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Statistical Tests for Detecting Crustal Movements Using Bayesian Inference

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K. R. Koch

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by K. R. Koch[1]
National Geodetic Survey
Charting and Geodetic Services
National Ocean Service, NOAA
Rockville, Md. 20852

ABSTRACT

The test of a general linear hypothesis in sampling theory, applied to the detection of recent crustal movements, is compared with Bayesian inference based on vague prior distributions. Both approaches give equivalent results. In addition, the test of inequality constraints can be readily derived by Bayesian inference. Also, less sensitive and more realistic tests than the ones of the sampling theory for the detection of crustal movements are obtained by Bayesian inference. These tests are simple to apply, since the distribution needed is the central F-distribution. As an example, two epochs of leveling data are analyzed in the Houston-Galveston, Tex., region, an area of marked land subsidence.

1. INTRODUCTION

For the detection of crustal movements by geodetic methods, networks are established by measurements repeated at different time epochs. Coordinates are then obtained at the different epochs for the points in the network, and coordinate differences can be formed. Since the geodetic observations are random quantities, not all coordinate differences can be attributed to crustal movements. By hypothesis testing a decision is reached whether a coordinate difference is significantly different from zero so that crustal movement can be assumed. The hypothesis to be formed equates the coordinates of certain points of one epoch to the coordinates of the following epoch. If the hypothesis is accepted, these points can be considered as fixed points between the two epochs; if the hypothesis is rejected, points which have moved are found. Thus by means of hypothesis testing the fixed points can be separated from the points which moved.

Unfortunately, assuming the identity of coordinates of points that did not move between two epochs is not realistic. First, the monumented points might have been subjected to short-period changes caused, for instance, by temperature effects. Then, centering errors in the placement of the instruments can occur when the observations are repeated at different time epochs. Also, different weather conditions at the epochs might influence the pointing of the instruments.

If the hypothesis of the identity of coordinates is applied in areas where there is adequate knowledge of the fixed points, the hypothesis will often be rejected for some points which it is otherwise reasonable to fix, thus finding fewer fixed points than expected. Hence, the test of the identity of the coordinates between different epochs is too sensitive. To overcome this deficiency and obtain less sensitive tests, Koch (1981a) proposed the use of inequalities rather than equalities for hypothesis testing. This means allowing intervals for the values of the coordinate differences under the hypothesis rather than constraining them to zero. However, by using sampling theory, it is difficult to derive the distribution of the test statistic for such a hypothesis. A simpler solution to this problem of obtaining less sensitive tests is derived by Bayesian inference.

For some time the Bayesian approach was considered subjective because it required the introduction of prior information. In the last two decades, however, the use of vague or noninformative priors has been shown to give results equivalent to the sampling theory. In the Bayesian approach the parameters are random variables whose distribution can be computed if the observations are given. The test of hypotheses in sampling theory corresponds to the determination of the probabilities for the subspaces of the parameter space defined by the hypotheses. Hypotheses that restrict the parameter space, for instance by inequalities, can therefore be handled more simply by the Bayesian approach than by sampling theory. Thus, the two methods do not compete with each other but complement each other.

The Bayesian approach was first introducted to geodesy by Bossler (1972) who investigated its use for geodetic problems, and applied point estimation under inequality constraints. Riesmeier (1984) showed the advantages of

[1] Permanent address: Institut für Theoretische Geodäsie der Universität Bonn, Nussallee 17, 5300 Bonn 1, Federal Republic of Germany.

This research was performed during July-Sept. 1983, while the author was a Senior Visiting Scientist at the National Geodetic Survey, under the auspices of the Committee on Geodesy, National Research Council, National Academy of Sciences, Washington, D.C.

the Bayesian approach over sampling theory in the case of the test of hypotheses with inequalities. He also derived less sensitive tests for the detection of crustal movements by Bayesian inference. These tests are applied in this report to study land subsidence from two epochs of leveling data in the Houston-Galveston area. The tests are simple to use since the probability of an F-distributed random variable has to be computed if the model for estimating the unknown parameters is the univariate model. Hence, the same distribution is used as for the hypothesis testing in sampling theory.

