stationary system. It seems questionable if the movable system can be used for the generation of "universal time" with satisfactory accuracy for a relativistic measuring system. Tests will be required for an evaluation.

The stationary VLBI instruments cannot be used in experiments where the clocks have to be moved. However, VLBI observations can still be used for showing how well the clocks can present proper time determined in the following way:

1. An extra movable hydrogen maser is used in a closed loop with the extreme points at the stationary instruments. Beat frequencies are recorded at each occupation of a stationary point.

2. With two movable hydrogen masers, the entire measuring program of the time trap can be established and additional information about drift can be obtained.

10. OUTLINE OF AN EXPERIMENT

Since the clocks are sensitive to the environment, it will be necessary to give special consideration to the following factors:

1. Gravity
2. Magnetic field
3. Temperature
4. Humidity
5. Pressure

The clocks might have slightly different correlations due to these factors, and individual calibrations may be needed.

It might be useful to mount the two clocks in the same van and then take complete records while transporting them on the road. If this cannot be done, then the transportation phase can be investigated by using two vans with hydrogen masers and simultaneous transportation.

The time-trap approach has meaningful application in combination with stationary VLBI, mobile VLBI, and GPS. The clocks used in combination with stationary VLBI

are most often mounted in a fixed position. Any application has to be made with a set of two independent, moving clocks or at least one moving clock.

The clocks in the mobile VLBI are movable and some of them are mounted in vans. They also have power equipment to allow measurement during transportation. The vans have a temperature-controlled room for the clocks. If temperature stabilization in the room is kept within 0.015°C, then satisfactory frequency stabilization can be expected.

A preliminary test should be of great interest. Two clocks should be mounted in the same van and records taken of the beat frequency during transportation. With only one moving clock available we can check the beat frequency against a stationary hydrogen maser before and after transportation.

11. BOUNDARY VALUE PROBLEMS AND RELATIVISTIC OBSERVATIONS

The free-boundary value problem lacks rigorous solutions for a realistic topography (Hormander 1975). This problem is improperly posed in a number of situations that deserve geodetic recognition. Useful solutions are considered mostly with the aid of an intermediate auxiliary surface, the telluroid. Originally, the geoid was used as a reference surface for the computation of the potentials at the physical surface and in space. The limiting accuracy of the geopotential differences obtained from geoid calculations is presently estimated as ±0.5 m over global distances (Rapp 1984).

The fixed boundary value problem was first considered by Koch and Pope (1972) who proved existence and uniqueness for the linear case, and uniqueness if there was a solution for the nonlinear case. Bjerhammar and Svensson (1983) proved uniqueness and existence for the nonlinear case. The observed quantity here is gravity, and the surface is considered known. The problem assumes that grad $|W| = g$ is given on the surface, where W is the geopotential. The harmonic potential V is unknown and has to be computed. This is an oblique derivative problem. This problem has interesting applications if VLBI, GPS, and LAGEOS are used to define the surface of the Earth. The number of useful points is still limited.

The mixed boundary value problem was solved for continents of limited size by Holota (1980), and Svensson (1983) gave the proof for existence and uniqueness for continents of arbitrary size. This problem corresponds to the case where gravity has been observed on the continents and sea surface topography has been measured by geometric means (altimetry). It should be noted that this problem will have an infinite number of singularities at the borders of the two sets of observations. Svensson overcomes these difficulties with the aid of operations in Sobolev space. Numerous geodetic solutions have been presented in the geodetic literature ignoring the existence of the mixed boundary value problem. These solutions might be useful for the actual purpose but they will not be uniformly convergent.

The mixed boundary value problem, according to Holota and Svensson, involves almost intractable mathematical difficulties in a geodetic approach. No practical application with a new kind of solution has yet been published. So far, there have been no realistic methods that can replace gravity observations.

