U. S. DEPARTMENT OF COMMERCE
CHARLES SAWYER, Secretary

COAST AND GEODETIC SURVEY
LEO OTIS COLBERT, Director

Special Publication No. 245

EQUAL-AREA PROJECTIONS FOR WORLD STATISTICAL MAPS

By
F. WEBSTER McBRYDE
Geographer-consultant
Office of the Coordinator, International Statistics
Bureau of the Census

and

PAUL D. THOMAS
Senior Mathematician
Coast and Geodetic Survey

UNITED STATES
GOVERNMENT PRINTING OFFICE
WASHINGTON : 1949

For sale by the Superintendent of Documents, U. S. Government Printing Office
Washington 25, D. C. - Price 50 cents (paper cover)

National Oceanic and Atmospheric Administration

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January 1, 2006

CONTENTS

TABLES

ILLUSTRATIONS

PREFACE

This publication presents results of a project which originated with efforts by Dr. F. Webster McBryde, of the Bureau of the Census, to produce a world projection for international census preparation and plottings, especially for use in the 1950 Census of the Americas, on the program of which he is geographer-consultant.[1]

McBryde considered Eckert's No. 6 to be the best of any existing type of projection for fulfilling the requirements,[2] but he desired to improve the scale and general proportions of regions in high and low latitudes.

For the mathematical formulas of flat-polar designs similar to Eckert's projections but with shorter polar lines and, in most cases, meridians of greater curvature, he is indebted to Mr. Paul D. Thomas of the Division of Geodesy, United States Coast and Geodetic Survey. The mathematics involved in deriving formulas for the correct poleward diminution of the intervals between parallels is outside of the general geographer's field of training.

The formulas developed by Thomas not only provide the specifications requested by McBryde for statistical use but also can be employed in producing any desired shapes and proportions in this type of projection. Thomas, in addition to deriving the formulas, computed the tables for construction and plotted some of the projections.

McBryde, as representative of the Inter-American Statistical Institute,[3] presented the flat-polar quartic authalic projection (fig. 26 in this publication) in Buenos Aires at the Fourth Pan American Consultation on Cartography, sponsored jointly by the Pan American Institute of Geography and History and the Government of Argentina, in October-November 1948, with the following proposal:

"It would seem desirable to adopt a common base for a general map of the Americas, especially for census purposes. Such a base map must be equal-area. It should show areas with minimum distortion of shape, and should also indicate true east and west for purposes of latitude comparisons and for ease of construction. For world comparisons, it would be well to use a single, uninterrupted world graticule. The use of a central meridian of 90° west longitude would distribute the distortion equitably through the Americas.

"On such a map a country could determine with considerable accuracy the territorial extent of its political subdivisions through the use of a planimeter. Population densities and other data could be plotted correctly. If each country were mapped on such a base, the maps could all be placed contiguously to form a single equal-area map of the Americas. If possible, it would be well to agree upon a common ultimate scale; preferably 1:1,000,000. Larger scales could be used for plotting minor civil divisions."

Though considerable interest was manifested in this proposal when it was presented before the Committee on Special Maps (Cartas), no formal action was taken by the Pan American Institute of Geography and History towards the adoption of a statistical

[1] Cooperation with the American Republics program of the Interdepartmental Committee on Scientific and Cultural Cooperation, operated by the United States Bureau of the Census.

[2] F. W. McBryde. A Map of the World in Perspective. Ohio Journal of Science, Vol. 42, No. 2, pp. 63–64, March 1942.

[3] Report in press, Estadística No. 23, Vol. 7, June 1949.

base map. The proposal, however, led to the formulation of Resolution 68 of the Final Act of the Fourth Pan American Consultation on Cartography, which is as follows:

"That the members of the Committee on Special Maps (Cartas) look into the matter of a special projection which in the future might be used as a Base Map for census purposes."

ACKNOWLEDGMENTS

The authors gratefully acknowledge the valuable assistance of the Drafting Section of the Geography Division, Bureau of the Census, in preparing certain of the original graticules, and of Miss Edna S. Kelley, University of Maryland cartographer, who helped McBryde with the plotting of the master drawing, on the flat-polar quartic (fig. 26). This served as the base for the drawings of the other new projections, which were made by Mr. R. K. Estes, Geographic Section, and by Mr. C. H. McLendon, Division of Geodesy, Coast and Geodetic Survey. The projections for figure 28 were drawn by Mr. A. J. Hoffman, Geographic Section, and the continents drawn on them by Mr. D. F. Kramer, Aeronautical Chart Branch, Coast and Geodetic Survey. McLendon assembled and finished figure 28 and also edited all the illustrations.

The authors are particularly grateful to Mr. Erwin Schmid, Division of Geodesy, who checked all the formulas and tables; to Mr. C. N. Claire and Mr. N. F. Braaten, Division of Geodesy, for editing of manuscript and assistance in computing table IX.

EQUAL-AREA PROJECTIONS FOR WORLD STATISTICAL MAPS

EVOLUTION OF WORLD STATISTICAL MAPS

F. Webster McBryde

PROJECTION REQUIREMENTS

A graticule for a world statistical base map should be authalic, or equivalent (equal-area, preserving areas in true proportions as to size), in order to bring out correct territorial relationships, isorithms, and all distributional aspects of regional phenomena. Data such as population densities and producing areas cannot be graphically depicted for comparative purposes unless the base map shows the correct relative sizes of political units.

A statistical map for the Americas, or any comparable portion of the earth, should be drawn on a projection having (1) world scope, to indicate world distributions and inter-relations when desired; (2) equivalence, for regional measurements and size comparisons; (3) equal-spaced meridians and straight parallels, so as to show true scale and directions along east-west lines, in addition to comparable latitudes, and to facilitate construction; (4) equitable distribution of scale and shape distortions over the entire map; (5) unbroken graticule lines, so that maps of different countries on the same scale can be placed side-by-side to form one continuous map.

A common base to satisfy each country would have to be one which represented good scales and shapes for equatorial as well as middle-latitude regions, since most of Latin America lies within the tropics. Tropical proportions comparable with those on the globe cannot ordinarily be obtained without greatly compressing and distorting high-latitude regions (as on the sinusoidal) or excessively shortening the longitudinal axis (as on Lambert's equivalent cylindric), or both.

THE PROBLEM OF WORLD REPRESENTATION ON A PLANE

The problem of representing the entire surface of the spheroidal earth on a plane surface is one of the oldest and most difficult with which geographers have had to contend. Few will question the importance of having some sort of flat map of the world on one continuous projection for plotting world data. Not only is the terrestrial globe limited of necessity to small size, but for navigation, for scientific and educational displays on wall, screen, or book page, and for similar purposes, a map on a plane surface is needed. Not even a full hemisphere can be shown in a photograph or can be seen at one time while viewing a globe at close range, and distortion due to foreshortening is great in all directions away from a small central area; this is also true on the orthographic projection (representing the globe as seen from infinity), as in figure 1.

No representation of the whole sphere on a plane is without various distortions, such as those of azimuth, scale in various directions, size of areas, and shape of areas, and no one flat map can show correctly both areal and angular data. Most errors increase with the size of the territory included, so that distortions reach a maximum on world maps, all of which show great aberrations of scale and over-all shape of surface features.

FIGURE 1.—Orthographic projection.

SPECIAL-PURPOSE PROJECTIONS NOT SUITABLE FOR GENERAL STATISTICAL USE

Mercator's chart (fig. 2), which alone shows rhumb bearings as straight lines throughout, is invaluable for navigation and is approximately correct for plotting angular data. It is not adaptable to maps of the entire earth for general statistical purposes, because large areal plottings, such as of continents which extend into high latitudes, are so distorted in shape and size as to be misleading.

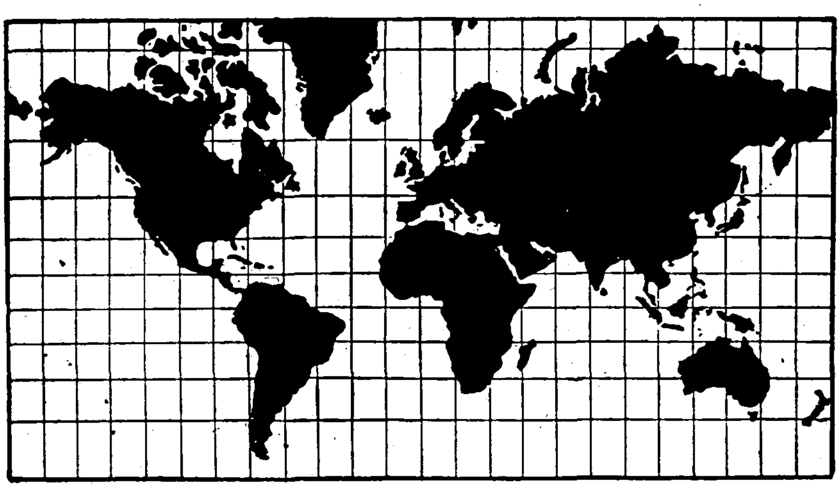

FIGURE 2.—Mercator projection.

The gnomonic projection (fig. 3) alone shows the orthodrome, or great-circle sailing route, as a straight line on any part of the map, but distortions of shape and size of areas are even far greater than on the Mercator, and it can be used only for portions of a hemisphere. Such special projections as the two mentioned above were not intended for world areal plottings and should not be used for general statistical maps.

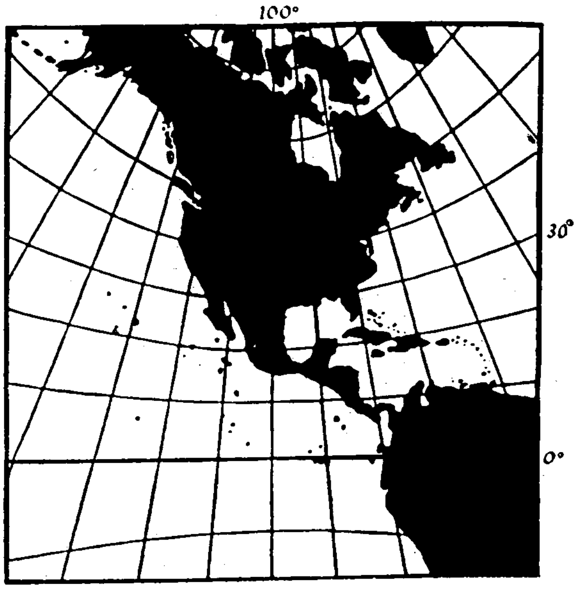

FIGURE 3.—Gnomonic projection.

WORLD PROJECTIONS WHICH ARE NOT EQUAL-AREA

Intermediate cylindrical projections such as Gall's and Miller's (figs. 4 and 5), on which meridians as well as parallels are straight parallel lines, have been devised for better representations of high-latitude regions, with parallels spaced much closer together than on Mercator's, to reduce excessive latitudinal expansion, and farther

FIGURE 4.—Gall's projection.

apart than on the Lambert's equal-area cylindrical, to avoid extreme polar flattening. Obviously these cannot be equivalent, and hence they are not suitable for general statistical use.

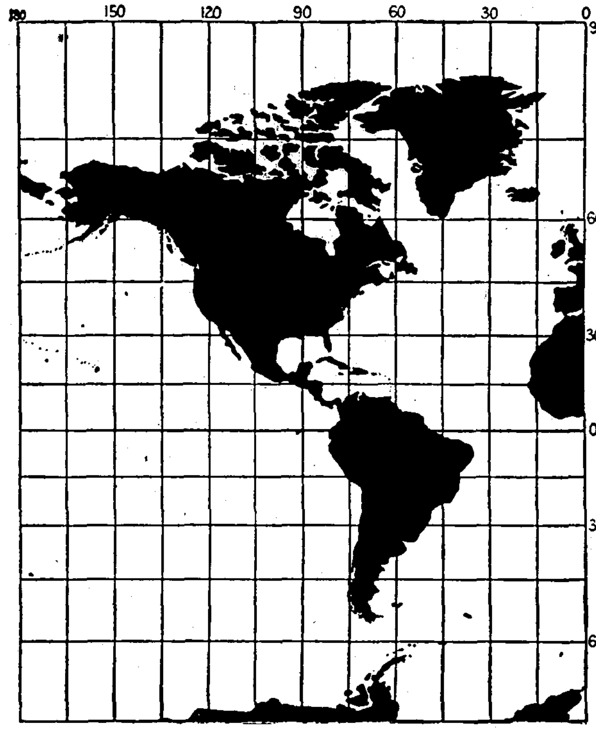

FIGURE 5.—Miller's projection.

Van der Grinten's projection (fig. 6) is another type which develops somewhat better shapes for small high-latitude areas, but it shows enormous distortions of size and over-all shape of continents, as well as scale and bearing, and it is more difficult to

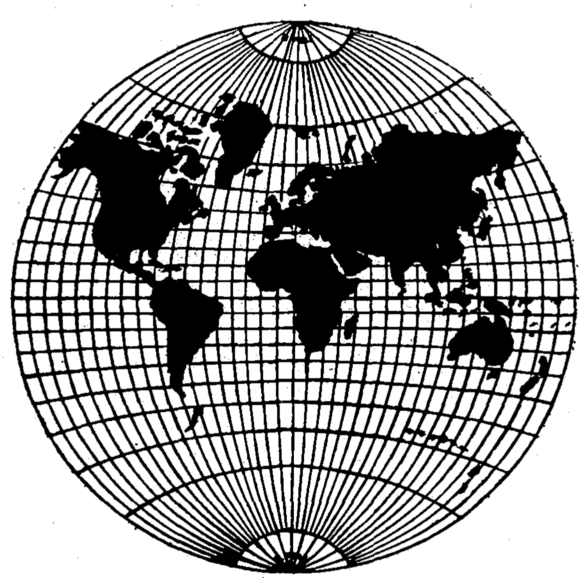

FIGURE 6.—Van der Grinten's projection.

construct than a straight-line graticule. It has none of the properties desired in map projections for scientific uses, and there is no real justification for its widespread popularity.

Denoyer's semielliptical projection (fig. 7) is a flat-polar parabolic type, somewhat similar to figures 22, 24, 26, and 27 in this publication. It has been used frequently in text books and for wall maps. The poles are shown as lines one-third the length of the equator, but parallel spacings do not diminish poleward to maintain equivalence, and meridian spacings widen toward the edges of the map in high latitudes. Though the representation of the continents on this projection is of pleasing appearance, the projection is not equivalent, and so is of limited value for statistical plottings.

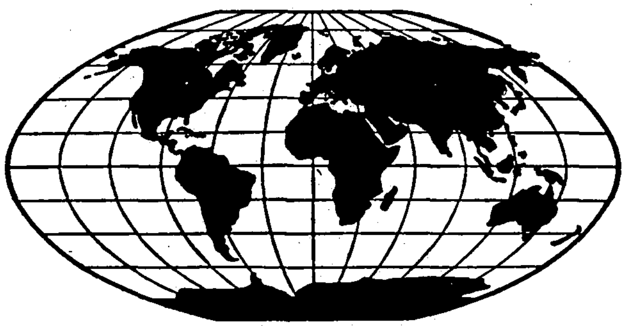

FIGURE 7.—Denoyer's semielliptical projection.

SOME COMMONLY USED EQUAL-AREA MAPS AND THEIR LIMITATIONS FOR WORLD STATISTICAL PLOTTING

A requisite property of any general world statistical map of the type discussed here is equivalence. On such a map, all areas appear in their true proportions as to size, though in order to have this quality they must be variously compressed in some directions and expanded in others. The number of square miles within a country's limits and the general relationships with other countries are, from a statistical standpoint, of greater value than approximation of the true form of the country; the latter characteristic is needed where esthetic interests and over-all scale are important, and it cannot be approached on any world equivalent projection without breaking the continuity of the graticule.

HEMISPHERICAL PROJECTIONS

A hemispherical equal-area projection, even though it may preserve excellent shapes for the continents, as Lambert's azimuthal (fig. 8) does, cannot be used satisfactorily to present distributions of world data, trade routes, and the like, for relationships between hemispheres are not well portrayed.

OVALOIDAL AND ANALAGOUS WORLD PROJECTIONS

The several equal-area ovaloidal and analagous world projections offer the most desirable compromises for general statistical purposes.

Oldest of this group, and still one of the best, is the sinusoidal (fig. 9), first used by Mercator in 1538, but usually credited to Sanson and Flamsteed, much later users of it. Since the midmeridian and all parallels are straight lines drawn to correct scale, map properties are excellent near the central portion along both axes, but maintenance of true spacing of all parallels results in the poles being pulled out to distinct points, with steep meridian sine curves joining them. This means excessive longitudinal crowding in high latitudes.

FIGURE 8.—Lambert's azimuthal projection.

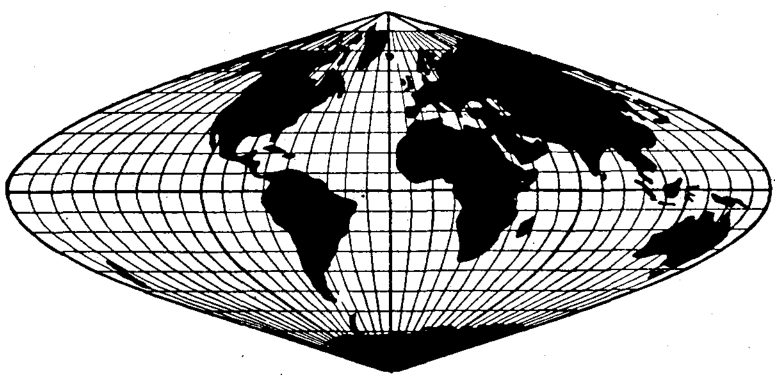

FIGURE 9.—Sinusoidal equal-area projection.

At the opposite extreme from the sinusoidal is Lambert's cylindrical equal-area projection (fig. 10), in which polar regions are shown with great latitudinal crowding, caused by successive narrowing of spaces between parallels.

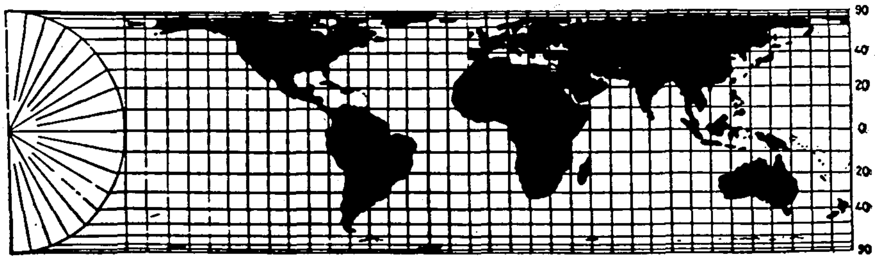

FIGURE 10.—Lambert's cylindrical equal-area projection.