The test of the identity of coordinates for the detection of crustal movements assumes that no information is available on the nature of the movements which might have occurred. But if the test, for instance, reveals equal shifts for a group of points, the problem of estimating the coordinates has to be reparameterized by replacing some unknown coordinates by translational or rotational parameters. Here again the problem of obtaining less sensitive tests for the parameters arises, which can be solved by the approach given below.

In the following sections, a short outline of hypothesis testing in the sampling theory for detecting crustal movements in a univariate model is given. Then the Bayesian approach leading to less sensitive tests is described, and finally the tests are applied to leveling data.

2. HYPOTHESIS TESTING IN SAMPLING THEORY

We start with the Gauss-Markof model for the estimation of unknown parameters

$$\mathbf{X}\,\beta = E(\mathbf{y}),\ D(\mathbf{y}) = \sigma^2\mathbf{I} \tag{2.1}$$

where β is the u x 1 vector of unknown fixed parameters, $\mathbf{y}$ the n x 1 vector of random observations, $\mathbf{X}$ the n x u matrix of known coefficients, σ^2 the variance of unit weight, and $\mathbf{I}$ the identity matrix. The operators leading to the expected values and to the variances and covariances of the random variables are denoted by E and D respectively. The special model (2.1) with the covariance matrix $\sigma^2\mathbf{I}$ for the observations can be obtained from a general model with the covariance matrix $\sigma^2\mathbf{P}^{-1}$, where $\mathbf{P}$ is the weight matrix, by a linear transformation. The model is assumed not to be of full rank, that is, $R(\mathbf{X}) = q < u$.

To analyze observations of different time epochs in order to detect recent crustal movements, the model (2.1) can be applied. If the configuration of the network and the design of the observations do not change over all epochs, the multivariate model for estimating parameters could be used. However, the leveling data to be analyzed here do not fulfill the requirements of the multivariate model, so that this model will not be considered. In addition, only the analysis of two epochs of observations is treated here, since this analysis is the most important one. It can be easily extended to more than two epochs of data (Koch 1981b).

Let $\mathbf{y}_1$ be the n_1 x 1 vector of observations of the first epoch to be analyzed, β_1 the u_1 x 1 vector of unknown coordinates of the network points for the first epoch, and $\mathbf{X}_1$ the n_1 x u_1 matrix of known coefficients with $R(\mathbf{X}_1) = q_1 < u_1$. Let $\mathbf{y}_2$ be the n_2 x 1 vector of observations of the second epoch to be compared with the first one, β_2 the u_2 x 1 vector of unknown coordinates of the network points for the second epoch, $\mathbf{X}_2$ the n_2 x u_2 matrix of known coefficients with $R(\mathbf{X}_2) = q_2 < u_2$ and σ^2 the variance of unit weight. In contrast to the multivariate model the observations of the first and second epoch are considered as independent so that instead of eq. (2.1) the following model is obtained:

$$\begin{vmatrix}\mathbf{X}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_2\end{vmatrix}\begin{vmatrix}\beta_1\\ \beta_2\end{vmatrix} = E\left(\begin{vmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{vmatrix}\right),\ D\left(\begin{vmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{vmatrix}\right) = \sigma^2\begin{vmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}\end{vmatrix}. \tag{2.2}$$

Free networks are assumed for which the datum is not defined. This is expressed by the rank deficiencies of the matrices $\mathbf{X}_1$ and $\mathbf{X}_2$. The coordinates β_1 and β_2 can, therefore, be only unbiasedly estimated if they are projected onto subspaces whose dimensions are reduced in comparison to the original space by the number $u_1 - q_1$ and $u_2 - q_2$ of constraints necessary to define the datum. If β_{b1} and β_{b2} denote the projected parameters, their best linear unbiased estimates are given by Koch (1980: 171)