Our present relativistic method of observing geopotential differences opens the possibility of overcoming the singularity problem when combining terrestrial and sea-surface observations. We can choose the following sets of observations:

1. Relativistic observations of geopotential differences over continents.

2. Altimetric observations of sea surface topography over the oceans.

Let W denote the geopotential at a point P, and V the harmonic potential. The surface is now considered known and we compute

$$V = W - (\omega^2 r^2 \cos^2\phi)/2$$

where r is the geocentric distance of the point P, and ϕ geocentric latitude. We can now compute the potential V* on an internal sphere with center at the center of gravity. The potential V is strictly harmonic and we can use the well-known Poisson integral formula

$$V_j = 4(\omega r_0)^{-1}(r_j^2-r_0^2) \iint_S V^*(r_j^2+r_0^2-2r_j r_0 \cos\omega_{ji})^{-3/2})dS$$

where V_j is the potential, and r_j the geocentric distance of a point j on the surface of the Earth. Furthermore, r_0 is the radius of the internal sphere, and ω_{ji} the geocentric angle between the point P_j and the moving point on the sphere S. In a discrete approach, we can use the matrix equation

$$V_j = A\ V^*$$

where

$$A_{ji} = (s-s^3)(1+s^2-2s\cos\omega_{ji})^{-3/2}$$

$$s = r_0/r_j.$$

In this approach V* represents the potential as a Dirac quantity on the internal sphere with the vertical through the given point on the physical surface. The justification for a downward continuation of this kind is given by the theorem of Walsh (1929).

Potentials outside the surface of the Earth can be computed after reversing the procedure and using the Poisson integral in the classical way. Potentials for points on the surface between the given points can also be computed from the obtained V*-values. These predicted potentials will have a much lower accuracy than the directly observed potential. However, the geoid is a smooth surface and the justification for the procedure is obvious.

If the determination of a vertical datum is of main importance, then a number of alternatives can be considered for the determination of geopotential differences and dynamic heights.

1. Geometric heights are determined from VLBI and GPS. Dynamic heights are computed through the free boundary value problem using a densified gravimetric network over the actual area. This procedure is closest to the classical technique.

Advantage: Fully developed technical procedure.

Disadvantage: The classical solutions of the free boundary value problem assuming that the geopotential of the boundary surface was known and the disturbance potentials as well as geometric heights were the unknowns. Furthermore the free boundary value problem is probably improperly posed when it comes to accuracies we now have in mind.

The theoretical justifications for a procedure of this kind are poor, but the technique will probably be successful in an area where local undulations of the geoid are small.

2. Geometric heights are determined from VLBI and GPS. Dynamic heights are computed through the fixed-boundary value problem. A densified gravimetric network over the actual network is also considered here.

Advantage: Fully developed technical procedure and a rigorous theory.

Disadvantage: The method is of an indirect nature and formally makes use of an infinite number of observations. Only a limited number of strictly useful observations can be included. Whenever new observations are available, then these might have an impact on the already computed potential difference. (These objections can also be made against the free boundary value problem.)

3. Direct observations of geopotential differences:

a. Leveling combined with gravity observations
b. Hydrostatic leveling
c. Relativistic approach

Meaningful comparisons can be made between these three dynamic methods. Traditional leveling is probably not as accurate as earlier estimates indicated. Systematic errors from the magnetic influence on automatic compensators remained undiscovered for 10 years, which means that the true accuracy in leveling can be questioned. However, there is little doubt that it is the most accurate measuring procedure for obtaining "potential differences" for distances up to 1,000 km.

The relativistic approach represents a measuring technique that never before has been applied in a technical procedure for estimating useful quantities. It has been mainly a question of applying corrections of relativistic nature. There seems to be good justification for an application of this new technique over large distances and perhaps between continents.

A comparison between classical and relativistic geodesy follows:

Classical geodesy (indirect approach by integral equations):
INPUT: Approximate boundary surface + gravity + integral equations.
OUTPUT: Geometry + potentials.

Relativistic geodesy (direct approach by observations):
INPUT: Fixed (known) boundary surface (VLBI, laser ranging, GPS) + proper time.
OUTPUT: Potentials.

