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GENERAL THEORY OF POLYCONIC PROJECTIONS

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PREFACE.

In this publication an attempt has been made to gather into one volume all of the investigations that apply to the system of polyconic projections. This was undertaken mainly for the reason that no such treatise has ever been produced in the English language. No adequate treatment even of the ordinary, or American, polyconic projection has been given in any separate publication. The work by Thomas Craig entitled "A Treatise on Projections," published by the United States Coast and Geodetic Survey, 1882, gives almost no treatment of the polyconic projection as used by the Coast and Geodetic Survey, but merely makes reference to the various yearly reports of the Superintendent of the Survey for information regarding it.

The subject of projections as a whole seems to have been considerably neglected by authors who employ the English language. A small work by Arthur R. Hinks, published by the Cambridge University Press in 1912, is an excellent introduction to the general subject, and gives promise of some awakened interest in this branch of applied mathematics.

In the preparation of this publication the following works were especially consulted: The most excellent work by M. A. Tissot, Mémoire sur la Représentation des Surfaces et les Projections des Cartes Géographiques, Paris, 1881; Traité des Projections des Cartes Géographiques, by A. Germain, Paris, 1866 (?); Lehrbuch der Landkartenprojektionen, by Norbert Herz, Leipzig, 1885; Notes on Stereographic Projection by Prof. W. W. Hendrickson, U. S. N.

It is hoped that the treatment of the various classes of polyconic projections may be found complete enough to serve all practical purposes.

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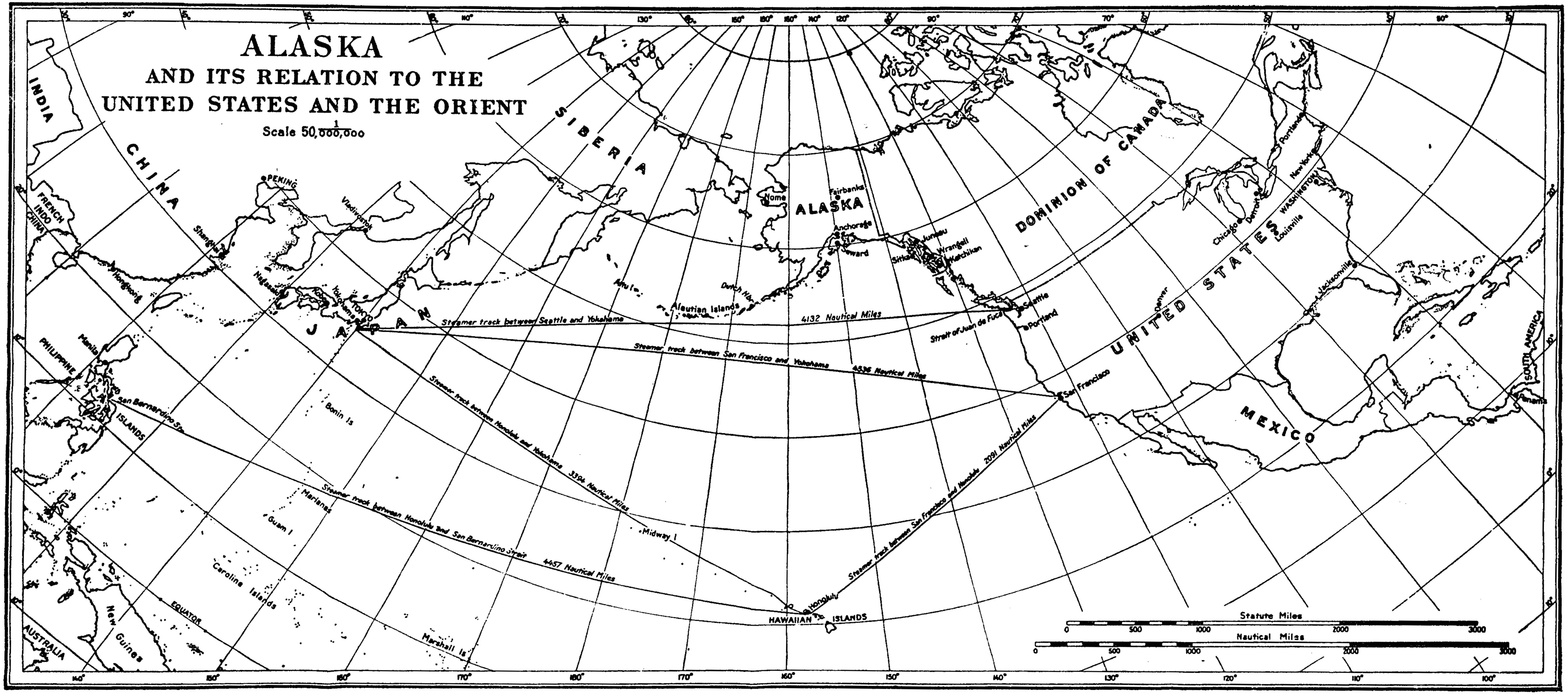