$$\begin{aligned}\begin{vmatrix}\hat{\beta}_{b1}\\ \hat{\beta}_{b2}\end{vmatrix} &= \begin{vmatrix}\mathbf{X}_1'\mathbf{X}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_2'\mathbf{X}_2\end{vmatrix}^-_{rs}\begin{vmatrix}\mathbf{X}_1'\mathbf{y}_1\\ \mathbf{X}_2'\mathbf{y}_2\end{vmatrix}\\ &= \begin{vmatrix}(\mathbf{X}_1'\mathbf{X}_1)^-_{rs} & \mathbf{O}\\ \mathbf{O} & (\mathbf{X}_2'\mathbf{X}_2)^-_{rs}\end{vmatrix}\begin{vmatrix}\mathbf{X}_1'\mathbf{y}_1\\ \mathbf{X}_2'\mathbf{y}_2\end{vmatrix}\end{aligned} \tag{2.3}$$

where $|\ldots|^-_{rs}$ denotes a symmetric reflexive generalized inverse. This inverse is computed by taking the points, which are considered as fixed between the two epochs of observations, to define the datum (Koch and Fritsch 1981, Koch 1983). It means that minimal constraints are introduced such that the centroid computed from the datum points is fixed from the first epoch to the second, in case a translation suffices to establish the datum, as in leveling nets, where only one constraint is needed. If a rotation or scale is also necessary to define the datum, additional minimal constraints are introduced (Pope 1971).

If the coordinates of the points assumed as fixed between the two epochs are denoted by the vectors β_{f1} and β_{f2}, the hypothesis to test this assumption is given by

$$H_o\colon \beta_{f1} = \beta_{f2} \text{ against } H_1\colon \beta_{f1} \neq \beta_{f2} \tag{2.4}$$

where H_o denotes the null hypothesis and H_1 the alternative hypothesis. The hypothesis is well suited to test for fixed points, but the more points included in the test, the more sensitive the test becomes.

A less sensitive test is obtained by the hypothesis that the coordinates of only one point are equal from one epoch to the next. If β_{i1} denotes the coordinates of the first epoch of the point i to be tested and β_{i2} the coordinates of the second epoch of the point i, the hypothesis is given by

$$H_o: \beta_{i1} = \beta_{i2} \text{ against } H_1: \beta_{i1} \neq \beta_{i2}. \quad (2.5)$$

Depending on the dimensions of the network, β_{i1} and β_{i2} contain one coordinate for a leveling net, two coordinates for a planar network, and three coordinates for a three-dimensional network.

The point i to be tested by hypothesis (2.5) should not belong to the points which define the datum and which are therefore considered as fixed points. Instead, the hypothesis (2.5) has to be used to find fixed points in addition to the points which define the datum. If fixed points are found by (2.5) they should be added to the points which define the datum. Thus, (2.5) is applied after each update of the datum points, so that in an iteration procedure the fixed points are separated from the points which have moved (Koch 1981b).

Using the maximum-likelihood criterion, the test statistic T of the test of a general linear hypothesis of the projected parameters

$$H_o: \mathbf{H}\beta_b = \mathbf{w} \text{ against } H_1: \mathbf{H}\beta_b \neq \mathbf{w}, \quad (2.6)$$

where **H**, is a known r x u matrix, and **w**, a known r x 1 vector, is given by

$$T = \frac{1}{r\hat{\sigma}^2}(\mathbf{H}\hat{\beta}_b - \mathbf{w})'(\mathbf{H}\,\mathbf{X}'\mathbf{X})^-_{rs}\mathbf{H}')^{-1}(\mathbf{H}\hat{\beta}_b - \mathbf{w}). \quad (2.7)$$

$\hat{\beta}_b$ and $\hat{\sigma}^2$ are the estimates of β_b and σ^2. If the null hypothesis is true, the test statistic T is distributed as the central F-distribution

$$T \sim F(r, n-q) \quad (2.8)$$

so that the hypothesis is rejected if

$$T > F_{1-\alpha;\, r,\, n-q} \quad (2.9)$$

where $F_{1-\alpha;\, r,\, n-q}$ is the upper α-percentage point of the F-distribution.