Classical geodesy used in a linear approach has an inherent accuracy of about 3×10^{-3}. The nonlinear approach is not strictly valid for arbitrary topography. Hörmander (1975) gave proofs for existence and uniqueness when assuming a "smooth surface" (Hölder class $H^{2+\varepsilon}$)

Relativistic geodesy includes a potential theory that is strictly valid to an accuracy of about 10^{-11} (if the Einstein metric is errorless).

The weakness in the classical approach is mainly related to the theoretical foundations. The weakness in the relativistic approach is at present linked to instrumental problems. However, with the development of new precise clocks, future applications seem very promising. Potential improvements in the technique are only limited by the Heisenberg uncertainty principle. Accuracies at the centimeter level should be completely realistic over continental distances. The geometric coordinates from VLBI, LAGEOS, and GPS have already reached these accuracies. However, geometric heights have only a limited application. Any use of water requires dynamic heights. The relativistic approach can serve here as the natural tool to exploit the precise geometric data from modern technology.

12. VALIDITY OF GENERAL RELATIVITY

The most early verifications of Einstein's theory of general relativity were presented as determinations of the bending of light when it approached the surface of the Sun. This bending can be expressed by (Misner et al. 1973)

$$d = (1+\gamma)GM_s(1+\cos\alpha)^{1/2}(1-\cos\alpha)^{-1/2}r^{-1}$$

where G is the Newtonian constant, M_s solar mass, r geocentric distance of the Sun, and α geocentric angle between the light source and the Sun. The constant γ should be equal to 1 if Einstein's theory is correct. If light is unaffected by gravity, then γ should be -1. The most accurate direct determination of the bending was obtained from VLBI observations, yielding a γ-value of 1.008±0.005 (Robertson and Carter 1984). Indirect determinations over velocity changes have given slightly more accurate determinations, but are not directly comparable.

Direct observations of the change in proper time with the geopotential were observed rather late. (See Pound and Rebka (1959, 1960) and Pound and Snider (1965).) These studies verify the Einstein approach for general relativity within ±0.01.

Vessot (1979) used a spacecraft with a hydrogen maser for comparison with another hydrogen maser on the ground. The spacecraft was launched in a nearly vertical orbit. The results confirmed the Einstein theory "to a precision of 70 parts per million." Experiments designed to replace the metric (Einstein) system in favor of a non-metric system have not yielded any significant findings.

It should finally be noted that our studies of the geopotential in a relativistic approach have to include the potentials from the Sun and the Moon. Since we are primarily concerned with observations over longer time intervals, we have not considered the inclusion of corrections for tides.

13. GEODETIC APPLICATION OF THE RELATIVISTIC APPROACH

Corrections for special relativity have already been applied frequently in astronomic and geodetic operations. Robertson and Carter (1984) showed that corrections for general relativity are important for successful application of the VLBI technique.

Our previous analysis has given the necessary theoretical foundation for geodetic application of relativistic estimates of geopotential differences. The state of the art for clock designs has also been covered to some extent. Techniques for the elimination of nonstochastic behavior of the clocks have been outlined. Appropriate geodetic applications still remain to be found.

Determinations of the geoid over the continents are mostly made after observing gravity anomalies for equal areas of 5° × 5° blocks or 1° × 1°blocks. Two of the latest publications of mean values for 1° × 1° blocks indicated differences for individual blocks of about 100 mgal. These very large differences are perhaps somewhat surprising. They correspond to about 300 m difference for the geopotential surface (free air). The main problem is not the observational accuracy, because good gravity meters yield standard errors below ±0.1 mgal.

The real problem is that the gravity anomaly represents very local information and is extremely difficult to evaluate for a global solution of the gravimetric boundary value problem.

A promising alternative can be outlined in the following way:

For any actual surface element (or surface element to be tested) we observe the geopotential difference by the relativistic technique and the geometric coordinates by GPS or VLBI.

A global solution is obtained from discrete observations on the continents (and the oceans) when using harmonicity down to an internal sphere. The smoothest solution satisfying the given observations should be the most logical intermediate solution. (Smoothness is referred to the physical surface.)