Mollweide's homalographic projection (1805), similar to the sinusoidal but with ellipses instead of sine curves to represent meridians (fig. 11), develops high-latitude quadrants much more amply, though there still is excessive crowding in polar margins. In addition to this, parallel intervals are arranged to diminish poleward and expand equatorward, so that the ratio of 10° of longitude to 10° of latitude at the Equator is

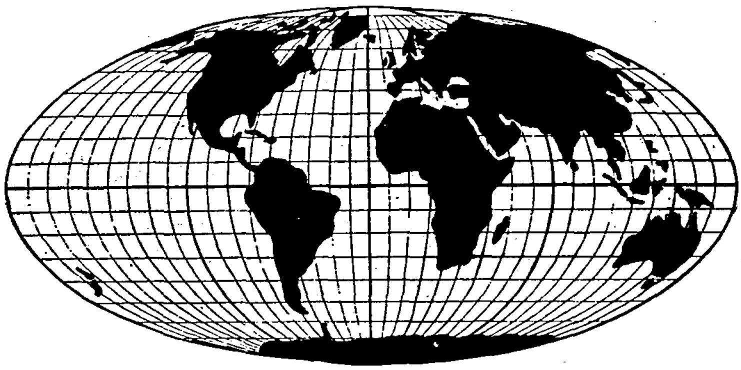

FIGURE 11.—Mollweide's homalographic equal-area projection.

only about 0.75 to 1.00 as against approximately 1 to 1 on the globe. For plotting countries within the Tropics, this means scale and shape distortion along the full length of the Equator. Africa, for example, is much elongated to the north and south, even when plotted in the center of the projection.

OTHER OVALOIDAL TYPES

In recent years improvements in this type of projection have been made through using various algebraic curves similar to the parabolic to represent meridians, with less high-latitude compression and better shape than on the older types. Craster, Boggs, and Adams have contributed notably in this regard. (See figs. 12, 13, and 14.)

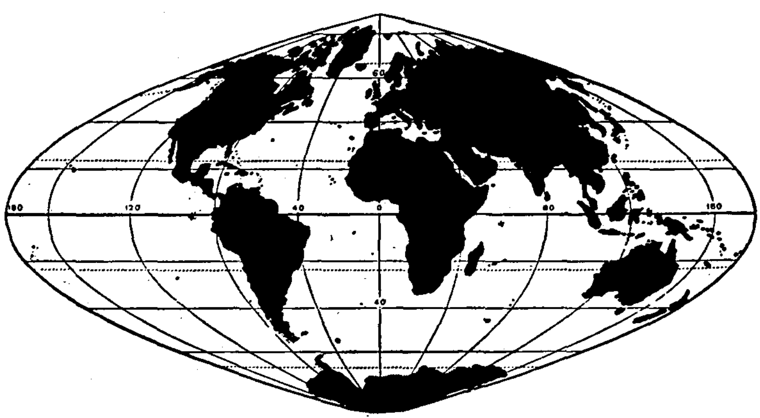

FIGURE 12.—Craster parabolic authalic projection.

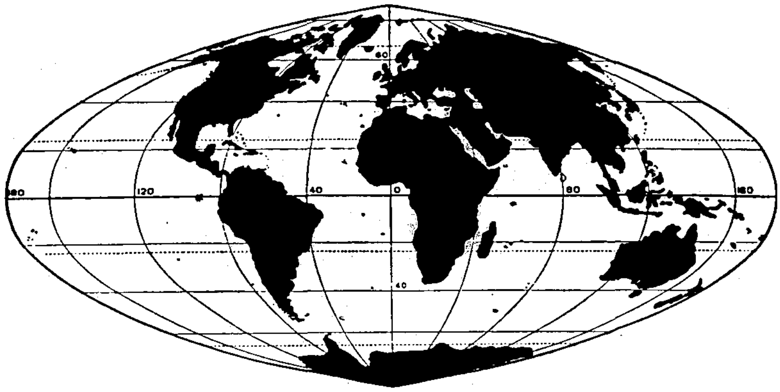

FIGURE 13.—Boggs eumorphic authalic projection.

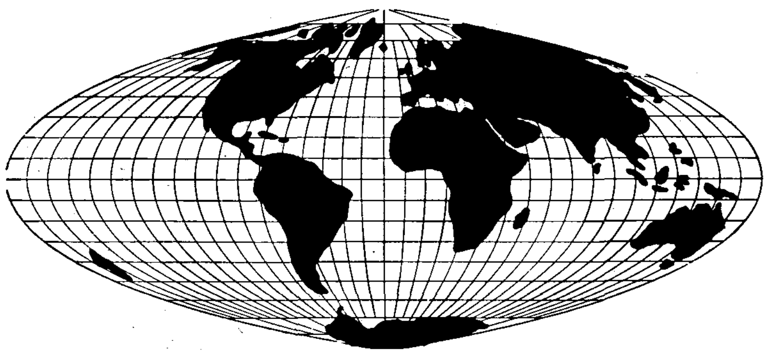

FIGURE 14.—Adams quartic authalic projection.

Another variation of the elliptical type of projection (fig. 15) was produced by Aitoff and Hammer, employing the same outer dimensions as on the Mollweide, but with parallels as curves derived from Lambert's azimuthal projection. Though this makes possible better representation of high-latitude land masses, while preserving equal-area, there is nevertheless excessive stretching of latitude along all margins, with the additional objection of curved parallels which are not true to scale, whereon comparisons are much more difficult and the construction is far more complicated.

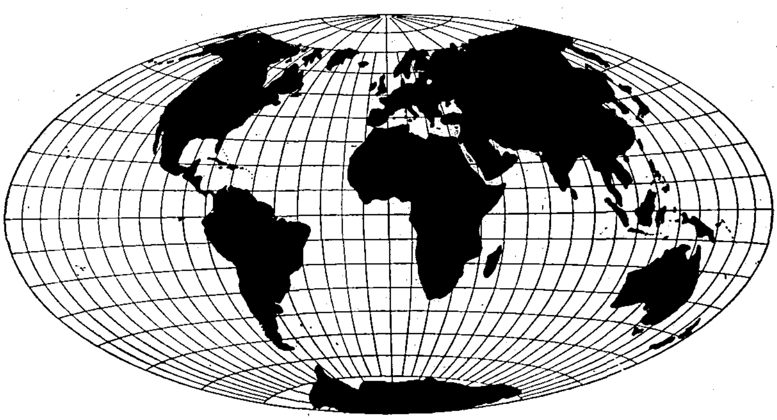

FIGURE 15.—Aitoff-Hammer authalic projection.

GOODE'S "INTERRUPTED" PROJECTIONS

Goode hit upon a technique for remedying the excessive compression of high-latitude margins. He repeated the pole at intervals, establishing two points at the north and four at the south, extending independent midmeridians to each, and drawing meridians on either side to converge at these points from breaks at the Equator, so that the earth was represented on six broad lune-like sections extending poleward. The first published example of this type was the "interrupted homalographic" (fig. 16),

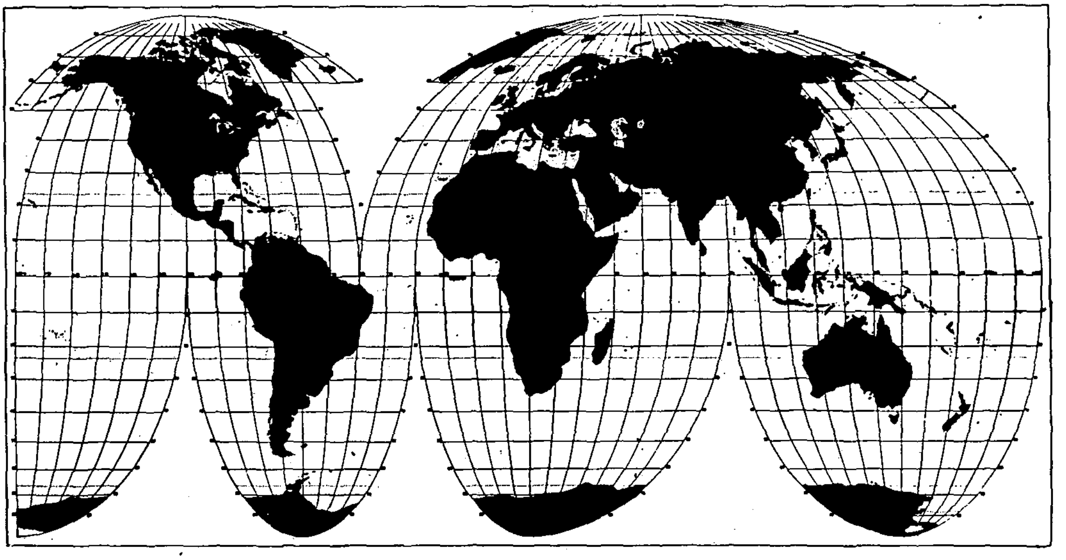

FIGURE 16.—Goode's interrupted homalographic projection.

which appeared in 1916, followed by the similarly interrupted "homalosine," a composite of the homalographic and sinusoidal designed to retain the best qualities of each. Meridians are sine curves from the Equator to 40° latitude, and ellipses from 40° to the poles.

Goode's basic idea of interrupting the graticule has been widely adopted by American geographers, who have made frequent use of the sinusoidal and of the Aitoff in breaking the entire graticule into the same six segments employed by Goode, but keeping uniform meridian curves throughout.

Though much better continental shapes are obtained by interrupting the graticule, the wide gaps in high latitudes cannot be effectively bridged by the eye.

INTERRUPTED AND CONDENSED MAPS

A common space-saving device consists in the partial deletion of oceans and piecing together of the major groups of land masses as in figure 17. This permits larger scale in a given area, but transoceanic relationships are destroyed so that no continuous distribution lines can be drawn, and a misleading picture results. Such a map is desirable only where local regional details are more important than entire world patterns, which require a continuous and unbroken world projection.

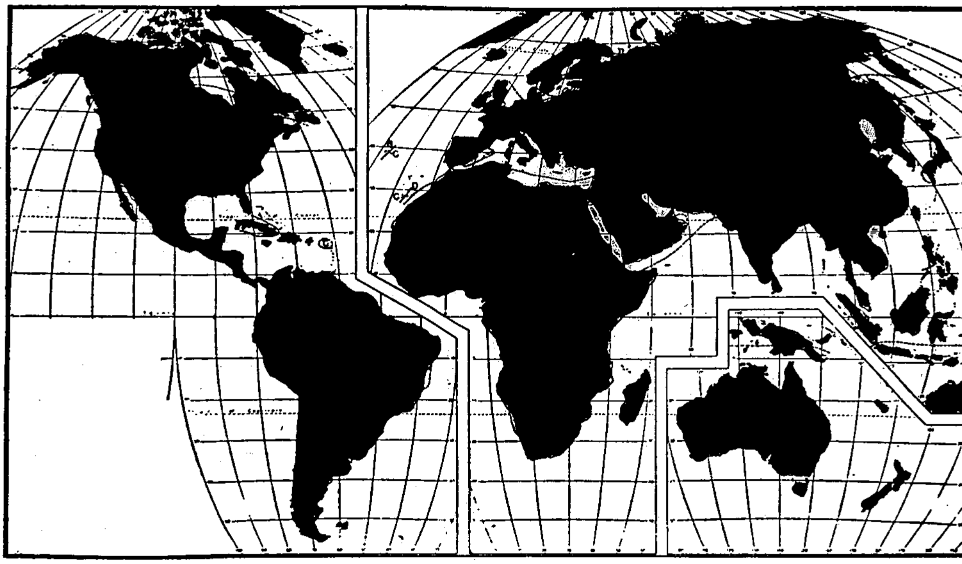

FIGURE 17.—An interrupted and condensed projection.

ECKERT'S PROJECTIONS

A different approach to the problem of developing high-latitude margins without sacrificing equivalence was that of Eckert, who in the early twentieth century used the principle of opening up the poles as straight lines one-half the length of the Equator. This represents a compromise between the ovaloidal types, with curved meridians converging at the poles, and the cylindrical graticules, on which the poles are lines as long as the Equator, and all lines are straight with right-angle intersections. The best known of Eckert's projections are his No. 4 (fig. 18), on which the bounding meridians are ellipses (a modification of the Mollweide) and No. 6 (fig. 19), derived from the sinusoidal, with meridians as sine curves.[4] On both, the midmeridian is one-half the

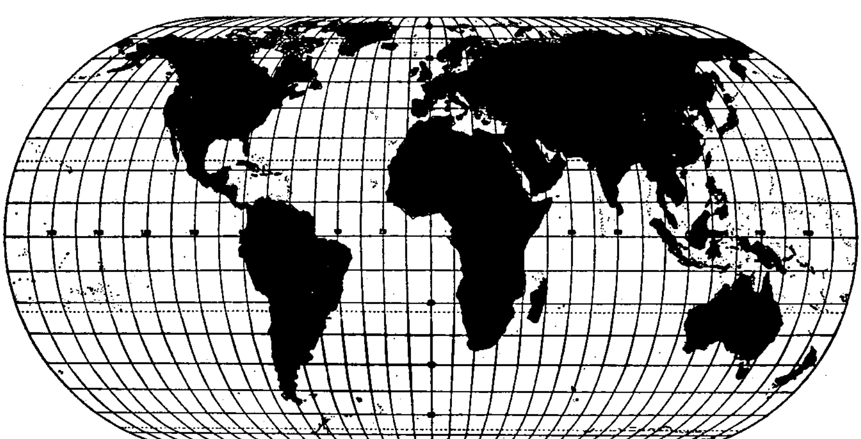

FIGURE 18.—Eckert's No. 4 authalic projection.

[4] M. Eckert, Neue Entwürfe für Erdkarten—Petermanns Mitteilungen Aus Justus Perthes' Geographischer Anstalt—52. Band 1906.

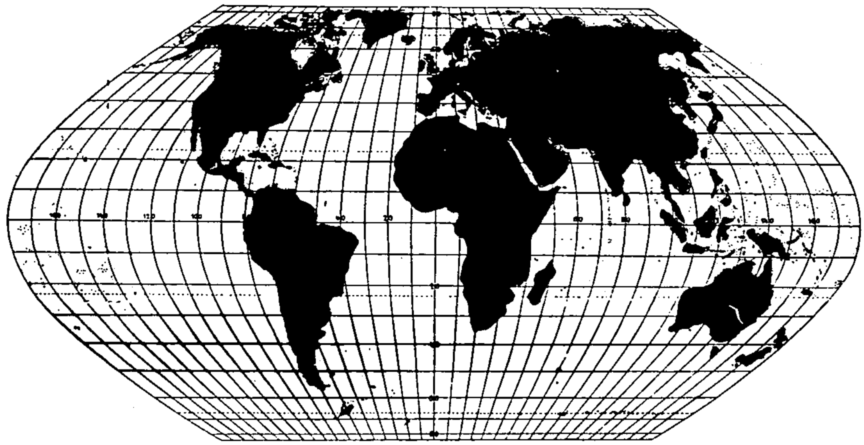

FIGURE 19.—Eckert's No. 6 authalic projection.

length of the Equator, as on the ovaloidal prototypes, and parallel spacings diminish poleward as they do on the Mollweide, at rates by which equivalence is maintained.

Eckert's No. 4 has been used widely in Europe, and also to some extent in the United States during recent years; No. 6 is somewhat less common in Europe, and occasionally appears in South American atlases, and as a base map in this country. Though good for the upper middle latitudes, as of Eurasia and North America, Eckert's projections show excessive longitudinal expansion in higher latitudes, because the poles are represented as extremely long lines and, owing to compensatory latitudinal stretching near the Equator, have equatorial distortions comparable with those of the Mollweide.

SOME NEW FLAT-POLAR PROJECTIONS BASED ON THE ECKERT PRINCIPLE

By inspection it seemed to the writer that a shortening of the lines representing the poles on Eckert's maps, to one-third the length of the Equator instead of one-half, might reduce the latitudinal distortion near the Equator by as much as one-half. At the same time it would diminish the longitudinal stretching in high latitudes. As the length of the line representing the poles is reduced, steep meridian curves should be avoided in order to prevent high-latitude crowding such as that seen on the sinusoidal. Sine curves would thus appear less desirable on world projections having narrow polar regions. Nevertheless, a modification of the Eckert 6 (itself a modified sinusoidal) was decided upon as a point of departure. It was evident at the outset that a similar projection with parabolic-type curves, preferably the fourth-degree (quartic) curves of Adams' projection, to represent meridians, would give a more satisfactory over-all compromise. A further reduction of north-south stretching in low latitudes seemed desirable through shortening the midmeridian and lengthening the map Equator, even though this increases the conventional true-to-scale 2-to-1 ratio of Equator to polar axis. It is not necessary to maintain this relationship by showing the Equator as twice the length of the midmeridian semicircle. Linear scale is incorrect on most parts of all equal-area world maps, whether or not the correct axis ratio is preserved. For a modification employing the meridian curves of the projection shown in figure 20 of this publication, polar lines one-fourth the length of the Equator were used, producing the flat polar graticule herein presented in figure 22.

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Although the flat-polar quartic (fig. 26) was selected as the base for plotting hemisphere census maps, it is believed that the other four new projections presented in this publication (figs. 20, 22, 24, and 27) will provide useful bases for world statistical maps.

Figure 28 shows the evolution of equal-area world projections as generated from the sine and tangent functions. Although this publication contains an independent mathematical development, these series are mentioned in an article by E. J. Baar.[5]

The following section, by P. D. Thomas, contains the mathematical development for the new projections.

DERIVATION OF FORMULAS

PAUL D. THOMAS

A complete mathematical discussion of authalic projections is presented by O. S. Adams in United States Coast and Geodetic Survey Special Publication No. 236, General Theory of Equivalent Projections. References in the following development and the bibliography indicate other sources. Since we are concerned here only with orthembadic (equal-area) projections whose parallels are straight lines, the subsequent mathematical development is restricted to this class of authalic projections.

The element of area on the sphere, when referred to its meridians, λ, and parallels, ϕ, is $R^2 \cos \phi d\phi d\lambda$ where R is the radius of the sphere. If $x=x(\lambda, \phi)$, $y=y(\lambda,\phi)$, then the corresponding element of area in the plane is $J\left(\frac{x,y}{\lambda,\phi}\right)d\phi d\lambda$, whence the condition for equivalence is the partial differential equation

$$J\left(\frac{x,y}{\lambda,\phi}\right)=R^2 \cos \phi, \tag{1}$$

where $J\left(\frac{x,y}{\lambda,\phi}\right)=\begin{vmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix}$ is the Jacobian functional determinant.