The null hypothesis of (2.5) expressed in the general form (2.6) is given by

$$H_o: |\mathbf{O}, \ldots, \mathbf{O}, \mathbf{I}, \mathbf{O}, \ldots, \mathbf{O}, -\mathbf{I}, \mathbf{O}, \ldots, \mathbf{O}| \begin{vmatrix} \beta_{b1} \\ \beta_{b2} \end{vmatrix} = \mathbf{O} \quad (2.10)$$

where the unit matrices **I** are positioned such that the coordinates of the point i in β_{b1} and β_{b2} are selected. By substituting (2.10) into (2.7) we get, together with (2.3),

$$T = \frac{1}{r\hat{\sigma}^2}(\hat{\beta}_{i1} - \hat{\beta}_{i2})'(((\mathbf{X}_1'\mathbf{X}_1)^-_{rs})_i + ((\mathbf{X}_2'\mathbf{X}_2)^-_{rs})_i)^{-1}(\hat{\beta}_{i1} - \hat{\beta}_{i2}) \quad (2.11)$$

where $((\mathbf{X}_1'\mathbf{X}_1)^-_{rs})_i$ contains the elements of $(\mathbf{X}_1'\mathbf{X}_1)^-_{rs}$ necessary to compute the variances and covariances of the coordinates of point i. For a leveling net, the vectors $\hat{\beta}_{i1}$ and $\hat{\beta}_{i2}$, have only one element so that with r = 1 the test statistic T is obtained by

$$T = (\hat{\beta}_{i1} - \hat{\beta}_{i2})^2 / (\hat{\sigma}^2(((\mathbf{X}_1'\mathbf{X}_1)^-_{rs})_{ii} + ((\mathbf{X}_2'\mathbf{X}_2)^-_{rs})_{ii})) \quad (2.12)$$

where $((\mathbf{X}_1'\mathbf{X}_1)^-_{rs})_{ii}$ is now the i-th diagonal element of the matrix $(\mathbf{X}_1'\mathbf{X}_1)^-_{rs}$.

The points most likely to be fixed points are the ones which give small values for the test statistic T. They are tested first, in order to find the fixed points.

3. BAYESIAN INFERENCE

We start with the linear model

$$\mathbf{X}\beta = \mathbf{y} + \mathbf{e},\ D(\mathbf{e}) = \sigma^2\mathbf{I} \quad (3.1)$$

where, in contrast to the Gauss-Markof model (2.1), the u x 1 vector β of unknown parameters is not a fixed vector but a random vector. The unknown variance of unit weight σ^2 is a random variable, too. Random vectors are also the n x 1 vector **y** of observations and the n x 1 vector **e** of errors, while **X** is a known, fixed n x u matrix.

The probability density $p(\beta, \sigma|\mathbf{y})$ of the unknown parameters β and σ, under the condition that the observations **y** are given, is obtained by Bayes' theorem

$$p(\beta, \sigma|\mathbf{y}) \propto p(\beta, \sigma)\, p(\mathbf{y}|\beta, \sigma) \quad (3.2)$$

where $\propto$ denotes proportionality. The density $p(\beta, \sigma|\mathbf{y})$ is also called the posterior density in contrast to the prior density $p(\beta, \sigma)$ of the parameters; and $p(\mathbf{y}|\beta, \sigma)$, interpreted as a function of β and σ, is the likelihood function.

To derive the posterior density for β, the multivariate normal distribution is assumed for the error vector **e**

$$\mathbf{e} \sim N(\mathbf{O}, \sigma^2\mathbf{I}) \quad (3.3)$$

By transforming **e** to **y** by means of (3.1), the likelihood function $p(\mathbf{y}|\beta, \sigma)$ is obtained, where the unknown parameters are replaced by the sufficient statistics