With a known surface and known geopotentials, we can compute the potential anywhere in space by the Dirichlet problem.

The geodetic problem will always be a "prediction problem" because we have information only at discrete points.

Another interesting application can be expected in the determination of the vertical datum for different continents or remote islands. There is also a related

problem for the determination of geopotential differences over large continents. Records from Canadian leveling indicate that errors of the order of 3 m can be suspected over continental distances. Probably still larger errors can be found in other parts of the world for leveling over similar distances.

The theoretical problems in the relativistic approach are not directly crucial. The instrumental problems still have to be carefully considered. This report indicates that present hydrogen masers seem to be the most convenient precise clocks now available for relativistic operations. Some restrictions or limitations should be noted:

1. Up to now, hydrogen masers were designed in such a way that the cavity had to be tuned exactly to the transition frequency of hydrogen. This technique is critical and leads mostly to a mismatch after a limited time interval. A more promising technique will be to use a tuning that utilizes equal detuning on both sides of the resonant frequency.

2. The frequency switching servosystem of Peters is expected to give an improvement in stability of up to 100 times compared to clocks of traditional design and using observation intervals greater than 10^5 seconds. It seems questionable if any competitive system is available today. Further improvements are expected to follow from the new technique.

3. Any relativistic estimation procedure will benefit from using several clocks at the observation site.

4. Operations with moving clocks will require improved magnetic shielding. This can be accomplished by using an external chamber fully shielded and large compared to the internal clock. The natural chamber will be the van used for the transportation of the clock. Peters (1984) states, "The only way to improve the shielding effectiveness in a maser is to make the outer shield as large as possible, the inner shield as small as possible, and to space the intermediate shields properly...." Traditional shielding has a shielding factor of 1,800 and a corresponding frequency dependence of $2 \times 10^{-13}/G$ (where G is the magnetic flux in Gauss). Two additional shields at large distance are expected to give at least 100 times better shielding. (See also Mattison and Vessot 1982.)

14. AN EVALUATION OF THE RELATIVISTIC APPROACH

The relativistic approach to geodesy differs so much from traditional geodetic techniques in Newtonian physics that a comparison cannot be applied in a straightforward manner. Let us make an evaluation against present Doppler methods based on the Transit system. The system has given submeter accuracy in spite of ionospheric errors up to 900 m. The relativistic approach should give better results with currently available instruments in the time-trap approach.

Systematic errors:

	Ionosphere at 40° lat. Error 12 hr (m)	Error/hr (m)	Atmosphere error (m)	Clock drift/day (m)	Clock drift/hr (m)
Doppler[1]					
400 MHz	120	10	5	not dominant	—
150 MHz	900	75	5	—	—
Relativity:					
Poor clocks[2]	—	0	0	< 100	<4
Good clocks	—	0	0	10	0.4
Excellent clocks	—	0	0	< 1	0.04

Stochastic errors:

Doppler in translocation mode with 150 MHz and 400 MHz signals: ±10 m/100 s

Relativity: ±10 m/1,000 s

[1]On the computation of ionospheric path delays for VLBI from satellite Doppler observations, see Campbell and Lohmar (1982).
[2]See Mattison and Vessot (1982).

15. CONCLUDING REMARKS

The relativistic approach offers completely new techniques for direct observation of potential differences over continental distances independent of line of sight and without involving satellites. Some experiences with relativistic experiments have been reported by Hafele and Keating (1972), Pound and Snider (1965), and Vessot (1979).

An early application of the discussed relativistic approach seems most promising in a simple use of two moving clocks in a proper time trap. Crude tests with present hydrogen masers are expected to give height differences with standard errors of about ±100 m. The hydrogen maser seems to be the most promising clock now available. (See Rueger (1981), Mattison and Vessot (1982), Peters (1984), and Walls and Persson (1984).) The stability of present clocks is limited to 10^{-15}. This kind of stability is mostly measured by the Allan variance (least squares). Optimum performance is obtained after about 1,000 s. This kind of measure indicates that stability declines with increasing observation time, and "flickering" is blamed for the unsatisfactory performance (Vessot 1976).