If the map parallels are to be straight lines, then y must be a function of ϕ alone, $y=y(\phi)$ and $\frac{\partial y}{\partial \phi}=y'(\phi)$, $\frac{\partial y}{\partial \lambda}=0$, so that equation (1) becomes

$$\frac{\partial x}{\partial \lambda}=\frac{R^2 \cos \phi}{y'(\phi)}. \tag{2}$$

Integrating (2) gives $x=\frac{R^2\lambda \cos \phi}{y'(\phi)}+A(\phi)$. If the central map meridian is to be the y-axis, then $x=0$ when $\lambda=0$, so that $A(\phi)=0$. The mapping equations are then, $x=\frac{R^2\lambda \cos \phi}{y'(\phi)}$, $y=y(\phi)$, which may be written in the equivalent form

$$x=\frac{R\lambda \cos \phi}{f'(\phi)},\ y=Rf(\phi). \tag{3}$$

[5] E. J. Baar, The Manipulation of Projections for World Maps. The Geographical Review, American Geographical Society of New York, January 1947.

From (3), if the x-axis is to be the map Equator, then $y=0$ when $\phi=0$, hence $f(0)=0$. For x to be finite $f'(\phi)\neq 0$. If the parameter ϕ is eliminated in (3), there results the equation of the map meridians which is of the form $F\left(\frac{x}{\lambda}, y\right)=0$.

From equations (3), with the pairs of values $\phi=\frac{\pi}{2}$, $\lambda=0$; $\phi=0$, $\lambda=\pi$ we have the ratio of the axes of the projection which is

$$\frac{y_0}{x_0}=\frac{f\left(\frac{\pi}{2}\right)\cdot f'(0)}{\pi}. \tag{4}$$

If equations (3) are to be modified, maintaining the equal-area property, so that the poles of the projection will be replaced by lines of given length parallel to the map Equator, the mapping equations are of the form

$$x=RC\left(k+\frac{\cos\alpha}{f'(\alpha)}\right)\lambda,\ y=Rf(\alpha), \tag{5}$$

where $f(0)=0$, $f'(\alpha)\neq 0$.

The area of the zone of the sphere from the Equator to latitude ϕ is $2\pi R^2\sin\phi$. From (5), with $\lambda=\pi$, we have $2\pi R^2\sin\phi=2\int_0^y x\,dy=2\pi CR^2\int_0^\alpha[kf'(\alpha)+\cos\alpha]d\alpha$, or

$$\sin\phi=C[kf(\alpha)+\sin\alpha]. \tag{6}$$

Placing $\phi=\alpha=\frac{\pi}{2}$ in equation (6) gives $C=\dfrac{1}{kf\left(\frac{\pi}{2}\right)+1}$ and (6) may then be written

$$n\sin\phi=kf(\alpha)+\sin\alpha,\ n=\frac{1}{C}=kf\left(\frac{\pi}{2}\right)+1. \tag{7}$$

Consequently equations (5) may be written as

$$x=\frac{R\lambda}{Mn}\left(k+\frac{\cos\alpha}{f'(\alpha)}\right),\ y=RMf(\alpha), \tag{8}$$

where M and k are arbitrary parameters, $f(\alpha)$ is the same function as $f(\phi)$, but the values of the auxiliary parameter α must be obtained by solving equation (7) for α.

For the pairs of values $\alpha=\frac{\pi}{2}$, $\lambda=0$; $\alpha=0$, $\lambda=\pi$ we have from (8) the ratio of the axes of the modified projection, namely

$$\frac{y_0}{x_0}=\frac{nM^2\cdot f\left(\frac{\pi}{2}\right)\cdot f'(0)}{m\pi}, \tag{9}$$

where $m=kf'(0)+1$.

If it is desired that the ratio of the axes in the modified projection be the same as in the original projection, we find from (4) and (9) that

$$M^2=\frac{m}{n}=\frac{kf'(0)+1}{kf\left(\frac{\pi}{2}\right)+1}. \tag{10}$$

With condition (10) we may write equations (8) as

$$x=\frac{RM\lambda}{m}\left(k+\frac{\cos\alpha}{f'(\alpha)}\right),\ y=RMf(\alpha). \tag{11}$$

From the x-coordinate in (11), with $\lambda=\pi$; $\alpha=0$, $\frac{\pi}{2}$ we have

$$\frac{x_{\frac{\pi}{2}}}{x_0}=\frac{kf'(0)}{m}, \tag{12}$$

which is the ratio of the line segment, which has replaced the pole, to the map-Equator length.

APPLICATION TO AUTHALIC PROJECTIONS

Consider the two functions $f(\phi)=p\sin\frac{\phi}{q}$, $F(\phi)=p\tan\frac{\phi}{q}$. We have $f'(\phi)=\frac{p}{q}\cos\frac{\phi}{q}$, $F'(\phi)=\frac{p}{q}\sec^2\frac{\phi}{q}$. These functions satisfy the requirements that $f'(\phi)\neq 0$, $f(0)=0$. However $F(\phi)\doteq\infty$ for $q=1$, $\phi=\frac{\pi}{2}$ hence if $q=1$ in $F(\phi)$ the map poles are infinitely distant.

Equations (3) become respectively

$$x=\frac{Rq}{p}\lambda\cos\phi\sec\frac{\phi}{q},\ y=Rp\sin\frac{\phi}{q} \tag{13}$$

$$x=\frac{Rq}{p}\lambda\cos\phi\cos^2\frac{\phi}{q},\ y=Rp\tan\frac{\phi}{q}. \tag{14}$$

From (4) the ratio of the axes in each case respectively is

$$\frac{y_0}{x_0}=\frac{p^2}{q\pi}\sin\frac{\pi}{2q}, \tag{15}$$

$$\frac{y_0}{x_0}=\frac{p^2}{q\pi}\tan\frac{\pi}{2q}. \tag{16}$$

Since we have two arbitrary parameters p and q in equations (13) and (14) it is seen that any number of projections may be constructed of these types. Imposing arbitrary desired conditions on certain of the elements of the projection will usually cause p and q to be irrational. Examples will be subsequently given of such projections as well as those in which p and q are rational or integral. These will then, in some cases, be modified according to the development of equations (7) through (12). Projections based on equations (13) will be designated the sine series; those based on equations (14) the tangent series.

As q becomes large, it is seen that for both types (13) and (14) a limiting case is obtained by placing $\sin\frac{\phi}{q}=\tan\frac{\phi}{q}=\frac{\phi}{q}$, $\cos\frac{\phi}{q}=1$ whence $x=R\frac{q}{p}\lambda\cos\phi$, $y=R\frac{p}{q}\phi$. If the ratio of the map axes, $\frac{y_0}{x_0}$, is to be ½, then from either (15) or (16) one finds that $p=q$, whence we have the mapping equations of the Mercator sinusoidal authalic projection,[6] $x=R\lambda\cos\phi, y=R\phi$. The equation of the meridians is $x=R\lambda\cos\frac{y}{R}$. In this case $f(\phi)=\phi$, $f'(\phi)=1$, $f'(0)=1$, $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$. For the modified projection, equations (7), (10), (11), and (12) become

$$n\sin\phi=k\alpha+\sin\alpha,\ n=\frac{1}{2}(k\pi+2). \tag{17}$$

$$M^2=\frac{k+1}{n}. \tag{18}$$

$$x=\frac{RM\lambda}{k+1}(k+\cos\alpha),\ y=RM\alpha. \tag{19}$$

$$\frac{x_{\frac{\pi}{2}}}{x_0}=\frac{k}{k+1}. \tag{20}$$

To solve equation (17) for α we may use the Newton-Raphson Method.[7] For the first approximation we may use the first few terms of the series expansion for α in terms of $\sin\phi$. This may be obtained by writing the series expansion of $\sin\alpha$ in (17), then reverting[8] the resulting series (see the appendix for the formulas) to obtain

$$\alpha=u+\frac{u^3}{3!(k+1)}+\frac{9-k}{5!(k+1)^2}u^5+\frac{225-54k+k^2}{7!(k+1)^3}u^7+\frac{11{,}025-4{,}131k+243k^2-k^3}{9!(k+1)^4}u^9+\ldots \tag{21}$$

where

$$u=\frac{n\sin\phi}{k+1}=\frac{\sin\phi}{M^2},\ n=\frac{1}{2}(k\pi+2).$$

An alternative method of obtaining a first approximation is to graph the simultaneous equations $y=-k\alpha+n\sin\phi=\sin\alpha$ on millimeter paper. This gives intersections of a family of parallel straight lines with the sine curve, three-significant-figure estimates of the abscissae of the intersection points being the estimates for α.

If we demand that the ratio (20) shall be ½, then $k=1$, $M^2=\frac{4}{\pi+2}$, $M=0.8820$, and equations (19) become $x=\frac{RM\lambda}{2}(1+\cos\alpha)$, $y=RM\alpha$ which are the mapping equations of Dr. Max Eckert's No. 6 authalic projection.[9]

Note: The appendix gives a special development for a general flat-polar sinusoidal authalic projection which avoids approximation methods.

THE SINE SERIES PROJECTIONS

Consider first equations (13) with $p=q=1$. We have $x=R\lambda$, $y=R\sin\phi$. The meridians and parallels are straight lines parallel to the coordinate axes. The resulting projection is the Lambert authalic cylindrical projection[10] and the modification

[6] Oscar S. Adams, General theory of equivalent projections, U. S. Coast and Geodetic Survey Special Publication No. 236, p. 17.
[7] Fr. A. Willers, Practical analysis, p. 222; J. B. Scarborough, Numerical mathematical analysis, p. 178.
[8] T. J. Bromwich, An introduction to the theory of infinite series, p. 156.
[9] Dr. A. Petermann, Mitteilungen Aus Justus Perthes' Geographischer Anstalt, 52. Band 1906, p. 106.
[10] Oscar S. Adams, General theory of equivalent projections, U. S. Coast and Geodetic Survey Special Publication No. 236, p. 9.

devised here does not apply. Since $-\frac{\pi}{2} \lesseqgtr \phi \lesseqgtr \frac{\pi}{2}$ this projection is the upper limiting case for the function $\sin \frac{\phi}{q}$. We have seen that for q very large (for $q=30$, $\sin \frac{\phi}{q}=\frac{\phi}{q}$ in radians to five decimals approximately), the form of the meridian is the sine curve. For $q=1$, the form of the meridian is a rectangle. Between these limits lie a number of curves of varying degree given by $A \geqq q > 1$, where $A \doteq \infty$. Hence it is seen that the modification devised here and represented by equations (7) through (12) actually accomplishes a combination of the Lambert authalic cylindrical projection and a given authalic projection whose parallels are straight lines.

Placing $p=q=3$, equations (13) become $x=R\lambda \cos \phi \sec \frac{\phi}{3}$, $y=3R \sin \frac{\phi}{3}$. The equation of the meridians is $\frac{x}{R\lambda}=1-\frac{4y^2}{9R^2}$ which represents a family of parabolas. A particular case is obtained by placing $q=3$ and demanding that the ratio given by (15) shall be ½. We find that $p=\sqrt{3\pi}$ and equations (13) become then $x=R\lambda \sqrt{\frac{3}{\pi}} \cos \phi \sec \frac{\phi}{3}$, $y=R\sqrt{3\pi} \sin \frac{\phi}{3}$ which are one form of the mapping equations of the Craster parabolic authalic projection.[11] Here $f(\phi)=\sqrt{3\pi} \sin \frac{\phi}{3}$, $f'(\phi)=\frac{1}{3}\sqrt{3\pi} \cos \frac{\phi}{3}$, $f\left(\frac{\pi}{2}\right)=\frac{1}{2}\sqrt{3\pi}$, $f'(0)=\frac{1}{3}\sqrt{3\pi}$. Equations (7), (10), (11), and (12), become

$$n \sin \phi=2(n-1) \sin \frac{\alpha}{3}+\sin \alpha,\ n=\frac{1}{2} k\sqrt{3\pi}+1. \tag{22}$$

$$M^2=\frac{2n+1}{3n}. \tag{23}$$

$$x=\frac{3RM\lambda}{2n+1}\left(k+\sqrt{\frac{3}{\pi}} \cos \alpha \sec \frac{\alpha}{3}\right),\ y=\sqrt{3\pi}\, RM \sin \frac{\alpha}{3}. \tag{24}$$

$$\frac{x_{\frac{\pi}{2}}}{x_0}=\frac{2(n-1)}{2n+1}. \tag{25}$$

With the substitution $\sin \alpha=3 \sin \frac{\alpha}{3}-4 \sin^3 \frac{\alpha}{3}$ placed in (22) one obtains the cubic

$$4x^3-(2n+1)x+n \sin \phi=0, \text{ where } x=\sin \frac{\alpha}{3}. \tag{26}$$

To solve (26) one may employ Horner's contracted method, or some other numerical method,[12] after three-significant-figure estimates are obtained from the abscissae of the intersection points of the graphed simultaneous equations

[11] Geographical Journal, November 1929.

[12] Mathematical tables and other aids to computation, National Research Council, Vol. I, p. 441f; Vol. 2, p. 28f.

$$y=\frac{2n+1}{4}x+\frac{n \sin \phi}{4}=x^3.$$

However, it is possible to avoid approximation methods in solving equation (26). The discriminant of equation (26) is

$$\frac{27n^2 \sin^2 \phi-(2n+1)^3}{1,728} \leq 0 \text{ for all } n>0 \text{ and for all } \phi.$$

Hence a trigonometric solution may be used. If we let

$$\sin u=\frac{3n}{(2n+1)^2} \sqrt{3(2n+1)} \sin \phi, \tag{27}$$

then

$$x=\sin \frac{\alpha}{3}=\frac{\sqrt{3(2n+1)}}{3} \cdot \sin \frac{u}{3} \tag{28}$$

is the required solution of equation (26).

If we demand that the ratio given by (25) be $\frac{1}{2}$, then $n=\frac{5}{2}$. With this value of n and placing $x=\frac{y}{2h}$ we find that equations (26), (27) and (28) become respectively $\left(\frac{y}{h}\right)^3-6\frac{y}{h}+5 \sin \phi=0$, $\sin u=\frac{5\sqrt{2}}{8} \sin \phi$, $\frac{y}{h}=2\sqrt{2} \sin \frac{u}{3}$.

These last three equations are given, for this particular case, on page 9 of a paper by W. Werenskiold entitled A Class of Equal Area Map Projections, Oslo 1945. However, Werenskiold credits R. V. Putnins with the invention of this particular case citing Putnins' two papers, Jaunas projekcijas pasaules kartem and Nouvelles projections pour les mappemondes, Geografiski Raksti, Folia Geographica, III un IV, Riga 1934.

If $p=q=2$, equations (13) become $x=R\lambda \cos \phi \sec \frac{\phi}{2}$, $y=2R \sin \frac{\phi}{2}$ which are the mapping equations of the Adams orthembadic projection.[13] The equation of the meridians is $\frac{x^2}{\lambda^2}=\frac{2R^2-y^2}{4R^2-y^2}$. Hence the meridians are curves of fourth degree.

For this projection $f(\phi)=2 \sin \frac{\phi}{2}$, $f'(\phi)=\cos \frac{\phi}{2}$, $f'(0)=1$, $f\left(\frac{\pi}{2}\right)=\sqrt{2}$. Equations (7), (10), (11) and (12) become respectively

$$n \sin \phi=2k \sin \frac{\alpha}{2}+\sin \alpha,\ n=k\sqrt{2}+1. \tag{29}$$

$$M^2=\frac{k+1}{n}. \tag{30}$$

$$x=\frac{RM\lambda}{k+1}\left(k+\cos \alpha \sec \frac{\alpha}{2}\right),\ y=2RM \sin \frac{\alpha}{2}. \tag{31}$$

$$\frac{x_{\frac{\pi}{2}}}{x_0}=\frac{k}{k+1}. \tag{32}$$

[13] Oscar S. Adams, General theory of equivalent projections, U. S. Coast and Geodetic Survey Special Publication No. 236, p. 46.

If $\sin \alpha = 2\sqrt{1-\sin^2\frac{\alpha}{2}} \cdot \sin\frac{\alpha}{2}$ is placed in (29) and the resulting equation is rationalized one obtains the quartic

$$x^4+(k^2-1)x^2-(nk \sin\phi)x+\left(\frac{n}{2}\sin\phi\right)^2=0,\ x=\sin\frac{\alpha}{2}. \tag{33}$$

Equation (33) may be solved by Horner's contracted method[14] (which is easily adapted to machine computation) or by other numerical methods,[15] after three-significant-figure estimates have been obtained by graphing the simultaneous equations $y=-kx+\frac{n}{2}\sin\phi=x\sqrt{1-x^2}$ analogously as described for the sinusoidal projection.

If the series estimate is desired, one may write (29) with $\sin\alpha=2x\sqrt{1-x^2}$, as $\frac{n}{2}\sin\phi=kx+x\sqrt{1-x^2}$, and expanding $\sqrt{1-x^2}$ by the binomial formula, reverting the resulting series in x, one obtains finally

$$x=\sin\frac{\alpha}{2}=u+\frac{u^3}{2(k+1)}+\frac{k+7}{2^3(k+1)^2}u^5+\frac{k^2+10k+33}{2^4(k+1)^3}u^7+\frac{5(k^3+13k^2+67k+143)}{2^7(k+1)^4}u^9+\ldots, \tag{34}$$

where $u=\frac{n\sin\phi}{2(k+1)}=\frac{\sin\phi}{2M^2};\ n=k\sqrt{2}+1.$

If $p=q=\frac{3}{2}$ in equations (13), one obtains the mapping equations $x=R\lambda\cos\phi\sec\frac{2\phi}{3}$, $y=\frac{3}{2}R\sin\frac{2\phi}{3}$. The equation of the meridians is $9R^2\left[\frac{2(9R^2-4y^2)x^2}{R^2\lambda^2}-9R^2\right]^2=(9R^2-4y^2)\cdot(9R^2-16y^2)^2$. Thus the meridians are curves of eighth degree. $f(\phi)=\frac{3}{2}\sin\frac{2\phi}{3}$, $f'(\phi)=\cos\frac{2\phi}{3}$; $f'(0)=1$, $f\left(\frac{\pi}{2}\right)=\frac{3\sqrt{3}}{4}$. Equations (7), (10), (11), and (12) become respectively

$$n\sin\phi=\frac{3}{2}k\sin\frac{2\alpha}{3}+\sin\alpha,\ n=\frac{3\sqrt{3}}{4}k+1. \tag{35}$$

$$M^2=\frac{k+1}{n}. \tag{36}$$

$$x=\frac{RM\lambda}{k+1}\left(k+\cos\alpha\sec\frac{2\alpha}{3}\right),\ y=\frac{3}{2}RM\sin\frac{2\alpha}{3}. \tag{37}$$

$$\frac{x_{\frac{\pi}{2}}}{x_0}=\frac{k}{k+1}. \tag{38}$$

With the identities $\sin\frac{2\alpha}{3}=2\sqrt{1-\sin^2\frac{\alpha}{3}}\cdot\sin\frac{\alpha}{3}$, $\sin\alpha=3\sin\frac{\alpha}{3}-4\sin^3\frac{\alpha}{3}$, equation (35)

[14] H. B. Fine, College algebra, p. 457.
[15] J. B. Scarborough, Numerical mathematical analysis, Ch. IX.

may be written

$$n \sin \phi - 3x + 4x^3 = 3kx\sqrt{1-x^2},\ x = \sin \frac{\alpha}{3}. \tag{39}$$

Rationalizing (39) leads to the sextic equation

$$16x^6 + 3(3k^2-8)x^4 + 8n \sin \phi \cdot x^3 + 9(1-k^2)x^2 - 6n \sin \phi \cdot x + n^2 \sin^2 \phi = 0. \tag{40}$$

Equation (40) may be solved by Horner's contracted method, or some other numerical method, after three-significant-figure estimates are obtained from the abscissae of the intersections of the graphed simultaneous equations $y = -3x + n \sin \phi = 3kx\sqrt{1-x^2} - 4x^3$.