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \text{ and } \hat{\sigma}^2 = (\mathbf{X}\hat{\beta} - \mathbf{y})'(\mathbf{X}\hat{\beta} - \mathbf{y})/(n-u) \quad (3.4)$$

if a model (3.1) of full rank is given. Vague priors, also called noninformative priors, are introduced for the unknown parameters β and σ. Then it can be shown that the posterior distribution of the unknown parameter β under the condition, that the observations **y** are given, is the multivariate t-distribution (Box and Tiao 1973: 117; Zellner 1971: 67)

$$\beta \sim t(\hat{\beta}, \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}, n-u) \quad (3.5)$$

with the density

$$p(\beta|\mathbf{y}) = \frac{\Gamma(n/2)\det(\mathbf{X}'\mathbf{X})^{1/2}\,\hat{\sigma}^{-u}}{\Gamma((n-u)/2)((n-u)\pi)^{u/2}} \times \{1 + (\hat{\beta} - \beta)'\mathbf{X}'\mathbf{X}(\hat{\beta} - \beta)/((n-u)\hat{\sigma}^2)\}^{-n/2} \quad (3.6)$$

A linear combination $\mathbf{H}\beta$ of the vector β of unknown parameters, with **H** being an r x u matrix of full row rank, is also distributed as the multivariate t-distribution

$$\mathbf{H}\beta \sim t(\mathbf{H}\hat{\beta}, \hat{\sigma}^2\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}', n-u) \quad (3.7)$$

In models not of full rank with $R(\mathbf{X}) = q < u$, the density of the unknown parameters β is still proportional to

$$p(\beta|\mathbf{y}) \propto \{1 + (\bar{\beta} - \beta)'\mathbf{X}'\mathbf{X}(\bar{\beta} - \beta)/((n-q)\hat{\sigma}^2)\}^{-n/2}$$

as in (3.6), with $\bar{\beta}$ being an estimate of β. But the density cannot be normalized because of $\det(\mathbf{X}'\mathbf{X}) = 0$. For linear transformations of the unknown parameters that lead to estimable functions, like the linear transformations $H\beta_b$ of the projected parameters β_b, we get from eq. (3.7) the distribution

$$H\beta_b \sim t(H\hat{\beta}_b, \hat{\sigma}^2 H(X'X)^-_{rs} H', n\text{-}q) \tag{3.8}$$

where $\det(H(X'X)^-_{rs} H') \neq 0$.

By means of the multivariate t-distributions (3.5), (3.7) or (3.8) any statistical inference concerning the unknown parameters or linear transformations of the parameters may be performed. For instance, if the parameter space is restricted by the inequality constraints $H\beta > w$, the probability associated with this restricted space is given by

$$P(H\beta > w|y) = \int_B p(\beta|y)\,d\beta \text{ with } B = \{\beta : H\beta > w\}. \tag{3.9}$$

Of course, the integration of the multivariate t-distribution is difficult, especially if the limits of the integration are not orthogonal (Riesmeier 1984). However, in general, the hypothesis testing with inequalities is solved by eq. (3.9), since the hypothesis $H\beta > w$ is rejected, if

$$P(H\beta > w|y) > 1 - \alpha$$

with α being the significance value of the test. This can be seen by assuming $H\beta > w$ as the region of acceptance for the test of the hypothesis, if it is true. Hence, by integrating over the region of rejection the probability of the error of the first kind is obtained to be smaller than the significance level α. However, for a test of a hypothesis the probability of the error of the first kind has to be smaller than or equal to α, so that (3.10) is valid.

4. LESS SENSITIVE TESTS

To avoid the integration of the multivariate t-distribution, a special case of eq. (3.9), which leads to the test (3.10) of the hypothesis $H\beta > w$, is now considered. This gives tests for detecting recent crustal movements which are less sensitive than the ones of the sampling theory. By means of Bayesian inference, results are obtained which are equivalent to the test of a general linear hypothesis in sampling theory because the quadratic form in the density of (3.8) is distributed as the central F-distribution (Box and Tiao 1973: 117; Zellner 1971: 385)

$$(H\hat{\beta}_b - H\beta_b)'(H(X'X)^-_{rs}H')^{-1}(H\hat{\beta}_b - H\beta_b)/(r\hat{\sigma}^2) \sim F(r, n\text{-}q). \tag{4.1}$$