Repeated operations in closed loops are expected to eliminate most of the bias from the clock drift. The measurements can be performed as phase (or time) observations with a resolution of ±10 ps. The corresponding ultimate performance will then be:

$s_0 = \pm 10$ ps $\qquad s_h = \pm 1.4$ m for a time interval of 1 day

$s_h = \pm 0.1$ " " " " " 10 days

where s_0 is the resolution, and s_h the standard deviation. Uncompensated bias must be added to these errors. Errors from the drift are given by the time misclosure of the closed loop (P-Q-P). The relative frequency drift is estimated from observations of the beat frequencies at three positions (P-Q-P). If the measuring time is 100 times longer than the transportation time, then there is a corresponding reduction of the transportation error. No estimates of the transportation errors can be made without field operations.

Most hydrogen clocks will probably not be operational while in transport. A new design by Peters (1984) appears to revolutionize the technology of stabilizing the frequency from a hydrogen maser. The frequency stabilization is accomplished by using a servoswitching technique which locks the frequency of the maser to the natural transition frequency of hydrogen. There is a forced detuning with equal amounts on both sides of the transition frequency. This means that the maser will be locked in a very stable position. A forced offset of 10^{-12} is compensated for within 2 hours. These results look very promising.

The principal error source for this new technology will presumably be second-order Zeeman effects. A relativistic design probably requires additional magnetic shielding of the entire transportation van. This should give an expected magnetic shielding of better than 100,000.

There is a very impressive development of new precise clocks that utilizes superconducting systems as well as trapped ions and laser cooling. (See Bollinger et al. (1983), Cutler et al. (1981), Dehmelt (1982), Itano et al. (1983), Wayne et al. (1983), and Wineland et al. (1981).)

Superconducting cavity resonators have been made with short-term stability of 10^{-16} and new designs promise a stability of 10^{-18}. (See Stein et al. (1975), Turneaure et al. (1983), and Dick and Strayer (1984).)

Cesium standards have been successfully deployed in the Global Positioning System (Hellwig and Levine 1984).

A geodetic clock might benefit from a hydrogen maser generating a primary signal, which is filtered by a superconducting cavity resonator with short-term stability on the order of 10^{-18}. A successful geodetic application will most certainly require a pair of specially matched clocks.

Relativistic networks should benefit from constraints based on the determination of geopotential differences from traditional precise leveling.

There is a possibility of using a geostationary satellite link in combination with the method. For approximately a 1,000 km distance and an observational interval of 10 days a stability of 3×10^{-16} was given by Knowles et al. (1982).

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APPENDIX.--CLOCK PARAMETERS

The values listed below give crude estimates of Q-values as well as ultimate stability (square root of Allan variance for 1,000 s).

Clocks	Q	Instability
RF clocks		
Quartz crystal	10^6	$\pm 10^{-10}$
Rubidium	10^7	$\pm 2 \times 10^{-11}$
Cesium	10^8	$\pm 2 \times 10^{-13}$
Hydrogen	10^9	$\pm 10^{-15}$
Laser clocks[1]		
Beryllium (Bollinger et al. 1983)	10^{10}	
Mercury (Wineland et al. 1981)	3×10^{12}	
Thallium (Dehmelt et al. 1982)	7×10^{14}	
Barium (Schneider and Werth 1979)	2×10^{16}	
Indium (Bollinger et al. 1983)		$\pm 10^{-19}$
Cavity resonators		
Room temperature	10^6	$\pm 10^{-9}$
Superconducting (Turneaure et al 1983)	4×10^{10}	$\pm 3 \times 10^{-16}$ (100s)

Definition:

$$Q = 2f/(f_2 - f_1)$$

where f is resonant frequency, f_1 lower frequency of half amplitude and f_2 upper frequency of half amplitude.

[1]Laser clocks are not available on the open market. Tests with beryllium clocks have been reported.

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