FRONTISPIECE—Transverse polyconic projection of the North Pacific Ocean.

GENERAL THEORY OF POLYCONIC PROJECTIONS.

By Oscar S. Adams,
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DETERMINATION OF ELLIPSOIDAL EXPRESSIONS.

In the consideration of the subject of map construction, the initial question to be decided is the manner in which the meridians and parallels are to be represented in an orderly way upon the plane surface of the map. This is done by the adoption of some mathematical expression that determines a one-to-one relation between the meridians and parallels and their corresponding curves in the plane. In the consideration of this determination, the earth can be looked upon either as a sphere or as an ellipsoid of revolution. When especial accuracy is desired, the eccentricity must be taken into account. If the formulas are determined for the ellipsoid, they can be reduced to those for the sphere by setting the expression for the eccentricity equal to zero. Since the ellipsoidal form is to be taken as the basis of most of the following discussions, a preliminary determination of the necessary lines will be given.

In figure 1 let EPS represent a quadrant of the generating ellipse. P and P' are contiguous points; PK is the normal at P and $P'K$ the same at P'. If the equation of the ellipse be given in the parametric form

$$x = a \cos \psi$$

$$y = b \sin \psi,$$

a will represent the equatorial radius or the semimajor axis, and b the polar radius or semiminor axis; ψ is the eccentric angle as indicated in figure 1 If φ is the latitude of the point P, it will be seen that

$$\tan \varphi = -\frac{dx}{dy};$$

but

$$dx = -a \sin \psi \, d\psi$$

$$dy = \quad b \cos \psi \, d\psi.$$

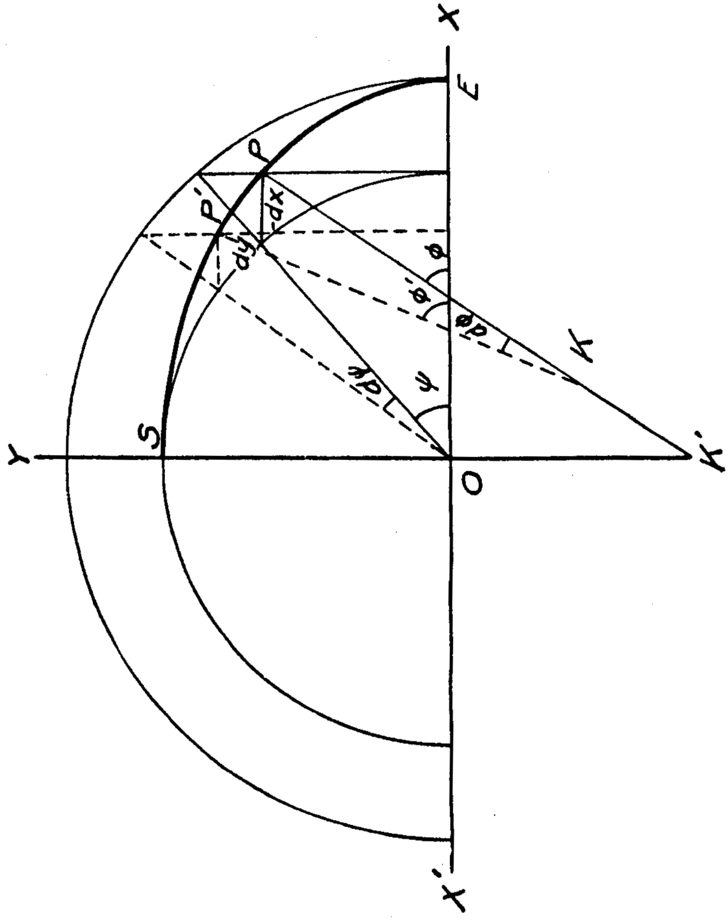

FIG. 1.—Generating ellipse with the radii of curvature of the earth.