If the left side is set equal to the upper α-percentage point $F_{1-\alpha;\, r,\, n-q}$ of the F-distribution, a hyperellipsoid B is obtained (Koch 1980: 259) with the probability

$$P(H\beta_b \in B|y) = 1 - \alpha \tag{4.2}$$

where

$$B = \{H\beta_b: (H\hat{\beta}_b - H\beta_b)'(H(X'X)^-_{rs}H')^{-1}(H\hat{\beta}_b - H\beta_b)/(r\hat{\sigma}^2) \sim F_{1-\alpha;\, r,\, n-q}\}$$

so that the midpoint of the hyperellipsoid B is $H\hat{\beta}_b$. Depending on the values we assign to $H\beta_b$, for instance $H\beta_b = w$, we get points w_i which lie inside the hyperellipsoid B and points w_a which lie outside the hyperellipsoid. For the hyperellipsoids B_i and B_a determined by w_i and w_a we have

$$P(H\beta_b \in B_i|y) < 1 - \alpha \tag{4.3}$$

and

$$P(H\beta_b \in B_a|y) > 1 - \alpha \tag{4.4}$$

Hence, the test of the general linear hypothesis (2.6) is equivalent in Bayesian inference to the check of whether the point $H\beta_b = w$ lies inside or outside the hyperellipsoid defined by eq. (4.2). If it lies outside, the hypothesis has to be rejected according to eqs. (3.10) and (4.4). The probability in eqs. (4.3) or (4.4) can be easily computed by means of the F-distribution (4.1). With the test statistic T from eq. (2.7), the hypothesis (2.6) has to be rejected, if

$$\int_0^T F(r, n\text{-}q)\,dF > 1 - \alpha \tag{4.5}$$

or if $T > F_{1-\alpha;\, r,\, r-q}$, as shown in (2.9), so that the Bayesian approach and the sampling theory give the same results.

The probability of the unknown parameters or their linear transformations confined to a restricted parameter space can easily be obtained in Bayesian inference by eq. (3.9). Now, less sensitive tests are found if we restrict the parameter space such that we introduce a region R for the parameters close to their expected values, which is excluded from the statistical inference. Applied to the test of the hypothesis (2.4) or (2.5) of the identity of coordinates, this is comparable to introducing intervals in which the coordinates are allowed to move. The probability density function of the parameters for this region R obtains the value zero. If R has the shape of a hyperellipsoid with the midpoint at $H\hat{\beta}_b$, as in eq. (4.2), all hypotheses with w inside of this region receive the probability zero, since the density of the parameters of their linear transformations in R is zero.

By introducing the zero values for the density, we truncate the density for the parameters in R so that the density for the remaining space has to be renormalized. If Ω denotes the entire space for $H\beta_b$, the space Ω_T, on which the transformed parameters are now restricted, is given by the difference of Ω and R

$$\Omega_T = \Omega \setminus R \tag{4.6}$$

Let the posterior density of the transformed parameters be given by $h(H\beta_b)$. Thus

$$p(H\beta_b|y) = h(H\beta_b) \text{ for } H\beta_b \in \Omega\,; \tag{4.7}$$

then the renormalized truncated density for the restricted space Ω_T is obtained by

$$p_T(H\beta_b|y) = \left(\int_{\Omega_T} h(H\beta_b)\,dH\beta_b\right)^{-1} h(H\beta_b) \text{ for } H\beta_b \in \Omega_T$$

or with eq. (4.6)

$$p_T(H\beta_b | y) = (1 - \int_R h(H\beta_b)\, dH\beta_b)^{-1}\, h(H\beta_b) \text{ for } H\beta_b \in \Omega_T. \quad (4.8)$$

For any region A with $A \subset \Omega_T$ we obtain the probability from eq. (4.7)

$$P(H\beta_b \in A) = \int_A h(H\beta_b)\, dH\beta_b \text{ with } A \subset \Omega_T \quad (4.9)$$

and from eq. (4.8)

$$P_T(H\beta_b \in A) = (1 - \int_R h(H\beta_b) dH\beta_b)^{-1} \int_A h(H\beta_b) dH\beta_b \quad (4.10)$$