For the series estimate, we may expand $\sqrt{1-x^2}$ by the binomial formula to obtain from (39) the series $u = \frac{n \sin \phi}{3(k+1)} = x - \frac{3k+8}{6(k+1)}x^3 - \frac{kx^5}{8(k+1)} - \frac{kx^7}{16(k+1)} - \frac{5kx^9}{128(k+1)} - \ldots$, and then revert this series to get the expansion

$$x = \sin \frac{\alpha}{3} = u + \frac{3k+8}{6(k+1)}u^3 + \frac{21k^2+99k+128}{24(k+1)^2}u^5 + \frac{297k^3+2010k^2+4809k+4096}{144(k+1)^3}u^7 + \ldots, \tag{41}$$

where $u = \frac{n \sin \phi}{3(k+1)} = \frac{\sin \phi}{3M^2}$.

With $p = q = \frac{4}{3}$, equations (13) give the mapping equations $x = R\lambda \cos \phi \sec \frac{3\phi}{4}$, $y = \frac{4}{3}R \sin \frac{3\phi}{4}$. The equation of the meridians is

$\frac{x^2}{R^2\lambda^2}\left(1 - \frac{9y^2}{16R^2}\right)\left[\frac{4x^2}{R^2\lambda^2}\left(1 - \frac{9y^2}{16R^2}\right) - 3\right]^2 = \left[8\left(\frac{1}{2} - \frac{9y^2}{16R^2}\right)^2 - 1\right]^2$. The meridians are thus seen to be curves of twelfth degree. $f(\phi) = \frac{4}{3}\sin \frac{3\phi}{4}$, $f'(\phi) = \cos \frac{3\phi}{4}$, $f'(0) = 1$, $f\left(\frac{\pi}{2}\right) = \frac{2}{3}\sqrt{2+\sqrt{2}}$. Equations (7), (10), (11), and (12) become respectively

$$n \sin \phi = \frac{4}{3}k \sin \frac{3\alpha}{4} + \sin \alpha,\ n = \frac{2}{3}\sqrt{2+\sqrt{2}}\,k + 1. \tag{42}$$

$$M^2 = \frac{k+1}{n}. \tag{43}$$

$$x = \frac{RM\lambda}{k+1}\left(k + \cos \alpha \sec \frac{3\alpha}{4}\right),\ y = \frac{4}{3}RM \sin \frac{3\alpha}{4}. \tag{44}$$

$$\frac{x_{\frac{\pi}{2}}}{x_0} = \frac{k}{k+1}. \tag{45}$$

With $x = \sin \frac{\alpha}{4}$, we write the following identities: $\sin \frac{3\alpha}{4} = 3 \sin \frac{\alpha}{4} - 4$ $3x - 4x^3$, $\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 4\sqrt{1 - \sin^2 \frac{\alpha}{4}}\left(1 - 2 \sin^2 \frac{\alpha}{4}\right) \sin \frac{\alpha}{4} = 4x(1-2x$

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With these values of $\sin \alpha$ and $\sin \frac{3\alpha}{4}$ placed in equation (42) one obtains

$$3n \sin \phi = 4kx(3-4x^2)+12x(1-2x^2)\sqrt{1-x^2},\ x=\sin \frac{\alpha}{4}. \tag{46}$$

Rationalizing (46) leads to the octic equation

$$576x^8+128(2k^2-9)x^6+48(15-8k^2)x^4+96(kn \sin \phi)x^3+$$
$$144(k^2-1)x^2-72(kn \sin \phi)x+9(n \sin \phi)^2=0. \tag{47}$$

Equation (47) may be solved by Horner's contracted method, after obtaining three-significant-figure estimates from the abscissae of the intersections of the graphed simultaneous equations $y=-12kx+3n \sin \phi=12x(1-2x^2)\sqrt{1-x^2}-16kx^3,\ x=\sin \frac{\alpha}{4}$.

For the series estimate we may expand $\sqrt{1-x^2}$ by the binomial formula to obtain from (46) the series

$$u=\frac{n \sin \phi}{4(k+1)}=x-\frac{8k+15}{6(k+1)}x^3+\frac{7x^5}{8(k+1)}+\frac{3x^7}{16(k+1)}+\cdot\cdot\cdot$$

Reverting this series one finds that

$$x=\sin \frac{\alpha}{4}=u+\frac{8k+15}{6(k+1)}u^3+\frac{128k^2+459k+429}{24(k+1)^2}u^5+$$
$$\frac{4096k^3+21669k^2+39282k+24453}{144(k+1)^3}u^7+\cdot\cdot\cdot, \tag{48}$$

where $u=\frac{n \sin \phi}{4(k+1)}=\frac{\sin \phi}{4M^2}$.

The evolution of the sine series from the Lambert authalic cylindrical projection to the Mercator authalic sinusoidal projection is shown in figure 28. The intermediate projections shown are those discussed above for $\frac{1}{p}=\frac{1}{q}=\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

THE TANGENT SERIES PROJECTIONS

The meridian curves for the tangent series have inflection points, and although partially concave toward the x-axis, the net result is a convex appearance which distorts the continents in higher latitudes. This would necessarily be so, since we have seen that the Mercator authalic sinusoidal projection is the upper limiting case.

We have arranged in a tabular manner the corresponding projections for the same values of p and q as used for the sine series. The mapping equations are obtained from equations (14). The modification, equations (7) to (12), will be subsequently illustrated on only one example of the tangent series.

$p=q$	Mapping equations	Axes	Equation of meridians	Degree
1	$x=R\lambda\cos^3\phi$ $y=R\tan\phi$	$x_0=R\pi$ $y_0\to\infty$	$\frac{x^2}{R^2\lambda^2}\left(1+\frac{y^2}{R^2}\right)^3=1$	8
4/3	$x=R\lambda\cos\phi\cos^2\frac{3\phi}{4}$ $y=\frac{4}{3}R\tan\frac{3\phi}{4}$	$x_0=R\pi$ $y_0=\frac{4}{3}R\tan\frac{3\pi}{8}$	$4\left(1+\frac{9y^2}{16R^2}\right)^5\frac{x^3}{R^3\lambda^3}-3\left(1+\frac{9y^2}{16R^2}\right)^3\frac{x}{R\lambda}+$ $\left(1+\frac{9y^2}{16R^2}\right)^2=2\left(1-\frac{9y^2}{16R^2}\right)^2$	13
3/2	$x=R\lambda\cos\phi\cos^2\frac{2\phi}{3}$ $y=\frac{3}{2}R\tan\frac{2\phi}{3}$	$x_0=R\pi$ $y_0=\frac{3}{2}\sqrt{3}R$	$\left(1+\frac{4y^2}{9R^2}\right)^3\left[\frac{2x^2}{R^2\lambda^2}\left(1+\frac{4y^2}{9R^2}\right)^2-1\right]^2=$ $\left(1-\frac{4y^2}{3R^2}\right)^2$	18
2	$x=R\lambda\cos\phi\cos^2\frac{\phi}{2}$ $y=2R\tan\frac{\phi}{2}$	$x_0=R\pi$ $y_0=2R$	$\left(1+\frac{y^2}{4R^2}\right)^2\frac{x}{R\lambda}=1-\frac{y^2}{4R^2}$	5
3	$x=R\lambda\cos\phi\cos^2\frac{\phi}{3}$ $y=3R\tan\frac{\phi}{3}$	$x_0=R\pi$ $y_0=\sqrt{3}R$	$\left(1+\frac{y^2}{9R^2}\right)^5\frac{x^2}{R^2\lambda^2}=\left(1-\frac{y^2}{3R^2}\right)^2$	12

As a special case of the above tabulated projection for $q=2$, let $p=1$ instead of $p=2$. The mapping equations (14) become then

$$x=2R\lambda\cos\phi\cos^2\frac{\phi}{2},\ y=R\tan\frac{\phi}{2}$$

which are the mapping equations of the Foucaut stereographic authalic projection.[16]

For this projection $F(\phi)=\tan\frac{\phi}{2}$, $F'(\phi)=\frac{1}{2}\sec^2\frac{\phi}{2}$, $F'(0)=\frac{1}{2}$, $F\left(\frac{\pi}{2}\right)=1$.

Equations (7), (10), (11), and (12) become respectively

$$(k+1)\sin\phi=k\tan\frac{\alpha}{2}+\sin\alpha \tag{49}$$

$$M^2=\frac{k+2}{2(k+1)} \tag{50}$$

$$x=\frac{2RM\lambda}{k+2}\left(k+2\cos\alpha\cos^2\frac{\alpha}{2}\right),\ y=RM\tan\frac{\alpha}{2}. \tag{51}$$

$$\frac{x_{\frac{\pi}{2}}}{x_0}=\frac{k}{k+2}. \tag{52}$$

[16] Norbert Herz, Lehrbuch der Landkartenprojektionen, p. 167.

Let $\sin\alpha=\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$ in (49) to obtain

$$\frac{(k+1)}{2}\sin\phi=\frac{kx}{2}+\frac{x}{1+x^2}, \tag{53}$$

which, clearing of fractions, becomes

$$kx^3-(k+1)\sin\phi\cdot x^2+(k+2)x-(k+1)\sin\phi=0,\ x=\tan\frac{\alpha}{2}. \tag{54}$$

Equation (54) may be solved by Horner's contracted method or by some other numerical method[17] after three-significant-figure estimates are obtained from the abscissae of the intersections of the graphed simultaneous equations

$$y=-\frac{kx}{2}+\frac{(k+1)}{2}\sin\phi=\frac{x}{1+x^2}.$$

If a series estimate is desired we may write from (53) $(k+1)\sin\phi=kx+2x(1-x^2+x^4-x^6+x^8-\ldots)$, and reverting this series obtain

$$x=\tan\frac{\alpha}{2}=u+\frac{2u^3}{k+2}+\frac{2(4-k)}{(k+2)^2}u^5+\frac{2(10-k)(2-k)}{(k+2)^3}u^7+\frac{2(112-112k+24k^2-k^3)}{(k+2)^4}u^9+\cdots, \tag{55}$$

where $u=\dfrac{(k+1)}{(k+2)}\sin\phi=\dfrac{\sin\phi}{2M^2}$.

The relation of the tangent series of projections to the sine series of projections through the Mercator authalic sinusoidal is shown in figure 28.

AUTHALIC PROJECTION OF THE SPHEROID

Thus far, only the authalic projection of the sphere has been considered. If it is desired to apply the modified projection to the oblate spheroid one has only to substitute for the geodetic latitude, ϕ, the authalic latitude β. The authalic latitude, β, is obtained by projecting the spheroid authalically upon a sphere of equivalent surface. The area of a zone of the oblate spheroid is $2\pi b^2\left[\dfrac{\sin\phi}{2(1-e^2\sin^2\phi)}+\dfrac{1}{4e}\ln\left(\dfrac{1+e\sin\phi}{1-e\sin\phi}\right)\right]$, where b and e are the semi-minor axis and eccentricity respectively of the meridian ellipse. The area of a zone of the authalic sphere is $2\pi R^2\sin\beta$. Demanding these two zones be equal gives

$$R^2\sin\beta=b^2\left[\frac{\sin\phi}{2(1-e^2\sin^2\phi)}+\frac{1}{4e}\ln\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)\right]. \tag{56}$$

Now we have $\dfrac{\sin\phi}{2(1-e^2\sin^2\phi)}=\dfrac{\sin\phi}{2}(1+e^2\sin^2\phi+e^4\sin^4\phi+e^6\sin^6\phi+\ldots)$, and $\dfrac{1}{4e}\ln\left(\dfrac{1+e\sin\phi}{1-e\sin\phi}\right)=\dfrac{\sin\phi}{2}\left(1+\dfrac{e^2}{3}\sin^2\phi+\dfrac{e^4}{5}\sin^4\phi+\dfrac{e^6}{7}\sin^6\phi+\ldots\right)$. These expansions placed in (56) give

$$R^2\sin\beta=b^2\left(1+\frac{2}{3}e^2\sin^2\phi+\frac{3}{5}e^4\sin^4\phi+\frac{4}{7}e^6\sin^6\phi+\ldots\right)\sin\phi. \tag{57}$$

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[17] See footnote 12 on p. 16.

Placing $\beta=\phi=\frac{\pi}{2}$ in (57) gives $\frac{b^2}{R^2}=\frac{1}{1+\frac{2}{3}e^2+\frac{3}{5}e^4+\frac{4}{7}e^6+\ \ldots}$ so that (57) may be written finally as $\sin\beta=\frac{1+\frac{2}{3}e^2\sin^2\phi+\frac{3}{5}e^4\sin^4\phi+\frac{4}{7}e^6\sin^6\phi+\ \ldots}{1+\frac{2}{3}e^2+\frac{3}{5}e^4+\frac{4}{7}e^6+\ \ldots\ldots}\sin\phi.$

For specific applications, the values of the authalic latitude do not have to be computed. It has been done and tabulated in 30-minute intervals.[18]

SUMMARY

The modification of authalic projections devised here and represented by equations (7) to (12) may be applied to any equal-area projection whose parallels are straight lines. It actually accomplishes a type of combination of the given projection with the Lambert authalic cylindrical projection,[19] whose parallels and meridians are straight lines. It may be considered a generalization of the method of Dr. Max Eckert in obtaining his No. 6 projection, although he derived mapping equations of this kind for only a particular case of the modified sinusoidal projection.

EXAMPLES OF MODIFIED AUTHALIC PROJECTIONS

PROJECTION NO. 1

From equations (13) we have the mapping equations of an authalic projection with two arbitrary parameters p and q. Let us compute the values of p and q for a projection where we impose the following conditions:

The x-coordinate at $\phi=80°$ shall be ⅓ the equatorial x-coordinate. The ratio x/y for $\phi=0$, $\lambda=20°$; $\phi=20°$, $\lambda=0$ shall be 0.85.

From equations (13) and the first given condition we have, for $\phi=0$, that $x=\frac{Rq}{p}\lambda$; for $\phi=80°$, that $x=\frac{Rq\lambda\cos 80°}{p\cos\frac{80°}{q}}$, whence $\frac{Rq\lambda\cos 80°}{p\cos\frac{80°}{q}}=\frac{1}{3}\frac{Rq}{p}\lambda$, which reduces to $3\cos 80°=\cos\frac{80°}{q}$, and solving for q

$$q=\frac{80°}{\cos^{-1}(3\cos 80°)}=1.365086. \tag{58}$$

Again from equations (13), with the second of the given conditions and the value of q from (58) we have (with the value of $20°=0.3490659$ radian) placing $x/y=0.85$, $0.85\ Rp\ \sin\frac{20°}{1.365086}=\frac{R}{p}\cdot 1.365086\times 0.3490659$, whence $p^2=\frac{1.365086\times 0.3490659}{0.85\sin 14.65109°}=$ 2.2163809, and

$$p=\sqrt{2.2163809}=1.488751. \tag{59}$$

With the values of q and p from (58) and (59) the mapping equations (13) become

$$x=0.9169337R\lambda\cos\phi\sec\frac{\phi}{1.365086},\ y=1.488751R\sin\frac{\phi}{1.365086}. \tag{60}$$

Coordinate table I and figure 20 correspond to mapping equations (60).

[18] Oscar S. Adams, Latitude developments connected with geodesy and cartography, U. S. Coast and Geodetic Survey Special Publication No. 67.

[19] See footnote 10 on p. 15.

TABLE I.—*Projection No. 1*

$$\left[x=9.169337\lambda \cos\phi \sec\frac{\phi}{1.365086},\ y=14.88751 \sin\frac{\phi}{1.365086}\right]$$

Area ratio 1 to the square of 60,000,000

ϕ	x	y
°	*cm.*	*cm.*
0	28. 806	0. 0000
5	28. 755	0. 9511
10	28. 602	1. 8983
15	28. 344	2. 8377
20	27. 979	3. 7655
25	27. 500	4. 6780
30	26. 902	5. 5713
35	26. 174	6. 4419
40	25. 305	7. 2862
45	24. 278	8. 1007
50	23. 072	8. 8821
55	21. 661	9. 6272
60	20. 007	10. 333
65	18. 060	10. 997
70	15. 750	11. 615
75	12. 980	12. 186
80	9. 6021	12. 708
85	5. 3952	13. 177
90	0. 0000	13. 593

PROJECTION NO. 2

We now apply the modification given by equations (7) through (12) to the authalic projection given by equations (60). From equations (60) it is seen that $f(\phi)=1.488751 \sin\phi/1.365086$. $f'(\phi)=1.090591 \cos\phi/1.365086$, $f'(0)=1.090591$, and $f\left(\frac{\pi}{2}\right)=1.359300$. We will demand that the ratio given by (12) shall be ¼, which gives $k=\frac{1}{3f'(0)}=\frac{1}{3.271773}=0.3056447$, whence $n=kf\left(\frac{\pi}{2}\right)+1=1.4154628$, and $m=kf'(0)+1=\frac{4}{3}$.