Hence

$$\tan \varphi = \frac{a}{b} \tan \psi.$$

We denote the eccentricity by ϵ and define it by the equation

$$\epsilon^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2},$$

hence

$$\frac{b}{a} = \sqrt{1 - \epsilon^2}.$$

By substituting this value, we obtain

$$\tan \psi = \sqrt{1-\epsilon^2} \tan \varphi.$$

$$\sin \psi = \frac{\tan \psi}{\sqrt{1+\tan^2\psi}} = \frac{\sqrt{1-\epsilon^2} \tan\varphi}{\sqrt{1+\tan^2\varphi - \epsilon^2 \tan^2\varphi}} = \frac{\sqrt{1-\epsilon^2} \sin \varphi}{\sqrt{1-\epsilon^2 \sin^2\varphi}}$$

$$\cos \psi = \frac{1}{\sqrt{1+\tan^2\psi}} = \frac{1}{\sqrt{1+\tan^2\varphi - \epsilon^2 \tan^2\varphi}} = \frac{\cos \varphi}{\sqrt{1-\epsilon^2 \sin^2\varphi}}$$

$$\sec^2\psi \, d\psi = \sqrt{1-\epsilon^2} \, \sec^2\varphi \, d\varphi$$

$$d\psi = \frac{\sqrt{1-\epsilon^2} \sec^2\varphi \, d\varphi}{1+\tan^2\varphi - \epsilon^2 \tan^2\varphi} = \frac{\sqrt{1-\epsilon^2} \, d\varphi}{1-\epsilon^2 \sin^2\varphi}.$$

If we denote the radius of curvature PK of the meridian by ρ_m, we have from the general theory of plane curves the relation $\rho_m d\varphi = ds$.

But

$$ds = \sqrt{dx^2+dy^2} = \sqrt{a^2 \sin^2\psi + b^2 \cos^2\psi} \, d\psi = a\sqrt{1-\epsilon^2 \cos^2\psi} \, d\psi.$$

Also

$$\sqrt{1-\epsilon^2 \cos^2\psi} = \frac{\sqrt{1-\epsilon^2}}{\sqrt{1-\epsilon^2 \sin^2\varphi}}$$

and

$$\sqrt{1-\epsilon^2 \cos^2\psi} \, d\psi = \frac{(1-\epsilon^2) \, d\varphi}{(1-\epsilon^2 \sin^2\varphi)^{3/2}}$$

or

$$ds = \frac{a\,(1-\epsilon^2) \, d\varphi}{(1-\epsilon^2 \sin^2\varphi)^{3/2}}.$$

Hence

$$\rho_m = \frac{a\,(1-\epsilon^2)}{(1-\epsilon^2 \sin^2\varphi)^{3/2}}.$$

The normals at any two points on the same parallel circle intersect in a point K' of the axis of rotation. If we pass a plane through these two normals and then let the normals approach each other until they finally coincide, we obtain a vertical plane tangent to the given parallel and perpendicular to the meridian at the point of tangency. The radius of curvature of a small arc in this direction is given by PK' because the normals of two contiguous points of this arc intersect in K'. If we denote this radius by ρ_n, we have

$$\rho_n = \frac{x}{\cos\varphi} = \frac{a\cos\psi}{\cos\varphi} = \frac{a}{(1-\epsilon^2\sin^2\varphi)^{1/2}}.$$

If the element of length of the meridian is denoted by dm, we obtain

$$dm = \frac{a(1-\epsilon^2)\,d\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}.$$

This is an elliptic integral that it is not necessary to evaluate in this place, since we shall have occasion to employ it only in the differential form.

DEVELOPMENT OF GENERAL FORMULAS FOR THE POLYCONIC PROJECTIONS.

Tissot defines a polyconic projection as one in which the parallels of latitude are represented by arcs of a non-concentric system of circles, with the centers of these various circles lying upon a straight line. This line of centers is generally called the central meridian; but it is not necessarily the central meridian of any given map and in cases does not appear upon the map at all.

In the following discussion the latitude will be denoted by φ, and the longitude out from the central meridian will be denoted by λ.