Since $A \cap R = \emptyset$ we get

$$P(H\beta_b \in (A \cup R)) = \int_A h(H\beta_b) dH\beta_b + \int_R h(H\beta_b)\, dH\beta_b$$

and therefore (Riesmeier 1984)

$$P_T(H\beta_b \in A) \leq P(H\beta_b \in (A \cup R)) \quad (4.11)$$

since, with

$$P_A = \int_A h(H\beta_b) dH\beta_b \text{ and } P_R = \int_R h(H\beta_b) dH\beta_b$$

we find

$$P_A \leq (1 - P_R)(P_A + P_R) = P_A + P_R(1 - (P_A + P_R))$$

or

$$O \leq P_R(1 - (P_A + P_R))$$

because $P_R \geq O$ and $P_A + P_R \leq 1$. The inequality (4.11) shows that less sensitive tests are obtained if the truncated density for the restricted parameter space is applied. The reason is that the probability obtained with this density for the region A, which will be associated with a hypothesis, is smaller than the probability obtained from the density of the unrestricted space.

Let the region R have the form of a hyperellipsoid with the midpoint at $H\hat{\beta}_b$, hence

$$R = \{H\beta_b: (H\hat{\beta}_b - H\beta_b)' (H(X'X)^-_{rs} H')^{-1} (H\hat{\beta}_b - H\beta_b)/(r\hat{\sigma}^2) < T_R\} \quad (4.12)$$

where

$$T_R = (H\hat{\beta}_b - w_R)' (H(X'X)^-_{rs} H')^{-1} (H\hat{\beta}_b - w_R)/(r\hat{\sigma}^2)$$

so that the size of this region is determined by the vector $H\hat{\beta}_b - w_R$. The density associated with R is given by the F-distribution because of (4.1)

$$\int_R h(H\beta_b) dH\beta_b = \int_0^{T_R} F(r,n\text{-}q) dF \quad (4.13)$$

Let the region $A \cup R$ be also defined by a hyperellipsoid, whose midpoint is $H\hat{\beta}_b$ and whose size is given by the general hypothesis $H\beta_b = w$ in (2.6); hence

$$A \cup R = \{H\beta_b: (H\hat{\beta}_b - H\beta_b)' (H(X'X)^-_{rs} H')^{-1} (H\hat{\beta}_b - H\beta_b)/(r\hat{\sigma}^2) < T\} \quad (4.14)$$

where

$$T = (H\hat{\beta}_b - w)' (H(X'X)_{rs} H')^{-1} (H\hat{\beta}_b - w)/(r\hat{\sigma}^2)$$

Because of $A = (A \cup R) \setminus R$ we get

$$\int_A h(H\beta_b) dH\beta_b = \int_{A \cup R} h(H\beta_b) dH\beta_b - \int_R h(H\beta_b) dH\beta \quad (4.15)$$

where

$$\int_{A \cup R} h(H\beta_b) dH\beta_b = \int_0^T F(r,n\text{-}q) dF$$

By substituting eqs. (4.13) and (4.15) into eq. (4.10) we get the probability for the less sensitive test

$$P_T(H\beta_b \in A) = \begin{cases} (1 - \int_0^{T_R} F(r,n\text{-}q) dF)^{-1} (\int_0^{T} F(r,n\text{-}q) dF - \int_0^{T_R} F(r,n\text{-}q) dF), & \text{for } T_R < T \\ 0 & \text{for } T_R \geq T \end{cases} \quad (4.16)$$

where the exterior of A is limited by the hyperellipsoid with the vector $H\hat{\beta}_b - w$ from the hypothesis (2.6) and the interior by the hyperellipsoid for the region R with the vector $H\hat{\beta}_b - w_R$ in eq. (4.12).

According to eq. (3.10) the hypothesis $H\beta_b = w$ in eq. (2.6) is rejected by the less sensitive test if

$$(1 - \int_0^{T_R} F(r,n\text{-}q) dF)^{-1} (\int_0^{T} F(r,n\text{-}q) dF - \int_0^{T_R} F(r,n\text{-}q) dF) > 1 - \alpha. \quad (4.17)$$

For this less sensitive test the F-distribution only is needed. The sensitivity of the test depends on the size of the region R, which means on the size of the vector $H\hat{\beta}_b - w_R$. The larger the size of this vector, the less sensitive the test becomes. If, on the other hand, the vector equals the zero vector, the less sensitive test gives the same result as the hypothesis testing of the sampling theory.