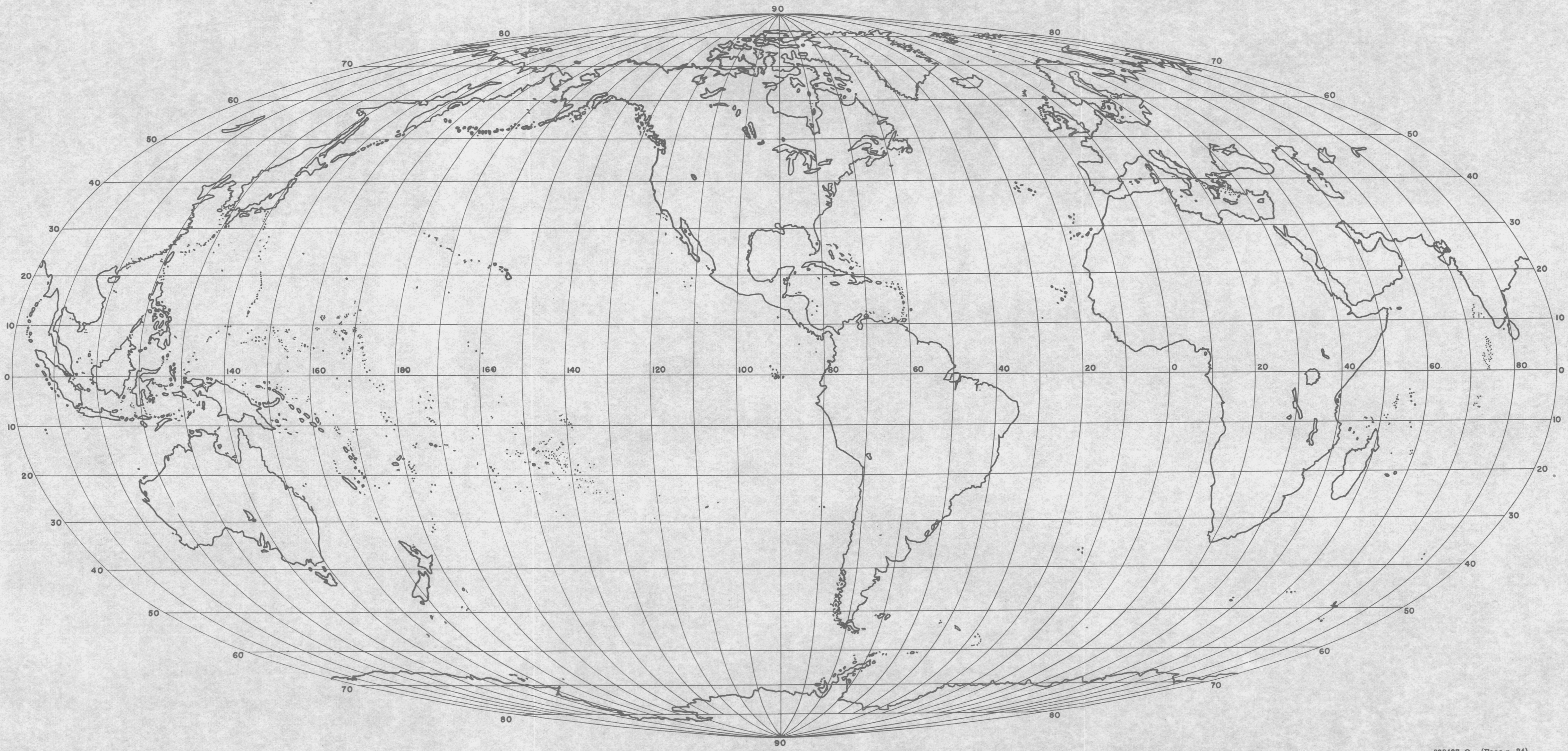


FIGURE 20.—Projection No. 1.

Demanding that the ratio of the axes in the modified projection shall be the same as in the original projection (60), we have from (10) that

$$M^2 = \frac{m}{n} = \frac{1.3333333}{1.4154628} = 0.94197693,$$

$$M = \sqrt{0.94197693} = 0.9705550.$$

The mapping equations (11) become with the above values

$$x = 0.2224837R\lambda\left(1 + 3\cos\alpha\sec\frac{\alpha}{1.365086}\right),\ y = 1.444915R\sin\frac{\alpha}{1.365086}. \tag{61}$$

The equation (7) becomes

$$1.415463\sin\phi = 0.4550289\sin\frac{\alpha}{1.365086} + \sin\alpha. \tag{62}$$

In order to solve equation (62) for α, we will use a numerical method, with the aid of a desk calculator, after obtaining estimates from the abscissae of the intersections of the graphed simultaneous equations

$$y = 1.415463\sin\phi - 0.4550289\sin\frac{\alpha}{1.365086} = \sin\alpha.$$

Figure 21 shows the graphical solution, the estimates being listed in table II with the computed values of α for 5° intervals of $0 < \phi < 90°$.

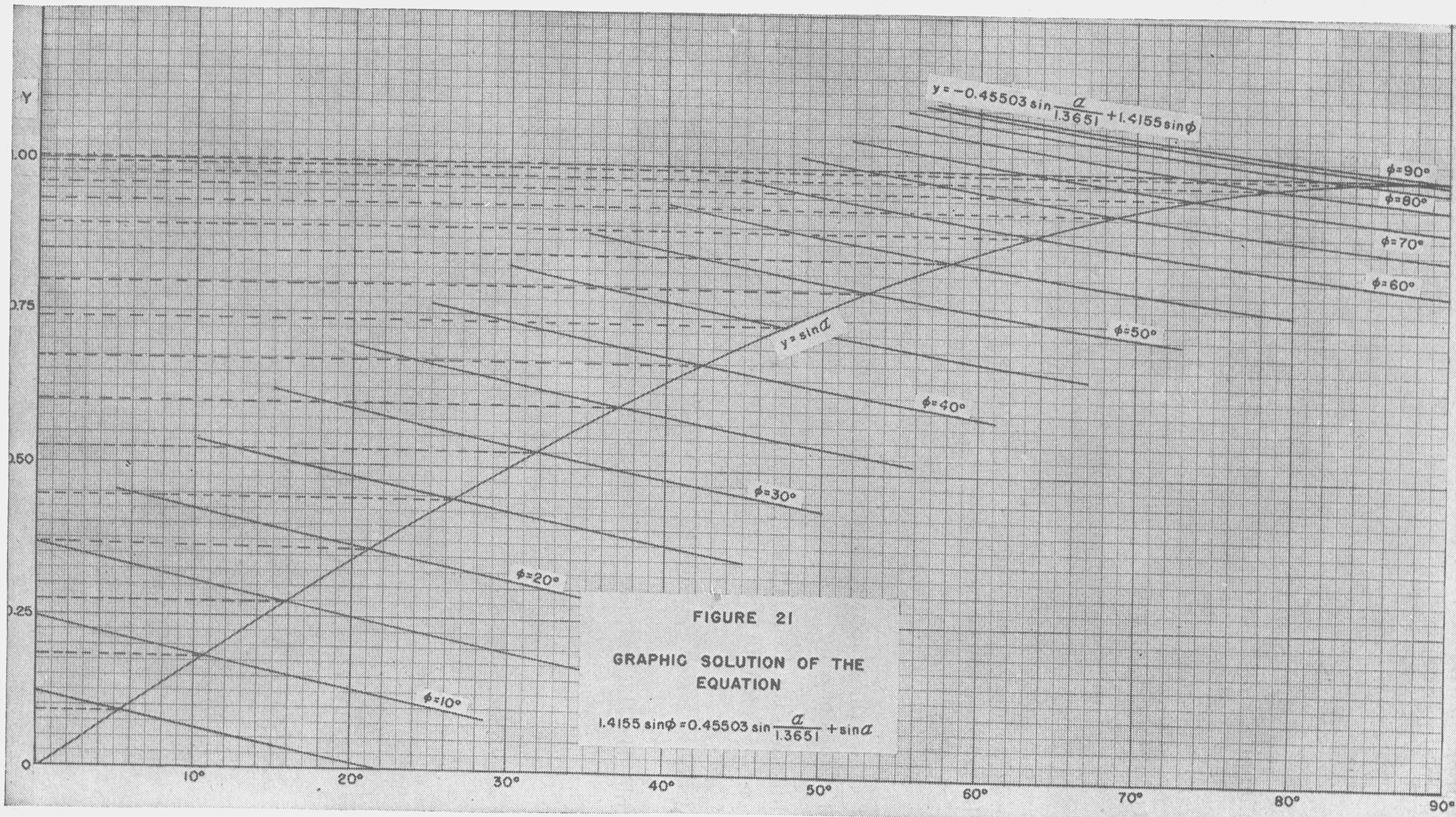
To illustrate the Regula Falsi[20] method used to compute α, we will compute its value for $\phi = 55°$. With this value of ϕ we may write equation (62) in the form

$$f(\alpha) = 1.159479 - 0.4550289\sin\frac{\alpha}{1.365086} - \sin\alpha. \tag{63}$$

From table II, the graphic estimate is $\alpha_1 = 58°18'$. Now $\sin 58°18' = 0.8508111$ and $\sin\frac{58°18'}{1.365086} = 0.6782614$. With these values equation (63) gives

$$f(\alpha_1) = 1.159479 - 1.159440 = 0.000039.$$

[20] Numerical Mathematical Analysis—J. B. Scarborough—p. 174.



Y
1.00
0.75
0.50
0.25
0
10°
20°
30°
40°
50°
60°
70°
80°
90°
y = −0.45503 sin α/1.3651 + 1.4155 sin φ
y = sin α
φ=10°
φ=20°
φ=30°
φ=40°
φ=50°
φ=60°
φ=70°
φ=80°
φ=90°
FIGURE 21
GRAPHIC SOLUTION OF THE EQUATION
1.4155 sin φ = 0.45503 sin α/1.3651 + sin α

FIGURE 21.

TABLE II.—*Graphic and computed values of α for Projection No. 2.*

ϕ	α (Graph)		α (Computed)		
°	°	′	°	′	″
0	0	00	0	00	00.0
5	5	18	5	18	28.7
10	10	36	10	36	56.8
15	15	54	15	55	23.7
20	21	12	21	13	48.7
25	26	24	26	32	10.5
30	31	48	31	50	27.4
35	37	00	37	08	37.0
40	42	24	42	26	35.5
45	47	36	47	44	17.5
50	53	00	53	01	34.0
55	58	18	58	18	10.5
60	63	30	63	33	43.6
65	68	42	68	47	30.1
70	73	54	73	58	05.5
75	78	48	79	02	25.4
80	84	00	83	52	27.1
85	88	00	88	02	01.0
90	90	00	90	00	00.0

From an examination of the differences in the trigonometric table being used, it is seen that a close value will be given by $\alpha_2=58°18'11''$. With the values $\sin 58°18'11''=0.8508391$ and $\sin\left(\frac{58°18'11''}{1.365086}\right)=0.6782900$, we have from (63) that $f(\alpha_2)=1.159479-1.159481=-0.000002$.

By the Regula Falsi method a better value of α is given by $\alpha_3=\alpha_1+\frac{(\alpha_2-\alpha_1)|f(\alpha_1)|}{|f(\alpha_1)|+|f(\alpha_2)|}$, the process being repeated until the desired accuracy is obtained. Using the above values of α_1, α_2, $f(\alpha_1)$, $f(\alpha_2)$ we obtain $\alpha_3=58°18'+11''\times\frac{0.000039}{0.000041}=58°18'10''.5$. Sin $58°18'10''.5=0.8508379$, $\sin\left(\frac{58°18'10''.5}{1.365086}\right)=0.6782887$, and with these values we have from (63) that $f(\alpha_3)=1.159479-1.159479=0$, hence the correct value is $\alpha=58°18'10''.5$. This accuracy is sufficient since we are computing our table for this projection to only five significant figures.

It will be noticed that only one Regula Falsi estimate was necessary since the graphic value, α_1, was close and a judicious choice of α_2, by use of differences in the trigonometric table, was made. Hence a careful graphic solution should be constructed, preferably on millimeter paper, and careful use should be made of the tables in choosing the next-best estimate for α.

Table III and figure 22 correspond to mapping equations (61) and the computed values of α from table II.

838437°—49——5

TABLE III.—*Projection No. 2*

$$\left[x=2.224837\left(1+3\cos\alpha\sec\frac{\alpha}{1.365086}\right)\lambda, y=14.44915\sin\frac{\alpha}{1.365086}\right]$$

Area ratio 1 to the square of 60,000,000

ϕ	α			x	y
°	°	′	″	*cm.*	*cm.*
0	0	00	00. 0	27. 958	0. 0000
5	5	18	28. 7	27. 916	0. 9798
10	10	36	56. 8	27. 791	1. 9552
15	15	55	23. 7	27. 579	2. 9214
20	21	13	48. 7	27. 278	3. 8741
25	26	32	10. 5	26. 883	4. 8088
30	31	50	27. 4	26. 388	5. 7211
35	37	08	37. 0	25. 784	6. 6069
40	42	26	35. 5	25. 059	7. 4617
45	47	44	17. 5	24. 199	8. 2817
50	53	01	34. 0	23. 182	9. 0626
55	58	18	10. 5	21. 983	9. 8007
60	63	33	43. 6	20. 568	10. 492
65	68	47	30. 1	18. 888	11. 132
70	73	58	05. 5	16. 886	11. 717
75	79	02	25. 4	14. 492	12. 240
80	83	52	27. 1	11. 670	12. 691
85	88	02	01. 0	8. 660	13. 040
90	90	00	00. 0	6. 990	13. 193

PROJECTION NO. 3

For an example of the modified sinusoidal authalic projection we demand that the ratio given by (20) be ⅓, whence $k=\frac{1}{2}$. Demanding also that the ratio of the axes be the same as in the original sinusoidal projection, we have from (18) that $M^2=6/(\pi+4)=0.84014872$, whence $M=\sqrt{0.84014872}=0.9165963$. The mapping equations (19) become

$$x=0.3055321R\lambda(1+2\cos\alpha),\ y=0.9165963R\alpha. \tag{64}$$

Equation (17) becomes

$$1.785398\sin\phi=0.5\alpha+\sin\alpha. \tag{65}$$

The series (21) becomes

$$\alpha=u+\frac{1}{9}u^3+\frac{17}{540}u^5+\frac{793}{68{,}040}u^7+\frac{72{,}161}{14{,}696{,}640}u^9+\ .\ .\ .\ . \tag{66}$$

where $u=\frac{\sin\phi}{M^2}=\frac{\pi+4}{6}\sin\phi=1.190265\sin\phi$.

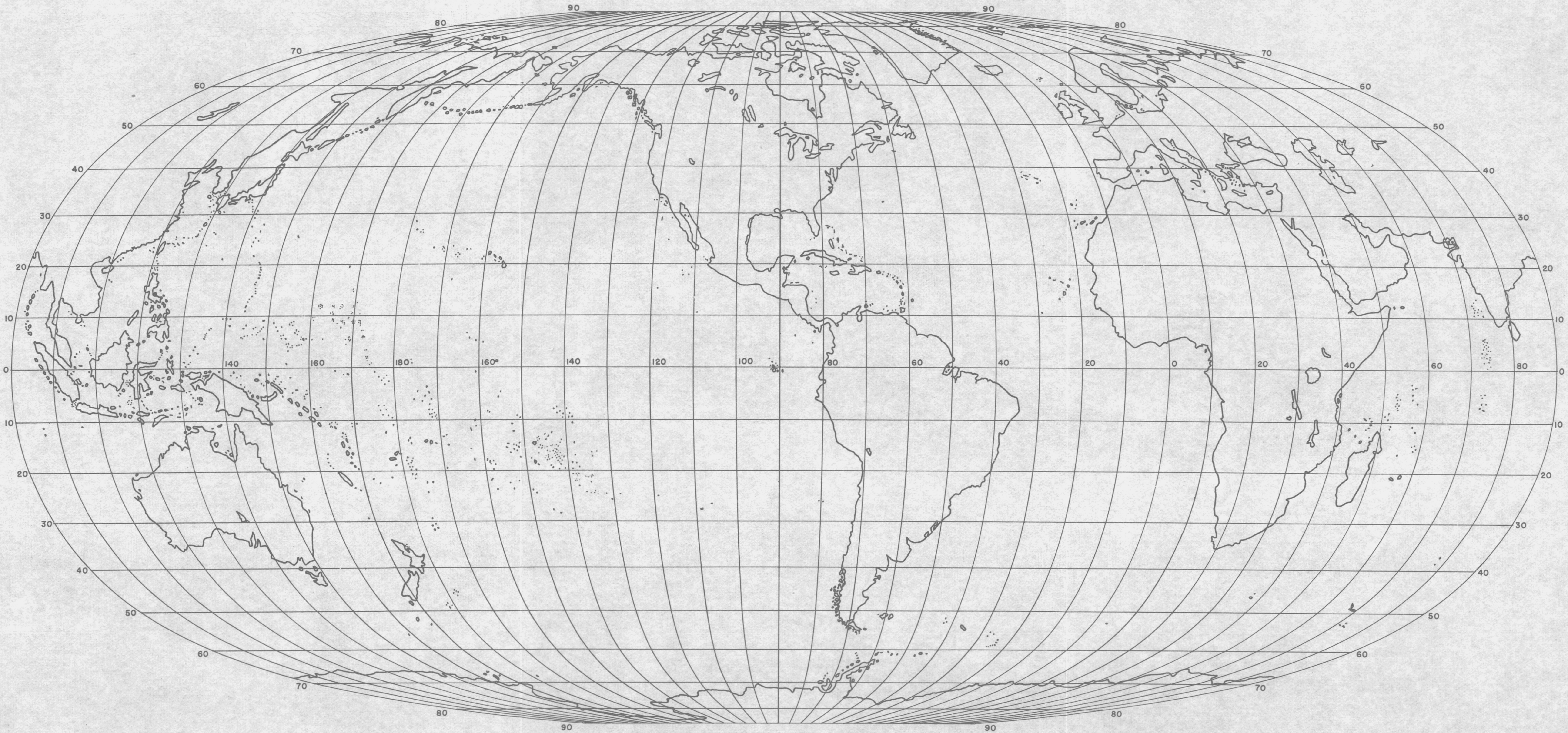


FIGURE 22.—Projection No. 2.

The series (66) may be used to obtain first estimates for α, or a graph of the simultaneous equations $y=-0.5\alpha+1.785398 \sin \phi=\sin \alpha$, as shown in figure 23, may be made, three-significant-figure estimates of the abscissae of the intersections being the first estimates for α. In either case a variation of the Newton-Raphson method may be advantageously employed, with the aid of a desk calculator, to complete the solution to the required accuracy.

The Newton-Raphson Method. If α is an approximation to the required value of the root of an equation $f(x)=0$, and $\Delta\alpha$ is a small correction which must be applied to α to give a more accurate approximation to the required accuracy of the root, then $x=\alpha+\Delta\alpha$ and $f(\alpha+\Delta\alpha)=0$, very nearly. Expanding $f(\alpha+\Delta\alpha)$ by Taylor's theorem one obtains

$$f(\alpha+\Delta\alpha)=f(\alpha)+f'(\alpha)\Delta\alpha+f''(\alpha)\frac{\Delta\alpha^2}{2!}+f'''(\alpha)\frac{\Delta\alpha^3}{3!}+\;.\;.\;=0. \tag{67}$$

Now consider $\Delta\alpha$ to be expanded in a power series in $f(\alpha)$, namely

$$\Delta\alpha=af(\alpha)+bf^2(\alpha)+cf^3(\alpha)+\;.\;.\;.\;.\;. \tag{68}$$

Substituting the value of $\Delta\alpha$ from (68) in (67) one obtains

$$f+(a+bf+cf^2+\;.\;.)ff'+(a+bf+cf^2+\;.\;.)^2\frac{f^2f''}{2!}+(a+bf+cf^2+\;.\;.)^3\frac{f^3f'''}{3!}+\;.\;.=0. \tag{69}$$

In equation (69) we now place the sums of the coefficients of like powers of f equal to zero, and solve for the values of a, b, c—as follows:

$$f:\quad 1+af'=0,\; a=-\frac{1}{f'}.$$

$$f^2:\quad bf'+\frac{a^2}{2}f''=0,\; b=-\frac{a^2f''}{2f'}=-\frac{1}{2}\frac{f''}{f'^3}$$

$$f^3:\quad cf'+abf''+\frac{a^3}{6}f'''=0,\; c=-\frac{a}{f'}\left(bf''+\frac{a^2}{6}f'''\right)=-\frac{1}{6f'^5}(3f''^2-f'f''')$$

. . . .

The values of a, b, c—returned to equation (68) give

$$\Delta\alpha=-\frac{f}{f'}-\frac{f^2f''}{2f'^3}-\frac{f^3}{6f'^5}(3f''^2-f'f''')-\text{----------},\text{ whence}$$

$$\alpha_1=\alpha+\Delta\alpha=\alpha-\frac{f}{f'}-\frac{f^2f''}{2f'^3}-\frac{f^3}{6f'^5}(3f''^2-f'f''')-\;.\;.\;.\;. \tag{70}$$

For practicable use of the higher order terms of (70), the derivatives f'', f'''— must be easily computed. Otherwise all terms involving higher-order derivatives than the first in equation (70) may be ignored and successive approximations made with the resulting formula, which is the Newton-Raphson Method.[21] Of course the accuracy required may not warrant use of the higher-order terms in any event.

[21] Practical Analysis, Fr. A. Willers, p. 222.

For the solution of equation (65), the successive derivatives are easily obtained. We may write (65) in the form $f(\alpha)=0.5\alpha+\sin\alpha-1.785398\sin\phi$, whence $f'(\alpha)=0.5+\cos\alpha$, $f''(\alpha)=-\sin\alpha$, etc. We will not need $f'''(\alpha)$ since we are going to use seven significant figures in the computations in order to obtain five-significant-figure values to be listed in the coordinate table for the projection.

With the above values of $f'(\alpha)$ and $f''(\alpha)$, ignoring all terms involving derivatives higher than the second, we may write equation (70) as

$$\alpha_1=\alpha-\frac{f}{0.5+\cos\alpha}+\frac{f^2\sin\alpha}{2(0.5+\cos\alpha)^3}, \tag{71}$$

where $f(\alpha)=0.5\alpha+\sin\alpha-1.785398\sin\phi$.

Table IV lists the graphical estimates from figure 23, and the computed values obtained by means of equation (71).

TABLE IV.—*Graphic and computed values of α for Projection No. 3*

ϕ	α (Graph)	α (Computed)	α (Radians)
°	° ′	° ′ ″	
0	0 00	0 00 00. 0	0. 0000000
5	6 00	5 57 03. 3	0. 1038629
10	11 54	11 53 57. 2	0. 2076805
15	17 48	17 50 31. 3	0. 3114023
20	23 42	23 46 32. 9	0. 4149662
25	29 42	29 41 45. 2	0. 5182910
30	35 30	35 35 45. 5	0. 6212667
35	41 30	41 28 02. 2	0. 7237403
40	47 18	47 17 50. 7	0. 8254955
45	53 00	53 04 07. 1	0. 9262224
50	58 42	58 45 18. 6	1. 0254711
55	64 12	64 19 09. 0	1. 1225813
60	69 42	69 42 15. 6	1. 2165699
65	74 54	74 49 34. 9	1. 3059662
70	79 36	79 33 32. 4	1. 3885665
75	83 36	83 43 01. 2	1. 4611375
80	87 00	87 02 39. 2	1. 5192084
85	89 12	89 13 54. 3	1. 5573879
90	90 00	90 00 00. 0	1. 5707963

As an example of the use of (71), we now compute the value of α for $\phi=50°$. Sin $50°=0.7660444$, and we have from (65)

$$f(\alpha)=0.5\alpha+\sin\alpha-1.3676941. \tag{72}$$

From table IV the graphic estimate is $\alpha_1=58°42'=1.0245083$ radians. Sin $58°42'=0.8544588$ and cos $58°42'=0.5195191$. Equation (72) gives with these values $f(\alpha_1)=-0.0009812=-9.812\times10^{-4}$. With this value of $f(\alpha_1)$ and the other above-needed values equation (71) becomes

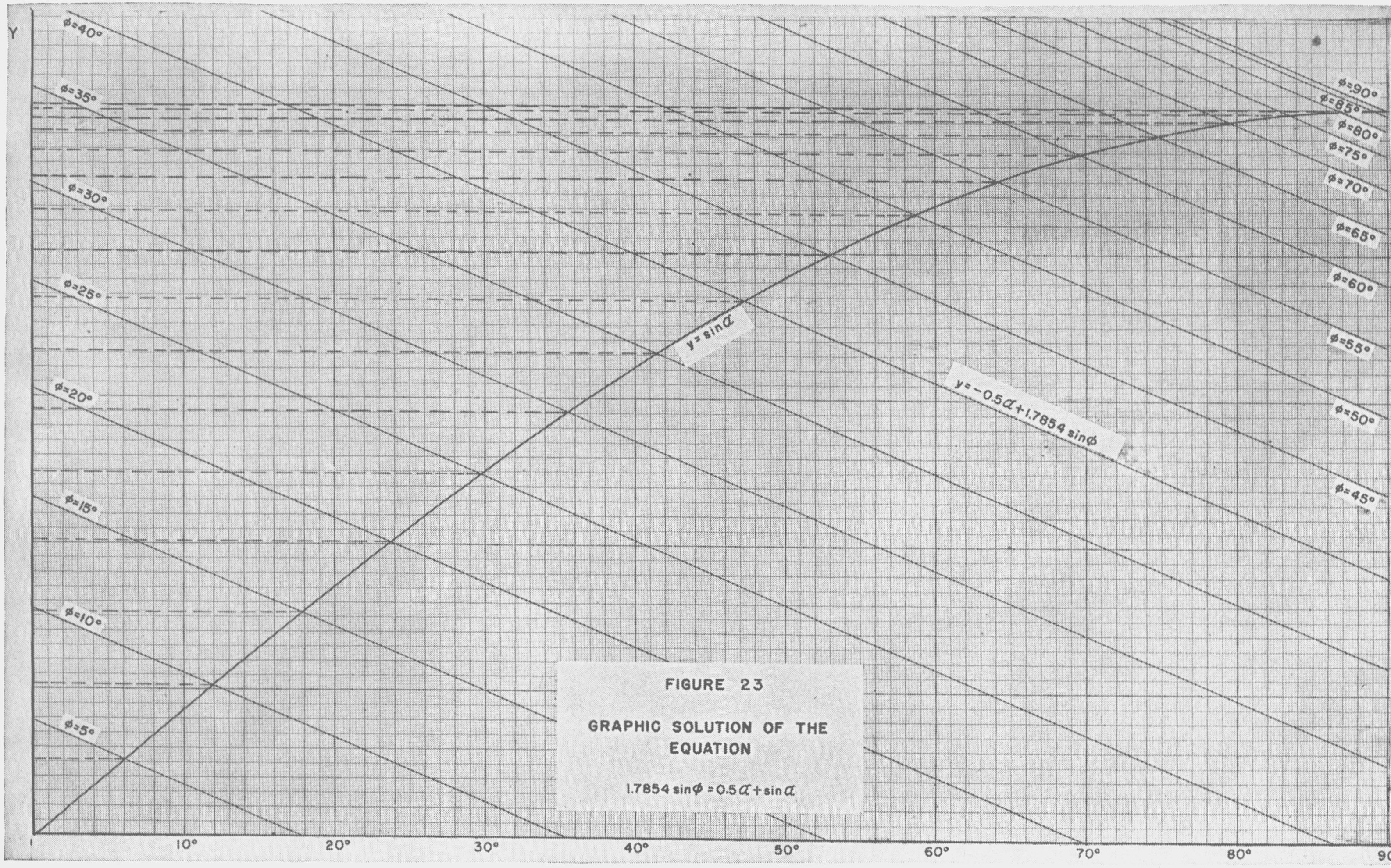

Y
φ=40°
φ=35°
φ=30°
φ=25°
φ=20°
φ=15°
φ=10°
φ=5°
φ=90°
φ=85°
φ=80°
φ=75°
φ=70°
φ=65°
φ=60°
φ=55°
φ=50°
φ=45°
y=sin α
y=−0.5α+1.7854 sin φ
FIGURE 23
GRAPHIC SOLUTION OF THE EQUATION
1.7854 sin φ = 0.5α + sin α
10°
20°
30°
40°
50°
60°
70°
80°
90°

Figure 23.

$$\alpha_2=1.0245083+\frac{9.812\times10^{-4}}{1.0195191}+\frac{0.8544588\times(9.812)^2\times10^{-8}}{2(1.0195191)^3}$$

$$\alpha_2=1.0245083+9.624\times10^{-4}+4.0\times10^{-7}$$

$$\alpha_2=1.0254711 \text{ radians}=58°45'18\rlap{.}''6.$$

Sin $58°45'18\rlap{.}''6=0.8549586$, and equation (72) gives $f(\alpha_2)=0.5127355+0.8549586-1.3676941=0$. Hence the correct value is $\alpha_2=58°45'18\rlap{.}''6$.

It will be found that no repetitions are necessary for $0<\phi<90°$ if equation (71) is used with the graphic estimates as listed in table IV. If available, a seven-place table giving the natural trigonometric functions for radian arguments is very useful. However, it is not difficult to convert degrees to radians by the formula $1°=0.0174532925$ radian, since it is assumed that a desk calculator will be used.

Table V and figure 24 correspond to the computed α values of Table IV and to the mapping equations (64) for this example of the modified sinusoidal projection which we will call the flat-polar sinusoidal authalic projection.

TABLE V.—*Projection No. 3, flat-polar sinusoidal authalic projection*

[$x=3.055321(1+2\cos\alpha)\lambda$; $y=9.165963\alpha$]
Area ratio 1 to the square of 60,000,000

ϕ	α				x	y
°	°	′	″	*radians*	*cm.*	*cm.*
0	0	00	00. 0	0. 0000000	28. 796	0. 0000
5	5	57	03. 3	0. 1038629	28. 692	0. 9520
10	11	53	57. 2	0. 2076805	28. 383	1. 9036
15	17	50	31. 3	0. 3114023	27. 872	2. 8543
20	23	46	32. 9	0. 4149662	27. 166	3. 8036
25	29	41	45. 2	0. 5182910	26. 275	4. 7506
30	35	35	45. 5	0. 6212667	25. 209	5. 6945
35	41	28	02. 2	0. 7237403	23. 984	6. 6338
40	47	17	50. 7	0. 8254955	22. 618	7. 5665
45	53	04	07. 1	0. 9262224	21. 133	8. 4897
50	58	45	18. 6	1. 0254711	19. 556	9. 3994
55	64	19	09. 0	1. 1225813	17. 918	10. 290
60	69	42	15. 6	1. 2165699	16. 257	11. 151
65	74	49	34. 9	1. 3059662	14. 623	11. 970
70	79	33	32. 4	1. 3885665	13. 078	12. 728
75	83	43	01. 2	1. 4611375	11. 699	13. 393
80	87	02	39. 2	1. 5192084	10. 588	13. 925
85	89	13	54. 3	1. 5573879	9. 8560	14. 275
90	90	00	00. 0	1. 5707963	9. 5986	14. 398

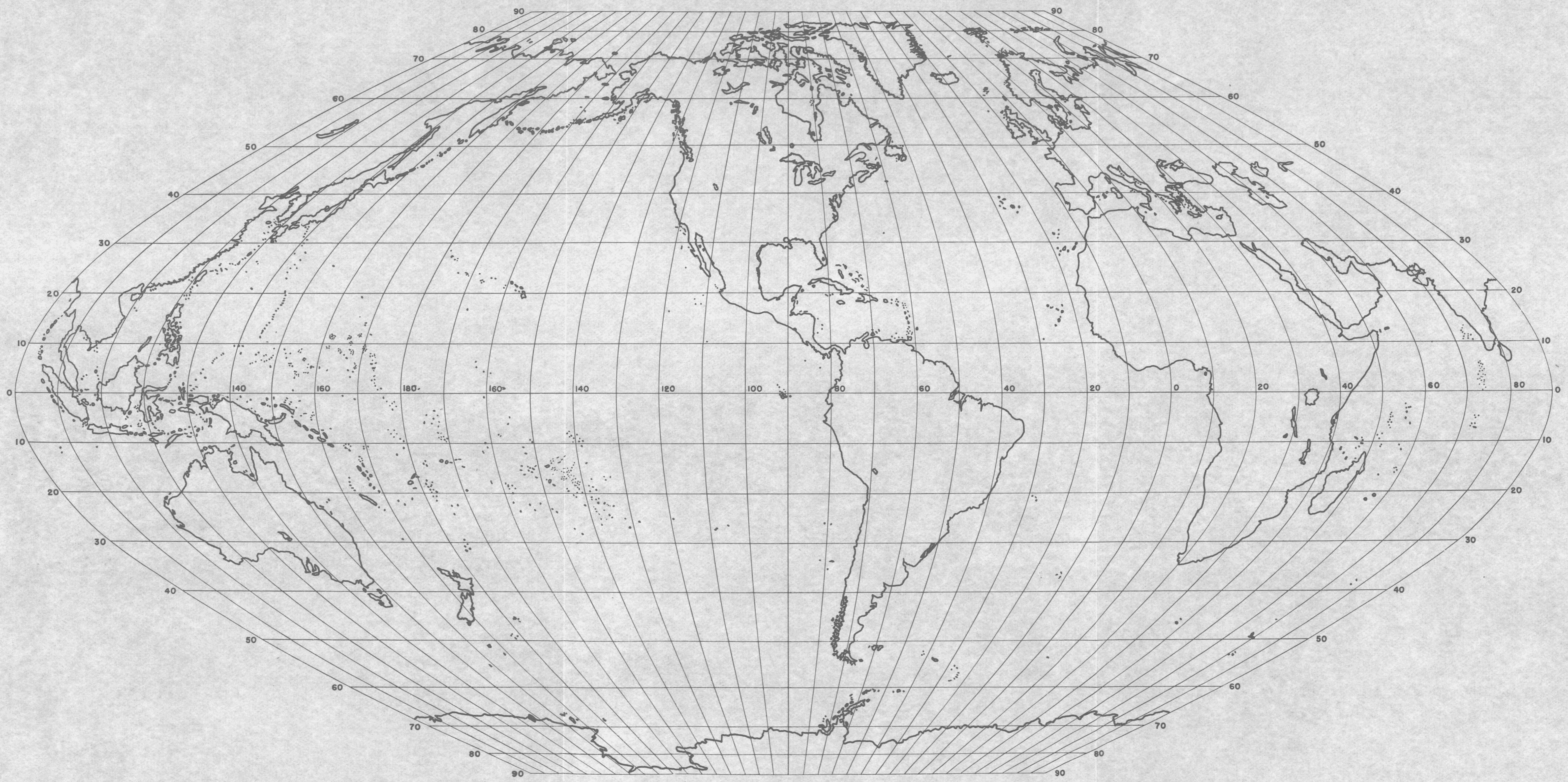


FIGURE 24.—Projection No. 3—Flat-polar sinusoidal authalic projection.

PROJECTION NO. 4

For an example of the modification of the Adams authalic projection, we make the same requirement as for the sinusoidal, namely, that the ratio given by (32) be ⅓ and that the ratio of the axes in the modified projection be the same as in the original Adams projection. Hence from (32) we find $k=\frac{1}{2}$ and with this value of k, equation (30) gives $M^2=3/(\sqrt{2}+2)=0.878679657$, whence $M=0.93737914$. The mapping equations (31) become

$$x=0.31245971\ R\lambda(1+2\cos\alpha\sec\alpha/2),\ y=1.87475828\ R\sin\alpha/2. \tag{73}$$

Equations (29) and (33) become respectively, since $n=\frac{\sqrt{2}}{2}+1=1.70710678$,

$$1.70710678\sin\phi=\sin\frac{\alpha}{2}+\sin\alpha, \tag{74}$$

$$x^4-0.75\ x^2-(0.85355339\sin\phi)x+0.72855339\sin^2\phi=0, \tag{75}$$

where $x=\sin\frac{\alpha}{2}$.

The series estimate from (34) is

$$x=\sin\frac{\alpha}{2}=u+\frac{1}{3}u^3+\frac{5}{12}u^5+\frac{17}{24}u^7+\frac{7195}{5184}u^9+\ldots, \tag{76}$$

where $u=(n\sin\phi)/3=0.56903559\sin\phi$.

Figure 25 is the graphic solution of the simultaneous equations $y=-0.5x+0.85355\cdot\sin\phi=x\sqrt{1-x^2}$. The curve $y=x\sqrt{1-x^2}$ is easily constructed from a table of natural trigonometric functions since $x=\sin\frac{\alpha}{2}$, $y=x\sqrt{1-x^2}=\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\frac{1}{2}\sin\alpha$. It needs to be constructed only from the origin to its maximum point $x=\frac{\sqrt{2}}{2}$, $y=\frac{1}{2}$.

While the series (76) will give good estimates for α for $\phi<45°$, it is evident that as ϕ becomes large and as $\phi\to 90°$, too many terms of the series would be needed. The graphic estimates are consistent over the entire range of ϕ, although only three-significant-figure estimates are obtainable from a graphic solution carefully constructed on millimeter paper.

In table VI are listed the graphic estimates, at 5° intervals, with the computed values of α.

Since Horner's contracted method is well known and described in most college algebra textbooks,[22] the mechanics of the method will not be described here. It is a good machine method for use with the graphic or series estimates to solve equation (75). We will describe a comparable method based on equation (70) to solve equation (74) for α. It is assumed that a desk calculator and eight-place tables of natural sines and cosines are available.