To apply the less sensitive test, the region R, i.e., the amount of the vector $H\hat{\beta}_b - w_R$, has to be chosen, in which the linear transformation $H\beta_b$ of the parameters are allowed to vary without associating a density with them. This choice, of course, is arbitrary. However, the vector should be selected so that it reflects the differences between the coordinates of the points in the network which may be expected between the two epochs of observations. In such a case the less sensitive test will give more realistic results than the hypothesis test of sampling theory.

5. APPLICATION TO A LEVELING NETWORK

Two epochs of the most recent leveling data in the Houston-Galveston area were analyzed. In this area land subsidence occurs, which is monitored by repeated levelings. The first epoch of the data was obtained in

1978, the second epoch in 1983. The network extends from Galveston in the south to Addicks, Houston, and Crosby in the north. The original observations for 1012 bench marks were condensed such that only 99 bench marks remained for the epoch 1978, and 94 bench marks for the epoch 1983. Eighty-nine of these bench marks are in common to both epochs.

The number n-q of degrees of freedom in the nets of both epochs is 6. Hence, the nets of both epochs are only weakly overdetermined. By assuming independent observations with variances $\sigma_v^2 = s\ [mm^2]$, where s is the distance measured in [km] between two bench marks, the estimate $\hat{\sigma}_1$ of the standard deviation of unit weight for epoch 1978 is $\hat{\sigma}_1 = 3.37$, and for epoch 1983 $\hat{\sigma}_2 = 1.57$. Thus, a considerable difference is encountered between the two epochs, which can be attributed to the small degree of freedom.

For the first combined adjustment of both epochs according to eq. (2.3), a deep seated bench mark common to both epochs was introduced to define the datum. Using a significance level of $\alpha = 0.05$ in the test of eq. (2.5), additional fixed points were found which were then introduced to define the datum. After three additional adjustments, 60 bench marks of the 89 bench marks common to both epochs had to be considered as fixed points. Only for the remaining 19 bench marks were the height differences significantly different from zero. These height differences are greater than 53 mm.

The less sensitive test defined by eq. (4.16) was applied by allowing for height differences of 4 mm, 5 mm, and 6 mm for the same point between the two epochs. These differences are excluded from the statistical inference. The amounts of 4 mm to 6 mm seem to be justified, taking into consideration the long time span of 5 years between observations. Hence, T_R in eq. (4.16) is obtained together with eq. (2.12) by

$$T_R = (\Delta h)^2/(\hat{\sigma}^2(((\mathbf{X}_1'\mathbf{X}_1)_{rs}^-)_{ii} + ((\mathbf{X}_2'\mathbf{X}_2)_{rs}^-)_{ii})) \qquad (5.1)$$

where

$$\Delta h = 4 \text{ mm}, \Delta h = 5 \text{ mm and } \Delta h = 6 \text{ mm}$$

For $\Delta h = 4$ mm, six additional points have to be considered as fixed with the seventh point very close to the boundary of rejection given by the significance level of $\alpha = 0.05$. Based on $\Delta h = 5$ mm and $\Delta h = 6$ mm in (5.1), seven additional points had to be considered as fixed points. Using these points also as datum points, the height differences which turn out to be statistically significant from zero are greater than 93 mm.

In the analysis of the data presented here only seven points in addition to the sixty points common to both epochs were found to be fixed points by the less sensitive test. However, the minimum height differences considered as being statistically significant from zero jumped from 53 mm for the hypothesis test of sampling theory to 93 mm for the Bayesian inference. The standard deviation $\hat{\sigma}$ of unit weight for the combined adjustment (2.3) of the first and second epoch is $\hat{\sigma} = 2.6$, so that the standard deviation of a height difference measured over a distance of 1 km length is 2.6 mm. This standard deviation is small in comparison to the minimum height difference of 93 mm, which is considered significantly different from zero.

Hence, using the standard deviation of unit weight as a measure for detecting height differences, which are statistically significant from zero, could be very misleading.

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