[22] H. B. Fine, College Algebra, pp. 456–457.

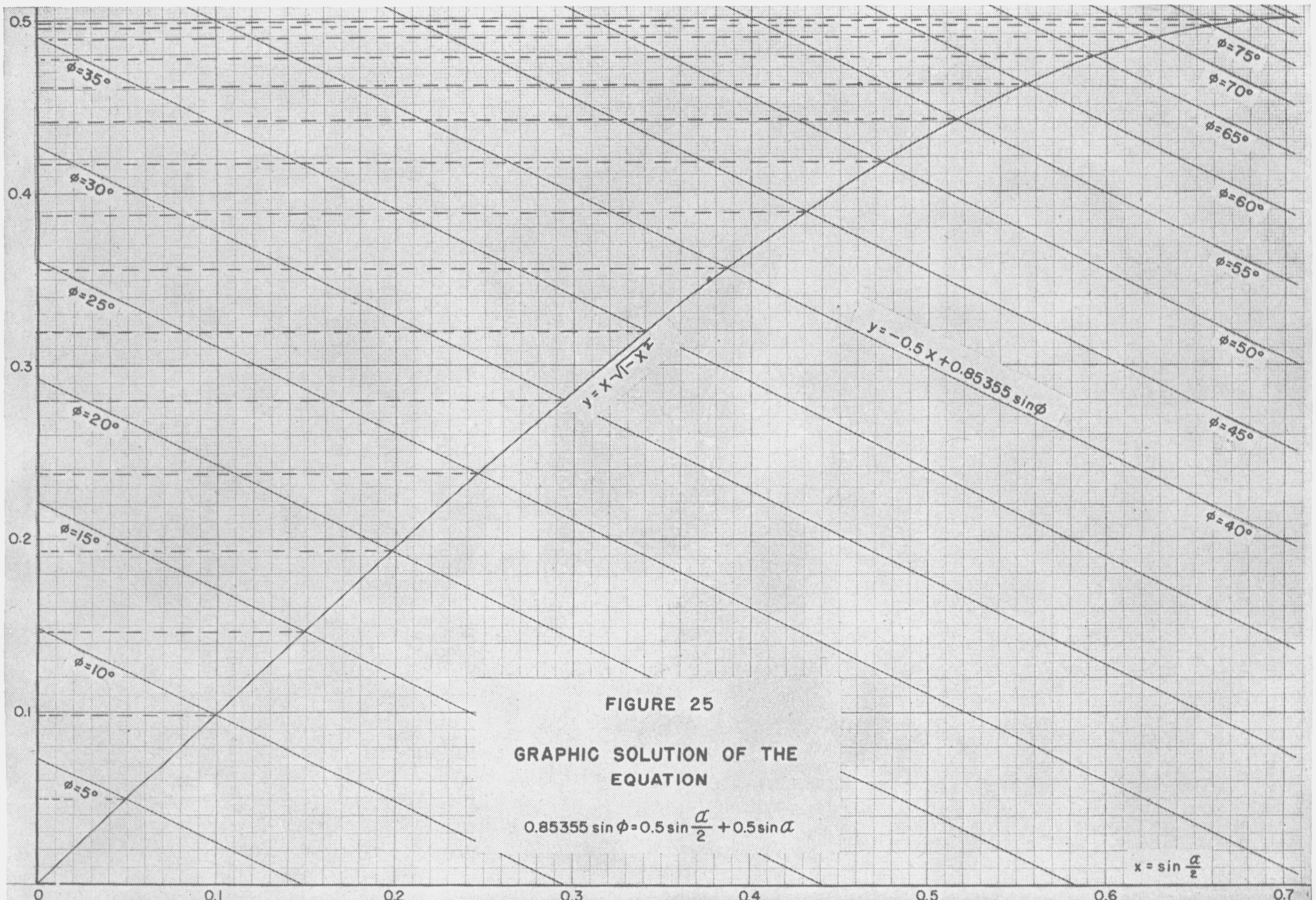


FIGURE 25
GRAPHIC SOLUTION OF THE EQUATION
0.85355 sin φ = 0.5 sin α/2 + 0.5 sin α
y = X√(1−X²)
y = −0.5 X + 0.85355 sin φ
x = sin α/2
φ=5°
φ=10°
φ=15°
φ=20°
φ=25°
φ=30°
φ=35°
φ=40°
φ=45°
φ=50°
φ=55°
φ=60°
φ=65°
φ=70°
φ=75°
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0

FIGURE 25.

From equation (74) we have $f(\alpha)=\sin\frac{\alpha}{2}+\sin\alpha-1.70710678\sin\phi$.

$f'(\alpha)=\frac{1}{2}\cos\frac{\alpha}{2}+\cos\alpha$, $f''(\alpha)=-\frac{1}{4}\sin\frac{\alpha}{2}-\sin\alpha$, $f'''(\alpha)=-\frac{1}{8}\cos\frac{\alpha}{2}-\cos\alpha$. With these values of f', f'', f''' we may write equation (70) as

$$\alpha_1=\alpha-\frac{f(\alpha)}{w}+\frac{vf^2(\alpha)}{2w^3}-\frac{(3v^2+wt)f^3(\alpha)}{6w^5}, \tag{77}$$

where $f(\alpha)=\sin\frac{\alpha}{2}+\sin\alpha-1.70710678\sin\phi$, $v=0.25\sin\frac{\alpha}{2}+\sin\alpha$,

$w=0.5\cos\frac{\alpha}{2}+\cos\alpha$, $t=0.125\cos\frac{\alpha}{2}+\cos\alpha$.

TABLE VI.—*Graphic and computed values of α for Projection No. 4.*

ϕ	$\sin\frac{\alpha}{2}$ (Graph)	$\sin\frac{\alpha}{2}$ (Computed)	$\alpha/2$			α		
			°	′	″	°	′	″
0	0.000	0.00000000	0	00	00.000	0	00	00.000
5	0.049	0.04963551	2	50	42.267	5	41	24.534
10	0.099	0.09913758	5	41	22.238	11	22	44.476
15	0.149	0.14837208	8	31	57.353	17	03	54.706
20	0.198	0.19720334	11	22	24.476	22	44	48.952
25	0.246	0.24549317	14	12	39.530	28	25	19.060
30	0.294	0.29309936	17	02	36.965	34	05	13.930
35	0.340	0.33987360	19	52	09.024	39	44	18.048
40	0.386	0.38565828	22	41	04.608	45	22	09.216
45	0.431	0.43028132	25	29	07.494	50	58	14.988
50	0.474	0.47354791	28	15	53.448	56	31	46.896
55	0.516	0.51522630	31	00	45.295	62	01	30.590
60	0.555	0.55502285	33	42	44.237	67	25	28.474
65	0.593	0.59253633	36	20	13.924	72	40	27.848
70	0.628	0.62717148	38	50	30.286	77	41	00.572
75	0.658	0.65797562	41	08	44.392	82	17	28.784
80	0.684	0.68335568	43	06	23.138	86	12	46.276
85	0.700	0.70076993	44	29	19.710	88	58	39.420
90	0.707	0.70710678	45	00	00.000	90	00	00.000

As an example of the use of (77), we compute the value of α for $\phi=45°$. Sin $45°=0.70710678$, whence $f(\alpha)=\sin\frac{\alpha}{2}+\sin\alpha-1.20710678$. From table VI, the graphic value is $\sin\frac{\alpha}{2}=0.431$, whence $\alpha=51°03'43''.484=0.89120140$ radians. Sin $\alpha=0.77782737$, $\cos\alpha=0.62847800$, $\cos\frac{\alpha}{2}=0.90235193$,

$f(\alpha)=0.431+0.77782737-1.20710678=+1.72059\times10^{-3}$,

$f^2(\alpha)=2.9604\times10^{-6}$, $f^3(\alpha)=5.0936\times10^{-9}$,

$w=0.5 \cos\frac{\alpha}{2}+\cos\alpha=1.07965396,\ w^3=1.259,\ w^5=1.5,$

$v=0.25 \sin\frac{\alpha}{2}+\sin\alpha=0.8856,\ 3v^2=2.353,$

$t=0.125 \cos\frac{\alpha}{2}+\cos\alpha=0.7413,$ and $3v^2+wt=3.2.$

With these values equation (77) becomes

$$\alpha_1=0.89120140-\frac{1.72059\times10^{-3}}{1.07965396}+\frac{0.8856\times2.960\times10^{-6}}{2\times1.259}-\frac{3.2\times5.1\times10^{-9}}{6\times1.5},$$

$$\alpha_1=0.89120140-1.59364\times10^{-3}+1.04\times10^{-6}-2\times10^{-9},$$

$$\alpha_1=0.88960880 \text{ radians}=50°58'14''.988,\ \frac{\alpha_1}{2}=25°29'07''.494.$$

$\sin\alpha_1=0.77682546$, $\sin\frac{\alpha_1}{2}=0.43028132$, and we have $f(\alpha_1)=0.43028132+0.77682546-1.20710678=0$. Note that the last term of (77) was not necessary to give an eight-place check. In general, if the graphic estimates from table VI are used, equation (77) will give a check to eight significant figures without the use of the last term. The values of α thus computed will give, when used in the mapping equations (73), coordinates to six significant figures.

After the 5° intervals have been computed, the estimates for the 1° intervals may be obtained by differencing and interpolation, the values then being computed to the required accuracy by equation (77).

TABLE VII.—*Projection No. 4, flat-polar quartic authalic projection*

[$x=3.1245971(1+2\cos\alpha\sec\frac{1}{2}\alpha)\lambda$; $y=18.7475826\sin\frac{1}{2}\alpha$]

Area ratio 1 to the square of 60,000,000.

ϕ	α			x	y
°	°	′	″	*cm.*	*cm.*
0	0	00	00. 000	29. 4486	0. 00000
5	5	41	24. 534	29. 3760	0. 93055
10	11	22	44. 476	29. 1580	1. 85859
15	17	03	54. 706	28. 7943	2. 78162
20	22	44	48. 952	28. 2843	3. 69709
25	28	25	19. 060	27. 6273	4. 60240
30	34	05	13. 930	26. 8224	5. 49490
35	39	44	18. 048	25. 8686	6. 37181
40	45	22	09. 216	24. 7651	7. 23016
45	50	58	14. 988	23. 5117	8. 06673
50	56	31	46. 896	22. 1093	8. 87788
55	62	01	30. 590	20. 5614	9. 65925
60	67	25	28. 474	18. 8767	10. 4053
65	72	40	27. 848	17. 0741	11. 1086
70	77	41	00. 572	15. 1929	11. 7579
75	82	17	28. 784	13. 3133	12. 3355
80	86	12	46. 276	11. 5923	12. 8113
85	88	58	39. 420	10. 3073	13. 1377
90	90	00	00. 000	9. 8162	13. 2565

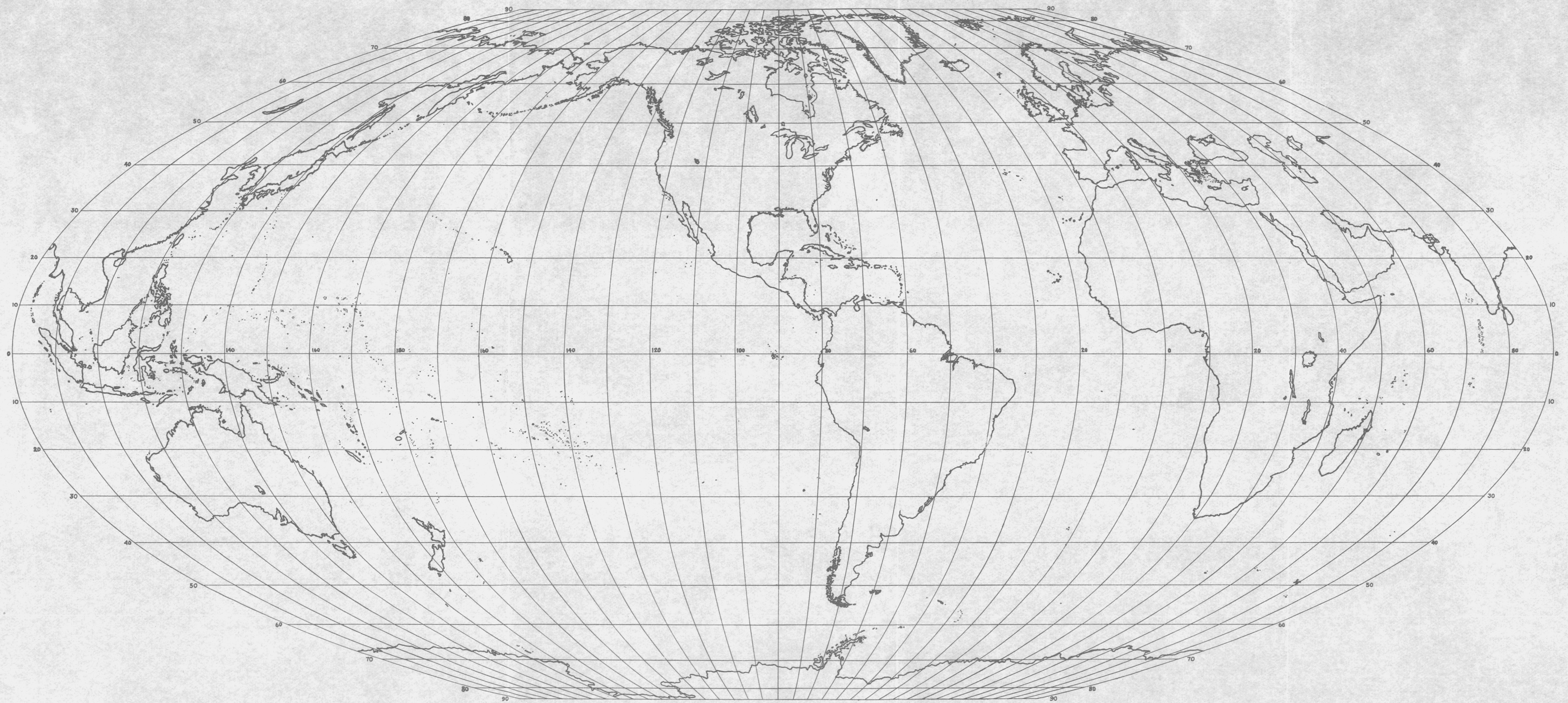


FIGURE 26.—Projection No. 4—Flat-polar quartic authalic projection.

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Table VII and figure 26 correspond to the computed values of α listed in table VI and to the mapping equations (73) for this example of the modified Adams projection which we will call the flat-polar quartic authalic projection. In table IX, computed in 1° intervals for this example of the modified Adams authalic projection, the corresponding authalic latitudes [23] were used, thus taking into account the spheroidal shape of the earth to improve the accuracy in compiling large-scale maps.

PROJECTION NO. 5

For a final example of the computation of the coordinates for a flat-polar projection we choose the parabolic type of the sine series given by $p=q=3$ in equations (13). This is not the Craster parabolic projection. In the Craster projection $q=3, p=\sqrt{3\pi}$ so that the ratio of the axes is ½. If the Craster projection is opened at the poles to produce the flat-polar type projection, it naturally shows greater north-south stretching in equatorial regions as did the sinusoidal whose axis ratio is also ½. (See fig. 24.) The sine series parabolic projection, whose mapping equations are $x=R\lambda \cos\phi \sec\frac{\phi}{3}$, $y=3R\sin\frac{\phi}{3}$, whose meridians are the parabolas $\frac{x}{R\lambda}=1-\frac{4}{9}\frac{y^2}{R^2}$, and whose axes ratio is $\frac{3}{2\pi}$, or $<\frac{1}{2}$, gives less north-south distortion in equatorial regions when opened at the poles.

For this projection $f(\phi)=3\sin\frac{\phi}{3}$, $f'(\phi)=\cos\frac{\phi}{3}$, $f\left(\frac{\pi}{2}\right)=\frac{3}{2}$, $f'(0)=1$. We may use equations (22) to (28) but with a different value of n. Requiring that the ratio given by (25) be ½ and with $n=kf\left(\frac{\pi}{2}\right)+1=\frac{3}{2}k+1$ we have $k=\frac{1}{2}$, $n=1.75$. From (23) $M^2=\frac{2n+1}{3n}=\frac{6}{7}=0.857142857$, whence $M=0.92582010$.

The mapping equations become

$$x=0.30860670R\lambda\left(1+\frac{2\cos\alpha}{\cos\frac{\alpha}{3}}\right),\ y=2.7774603R\sin\frac{\alpha}{3}. \tag{78}$$

Equations (26), (27), and (28) become respectively

$$x^3-1.125x+0.4375\sin\phi=0,\ x=\sin\frac{\alpha}{3}. \tag{79}$$

$$\sin u=\frac{7}{18}\sqrt{6}\sin\phi, \tag{80}$$

$$x=\sin\frac{\alpha}{3}=\frac{\sqrt{6}}{2}\sin\frac{u}{3}. \tag{81}$$

To illustrate the computations by means of equations (80) and (81), the value of $\sin\frac{\alpha}{3}$ for $\phi=65°$ will be computed. From (80), with $\sqrt{6}=2.44948974$ and $\sin 65°=0.90630779$, we have $\sin u=\frac{7}{18}\times 2.44948974\times 0.90630779=0.86333008$, whence $u=59°\ 41'\ 33''.237$, $\frac{u}{3}=19°\ 53'\ 51''.079$, $\sin\frac{u}{3}=0.34033888$. From (81) $x=\sin\frac{\alpha}{3}=1.22474487\times 0.34033888=0.41682830$.

[23] See footnote 18 on p. 23.

Table VIII gives the computed values of α and the mapping coordinates computed from (78) for this modified parabolic projection which we will call the flat-polar parabolic authalic projection. Figure 27 illustrates this projection.

TABLE VIII.—*Projection No. 5, flat-polar parabolic authalic projection*

$$\left[x=3.0860670\left(1+2\cos\alpha\sec\frac{\alpha}{3}\right)\lambda,\ y=27.774603\sin\frac{\alpha}{3}\right]$$

Area ratio 1 to the square of 60,000,000

ϕ	$\sin\frac{\alpha}{3}$	$\alpha/3$			α			x	y
°		°	′	″	°	′	″	*cm.*	*cm.*
0	0. 00000000	0	00	00. 000	0	00	00. 000	29. 0855	0. 00000
5	0. 03392862	1	56	39. 623	5	49	58. 869	28. 9962	0. 94235
10	0. 06780697	3	53	16. 930	11	39	50. 790	28. 7289	1. 88331
15	0. 10158364	5	49	49. 336	17	29	28. 008	28. 2851	2. 82145
20	0. 13520480	7	46	13. 667	23	18	41. 001	27. 6676	3. 75526
25	0. 16861261	9	42	25. 787	29	07	17. 361	26. 8804	4. 68315
30	0. 20174312	11	38	20. 078	34	55	00. 234	25. 9287	5. 60334
35	0. 23452334	13	33	48. 699	40	41	26. 097	24. 8195	6. 51379
40	0. 26686694	15	28	40. 499	46	26	01. 497	23. 5617	7. 41212
45	0. 29866761	17	22	39. 339	52	07	58. 017	22. 1668	8. 29537
50	0. 32978893	19	15	21. 474	57	46	04. 422	20. 6499	9. 15976
55	0. 36004757	21	06	11. 223	63	18	33. 669	19. 0309	10. 0002
60	0. 38918631	22	54	13. 962	68	42	41. 886	17. 3376	10. 8095
65	0. 41682830	24	38	04. 222	73	54	12. 666	15. 6095	11. 5772
70	0. 44240196	26	15	26. 051	78	46	18. 153	13. 9052	12. 2875
75	0. 46502555	27	42	42. 753	83	08	08. 259	12. 3130	12. 9159
80	0. 48336972	28	54	20. 578	86	43	01. 734	10. 9636	13. 4254
85	0. 49563642	29	42	42. 211	89	08	06. 633	10. 0321	13. 7661
90	0. 50000000	30	00	00. 000	90	00	00. 000	9. 6952	13. 8873

It will be noted that the computed values of the coordinates, in all the tables except table IX, have been multiplied by 10. Then the coordinates as listed in the tables if taken in centimeters give a projection whose area ratio is 1 to the square of 60,000,000. This is assuming that the radius of the earth, considered a sphere, is $R=6{,}000{,}000$ meters. In table IX, since we used the authalic latitudes in the computations, we should in scaling use the radius of the authalic sphere which is $R=6{,}370{,}997.2$ meters with respect to the Clarke 1866 spheroid. In addition, it should be noted that for all the tables, except table IX, only the coordinates of the bounding meridian have been computed, since the meridians are equally spaced for each parallel of latitude in all the authalic projections for which tables are given.

The choice of numerical method used above for any particular projection was of course arbitrary and we have by no means exhausted the methods by which coordinates may be computed for modified projections as described here. Obviously there are functions other than the sine and tangent which may be employed to generate series of such projections.

FIGURE 27.—Projection No. 5—Flat-polar parabolic authalic projection.

FLAT-POLAR QUARTIC AUTHALIC PROJECTION TABLE FOR WORLD OR SECTIONAL MAPPING

Table VII was computed for 5-degree intervals only, without using authalic latitudes, the earth being considered a sphere whose radius is 6,000,000 meters.

The following table, in 1-degree intervals, for the construction of the flat-polar quartic authalic projection (Projection No. 4, fig. 26) for world or sectional mapping was computed with the corresponding authalic latitudes [24] to take into account the spheroidal shape of the earth, the spheroid of reference being the Clarke 1866. For this spheroid the radius of the corresponding authalic sphere is 6,370,997.2 meters. The coordinates as listed are in centimeters and give a map whose area ratio is 1 to the square of 1,000,000.

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[24] See footnote 18 on p. 23.

TABLE IX.—*Flat-polar quartic authalic projection (1° interval).*

[$x=199.06799\ (1+2\cos\alpha\sec\alpha/2)\lambda$; $y=1{,}194.4080\sin\alpha/2$]

Latitude ϕ	α	x coordinate						y coordinate
		Longitude from central meridian						
		180°	90°	60°	30°	5°	1°	
°	° ′ ″	cm.	cm.	cm.	cm.	cm.	cm.	cm.
0	0 00 00.000	1,876.172	938.086	625.391	312.695	52.116	10.423	0.000
1	1 07 58.538	1,875.988	937.994	625.329	312.665	52.111	10.422	11.809
2	2 15 57.064	1,875.438	937.719	625.146	312.573	52.096	10.419	23.616
3	3 23 55.558	1,874.521	937.261	624.840	312.420	52.070	10.414	35.421
4	4 31 54.018	1,873.237	936.619	624.412	312.206	52.034	10.407	47.222
5	5 39. 52. 427	1,871.586	935.793	623.862	311.931	51.989	10.398	59.019
6	6 47 50.764	1,869.568	934.784	623.189	311.595	51.932	10.386	70.809
7	7 55 49.020	1,867.182	933.591	622.394	311.197	51.866	10.373	82.593
8	9 03 47.176	1,864.429	932.215	621.476	310.738	51.790	10.358	94.368
9	10 11 45.210	1,861.309	930.654	620.436	310.218	51.703	10.341	106.133
10	11 19 43.106	1,857.820	928.910	619.273	309.637	51.606	10.321	117.888
11	12 27 40.842	1,853.963	926.981	617.988	308.994	51.499	10.300	129.631
12	13 35 38.394	1,849.737	924.868	616.579	308.289	51.382	10.276	141.361
13	14 43 35.744	1,845.142	922.571	615.047	307.524	51.254	10.251	153.076
14	15 51 32.848	1,840.178	920.089	613.393	306.696	51.116	10.223	164.775
15	16 59 29.678	1,834.844	917.422	611.615	305.807	50.968	10.194	176.458
16	18 07 26.216	1,829.140	914.570	609.713	304.857	50.809	10.162	188.123
17	19 15 22.406	1,823.065	911.532	607.688	303.844	50.641	10.128	199.768
18	20 23 18.218	1,816.619	908.309	605.540	302.770	50.462	10.092	211.392
19	21 31 13.602	1,809.801	904.900	603.267	301.634	50.272	10.054	222.995
20	22 39 08.506	1,802.611	901.306	600.870	300.435	50.073	10.015	234.575
21	23 47 02.882	1,795.048	897.524	598.349	299.175	49.862	9.972	246.130
22	24 54 56.664	1,787.112	893.556	595.704	297.852	49.642	9.928	257.660
23	26 02 49.800	1,778.802	889.401	592.934	296.467	49.411	9.882	269.162
24	27 10 42.204	1,770.118	885.059	590.039	295.020	49.170	9.834	280.637
25	28 18 33.812	1,761.058	880.529	587.019	293.510	48.918	9.784	292.081
26	29 26 24.526	1,751.622	875.811	583.874	291.937	48.656	9.731	303.495
27	30 34 14.272	1,741.810	870.905	580.603	290.302	48.384	9.677	314.877
28	31 42 02.940	1,731.621	865.810	577.207	288.603	48.101	9.620	326.225
29	32 49 50.422	1,721.054	860.527	573.685	286.842	47.807	9.561	337.538
30	33 57 36.602	1,710.108	855.054	570.036	285.018	47.503	9.501	348.814
31	35 05 21.356	1,698.783	849.392	566.261	283.131	47.188	9.438	360.053
32	36 13 04.550	1,687.079	843.539	562.360	281.180	46.863	9.373	371.252
33	37 20 46.016	1,674.993	837.497	558.331	279.166	46.528	9.306	382.411
34	38 28 25.606	1,662.527	831.264	554.176	277.088	46.181	9.236	393.527
35	39 36 03.136	1,649.680	824.840	549.893	274.947	45.824	9.165	404.600
36	40 43 38.420	1,636.450	818.225	545.483	272.742	45.457	9.091	415.627
37	41 51 11.244	1,622.837	811.418	540.946	270.473	45.079	9.016	426.608
38	42 58 41.376	1,608.841	804.421	536.280	268.140	44.690	8.938	437.540
39	44 06 08.580	1,594.462	797.231	531.487	265.744	44.291	8.858	448.422
40	45 13 32.584	1,579.699	789.849	526.566	263.283	43.881	8.776	459.253
41	46 20 53.096	1,564.551	782.276	521.517	260.759	43.460	8.692	470.030
42	47 28 09.800	1,549.020	774.510	516.340	258.170	43.028	8.606	480.752
43	48 35 22.356	1,533.105	766.552	511.035	255.517	42.586	8.517	491.417
44	49 42 30.380	1,516.805	758.403	505.602	252.801	42.133	8.427	502.023
45	50 49 33.474	1,500.122	750.061	500.041	250.020	41.670	8.334	512.568

TABLE IX.—*Flat-polar quartic authalic projection (1° interval)*—Continued

Latitude ϕ	α	x coordinate: Longitude from central meridian 180°	90°	60°	30°	5°	1°	y coordinate
°	° ′ ″	cm.	cm.	cm.	cm.	cm.	cm.	cm.
45	50 49 33. 474	1, 500. 122	750. 061	500. 041	250. 020	41. 670	8. 334	512. 568
46	51 56 31. 188	1, 483. 056	741. 528	494. 352	247. 176	41. 196	8. 239	523. 051
47	53 03 23. 036	1, 465. 608	732. 804	488. 536	244. 268	40. 711	8. 142	533. 468
48	54 10 08. 490	1, 447. 779	723. 890	482. 593	241. 297	40. 216	8. 043	543. 819
49	55 16 46. 978	1, 429. 571	714. 785	476. 524	238. 262	39. 710	7. 942	554. 101
50	56 23 17. 858	1, 410. 984	705. 492	470. 328	235. 164	39. 194	7. 839	564. 311
51	57 29 40. 438	1, 392. 023	696. 011	464. 007	232. 004	38. 667	7. 733	574. 447
52	58 35 53. 962	1, 372. 688	686. 344	457. 563	228. 781	38. 130	7. 626	584. 507
53	59 41 57. 604	1, 352. 983	676. 492	450. 994	225. 497	37. 583	7. 517	594. 488
54	60 47 50. 436	1, 332. 913	666. 457	444. 304	222. 152	37. 025	7. 405	604. 387
55	61 53 31. 464	1, 312. 482	656. 241	437. 494	218. 747	36. 458	7. 292	614. 201
56	62 58 59. 572	1, 291. 695	645. 848	430. 565	215. 283	35. 880	7. 176	623. 927
57	64 04 13. 522	1, 270. 560	635. 280	423. 520	211. 760	35. 293	7. 059	633. 562
58	65 09 12. 000	1, 249. 082	624. 541	416. 361	208. 180	34. 697	6. 939	643. 102
59	66 13 53. 476	1, 227. 272	613. 636	409. 091	204. 545	34. 091	6. 818	652. 544
60	67 18 16. 316	1, 205. 140	602. 570	401. 713	200. 857	33. 476	6. 695	661. 883
61	68 22 18. 672	1, 182. 697	591. 349	394. 232	197. 116	32. 853	6. 571	671. 114
62	69 25 58. 508	1, 159. 958	579. 979	386. 653	193. 326	32. 221	6. 444	680. 234
63	70 29 13. 546	1, 136. 939	568. 469	378. 980	189. 490	31. 582	6. 316	689. 237
64	71 32 01. 256	1, 113. 658	556. 829	371. 219	185. 610	30. 935	6. 187	698. 117
65	72 34 18. 814	1, 090. 137	545. 069	363. 379	181. 690	30. 282	6. 056	706. 869
66	73 36 03. 040	1, 066. 402	533. 201	355. 467	177. 734	29. 622	5. 924	715. 486
67	74 37 10. 412	1, 042. 481	521. 240	347. 494	173. 747	28. 958	5. 792	723. 960
68	75 37 36. 960	1, 018. 407	509. 203	339. 469	169. 734	28. 289	5. 658	732. 283
69	76 37 18. 206	994. 219	497. 110	331. 406	165. 703	27. 617	5. 523	740. 447
70	77 36 09. 166	969. 962	484. 981	323. 321	161. 660	26. 943	5. 389	748. 441
71	78 34 04. 214	945. 686	472. 843	315. 229	157. 614	26. 269	5. 254	756. 256
72	79 30 56. 992	921. 450	460. 725	307. 150	153. 575	25. 596	5. 119	763. 878
73	80 26 40. 410	897. 322	448. 661	299. 107	149. 554	24. 926	4. 985	771. 295
74	81 21 06. 486	873. 379	436. 689	291. 126	145. 563	24. 261	4. 852	778. 491
75	82 14 06. 202	849. 708	424. 854	283. 236	141. 618	23. 603	4. 721	785. 450
76	83 05 29. 552	826. 410	413. 205	275. 470	137. 735	22. 956	4. 591	792. 153
77	83 55 05. 312	803. 598	401. 799	267. 866	133. 933	22. 322	4. 464	798. 581
78	84 42 41. 032	781. 400	390. 700	260. 467	130. 233	21. 706	4. 341	804. 710
79	85 28 02. 962	759. 959	379. 980	253. 320	126. 660	21. 110	4. 222	810. 516
80	86 10 56. 088	739. 432	369. 716	246. 477	123. 239	20. 540	4. 108	815. 973
81	86 51 04. 098	719. 992	359. 996	239. 997	119. 999	20. 000	4. 000	821. 050
82	87 28 09. 596	701. 824	350. 912	233. 941	116. 971	19. 495	3. 899	825. 718
83	88 01 54. 388	685. 126	342. 563	228. 375	114. 188	19. 031	3. 806	829. 944
84	88 31 59. 762	670. 099	335. 049	223. 366	111. 683	18. 614	3. 723	833. 695
85	88 58 07. 078	656. 947	328. 474	218. 982	109. 491	18. 249	3. 650	836. 938
86	89 19 58. 488	645. 866	322. 933	215. 289	107. 644	17. 941	3. 588	839. 643
87	89 37 17. 678	637. 035	318. 518	212. 345	106. 172	17. 695	3. 539	841. 780
88	89 49 50. 784	630. 607	315. 304	210. 202	105. 101	17. 517	3. 503	843. 326
89	89 57 27. 126	626. 701	313. 351	208. 900	104. 450	17. 408	3. 482	844. 261
90	90 00 00. 000	625. 391	312. 695	208. 464	104. 232	17. 372	3. 474	844. 574

APPENDIX

ALTERNATIVE DEVELOPMENT FOR THE FLAT-POLAR SINUSOIDAL AUTHALIC PROJECTION

The following development for a general flat-polar sinusoidal authalic projection was suggested by a particular case developed by W. Werenskiold in his paper, A Class of Equal Area Projections, Oslo 1945.

The meridians of the Mercator sinusoidal authalic projection are the sine curves given by the equation $x=R\lambda \cos \frac{y}{R}$. (See p. 15.) Let us introduce an arbitrary parameter c into this equation and write

$$x=R\lambda \cos \frac{cy}{R}. \tag{82}$$

From (82), with $\lambda=\pi$ and $y=0$, it is seen that the map equatorial semiaxis is $x_o=R\pi$. If we desire the length of the map polar semiaxis to be some factor $m(0<m<1)$ times the length of the map equatorial semiaxis we must have

$$y_o=mx_o=mR\pi. \tag{83}$$

If we desire the x-coordinate at $\phi=90°$, $\lambda=\pi$, to be some factor $n(0<n<1)$ times the length of the map equatorial semiaxis, then placing the value of y_o from (83) in (82) we must have

$$x_{\frac{\pi}{2}}=R\pi \cos (cm\pi)=nx_o=nR\pi. \tag{84}$$

Solving (84) for c we find

$$c=\frac{\cos^{-1}n}{m\pi}. \tag{85}$$

The area of the sphere from the Equator to latitude ϕ is known to be $2\pi R^2 \sin \phi$. Hence to maintain the equal-area property we must have from (82) and (85)

$$2\pi R^2A^2 \sin \phi=2R\pi\int_0^y \cos\left(\frac{y \cos^{-1}n}{Rm\pi}\right) dy=\frac{2R^2m\pi^2}{\cos^{-1}n} \sin\left(\frac{y \cos^{-1}n}{Rm\pi}\right),$$

or

$$A^2 \sin \phi=\frac{m\pi}{\cos^{-1}n} \sin\left(\frac{y \cos^{-1}n}{Rm\pi}\right), \tag{86}$$

where A^2 is the area scale factor.

Now when $y=y_o=mR\pi$, then $\phi=\frac{\pi}{2}$ and with these values equation (86) gives

$$A^2=\frac{m\pi}{\cos^{-1}n} \sin (\cos^{-1}n). \tag{87}$$

With the value of A^2 from (87), equation (86) becomes

$$\sin\left(\frac{y \cos^{-1}n}{Rm\pi}\right)=\sin (\cos^{-1}n) \sin \phi. \tag{88}$$

Now let

$$\alpha=\frac{y \cos^{-1}n}{Rm\pi}, \tag{89}$$

and (88) becomes

$$\sin \alpha=\sin (\cos^{-1}n) \sin \phi. \tag{90}$$

From (82), (83), (85) and (89) the mapping equations become

$$\frac{x}{y_o}=\frac{\lambda}{m\pi}\cos\alpha,\ \frac{y}{y_o}=\frac{\alpha}{\cos^{-1}n}. \qquad (91)$$

Consider the particular case $m=n=1/2$. Equations (87), (90), and (91) become respectively $A^2=\frac{3\sqrt{3}}{4}$, $\sin\alpha=\sin 60^\circ \sin\phi$, $\frac{x}{y_o}=\frac{\lambda}{90^\circ}\cos\alpha$, $\frac{y}{y_o}=\frac{\alpha}{60^\circ}$. This is the case discussed on pages 9 and 10 of Werenskiold's paper.

The obvious advantage of equations (90) and (91) is that the parameter α may be computed without recourse to approximation methods.

The coordinates as computed by equations (90) and (91) will vary slightly from those found by using equations (17) and (19) where the same conditions are imposed on each set of equations with regard to axes ratio, etc. However, the difference in shape of the two projections so produced is negligible.

Werenskiold uses advantageously the above method of beginning with the algebraic equation of the meridian curves to obtain particular cases of the flat-polar parabolic and flat-polar ellipsoidal projections, but does not avoid approximation methods for the ellipsoidal type. Where the degree of the meridian curves is greater than the second, the method would be impracticable except for some special cases.

FORMULAS FOR REVERSION OF SERIES

The formulas for reversion of series whose terms are in ascending order of odd powers of the variable are included here for convenience.

If the series $u=x-b_3x^3-b_5x^5-b_7x^7-b_9x^9-\ \ldots$
is reverted to obtain the series

$$x=u+a_3u^3+a_5u^5+a_7u^7+a_9u^9+\ \ldots$$

then the formulas for the a's in terms of the b's are

$$a_3=b_3$$
$$a_5=3b_3^2+b_5$$
$$a_7=12b_3^3+8b_3b_5+b_7$$
$$a_9=55b_3^2(b_3^2+b_5)+10b_3b_7+5b_5^2+b_9.$$

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FIG. 28.—GENERATION OF EQUAL-AREA PROJECTIONS FROM THE SINE AND TANGENT FUNCTIONS.

**Note that the names Craster and Foucaut have been used to show the positions of their projections in the series, the actual mapping equations differing slightly from those given above.*

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