In figure 2 let QM be the arc of a circle that represents a given λ on the parallel of latitude φ, with radius SQ and center at S. Let RM' be an arc of equal λ on the parallel of latitude $\varphi + d\varphi$, with radius $S'R$ and center at S'. O is the point of intersection of the central meridian and the Equator. Let OS be denoted by s. Then since s is a decreasing function of φ, SS' is equal to $-ds$. If the angle QSM is denoted by θ, we have

$$SP = -ds\cos\theta.$$

$$S'P = -ds\sin\theta.$$

$$M'N = S'M' \times \angle M'S'N.$$

But

$$\angle M'S'N = \angle OS'M' - \angle OS'N$$

$$= \angle OS'M' - \angle OSN - \angle S'NS,$$

since

$$\angle OS'N = \angle OSN + \angle S'NS.$$

But

$$\angle OS'M' - \angle OSN = \frac{\partial\theta}{\partial\varphi}d\varphi.$$

$$S'M' = S'N = \rho + d\rho,$$

at the limit

$$\angle S'NS = \frac{S'P}{S'N} = \frac{-ds \sin\theta}{\rho + d\rho}.$$

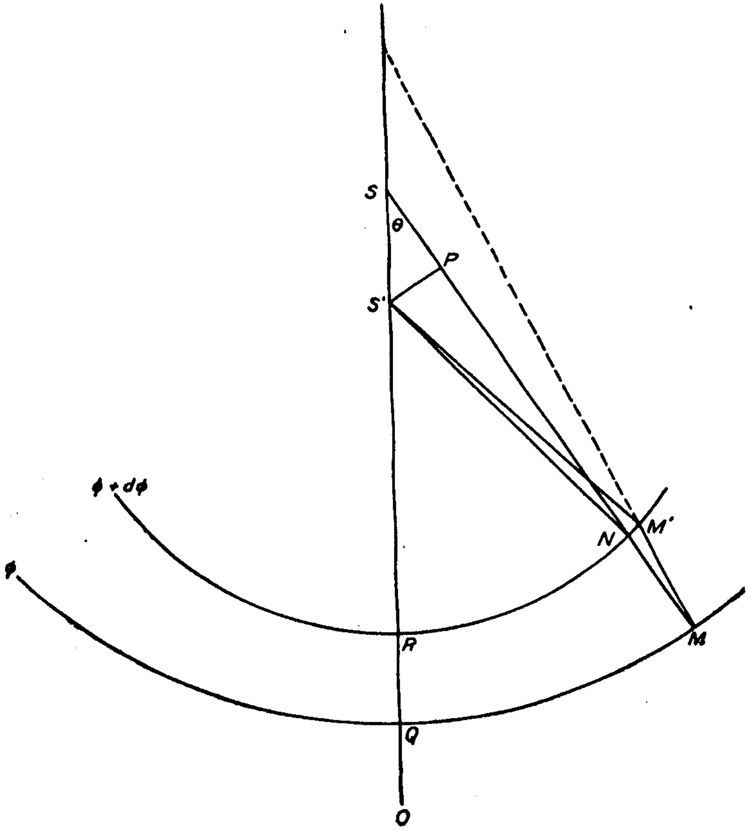

FIG. 2.—Differential elements of a polyconic projection.

Therefore

$$M'N=(\rho+d\rho)\left[\left(\frac{\partial\theta}{\partial\varphi}\right)d\varphi+\frac{ds\sin\theta}{\rho+d\rho}\right],$$

or, at the limit

$$M'N=\rho\left(\frac{\partial\theta}{\partial\varphi}\right)d\varphi+ds\sin\theta.$$

$$MN=SM-SN=SM-S'N-SP,$$

since at the limit

$$S'N=PN.$$

But

$$SM-S'N=-d\rho.$$

By substituting this value and the value of SP, we obtain

$$MN=-d\rho+ds\cos\theta.$$

If we denote $\angle M'MN$ by ψ, we have at the limit

$$\tan\psi=\frac{M'N}{MN}=\frac{\rho\frac{\partial\theta}{\partial\varphi}+\frac{ds}{d\varphi}\sin\theta}{\frac{ds}{d\varphi}\cos\theta-\frac{d\rho}{d\varphi}}.$$

If we denote the change in scale or the magnification along the meridian by k_m and that along the parallel by k_p, we shall obtain the following expressions for these quantities:

$$M'M=MN\sec\psi=(ds\cos\theta-d\rho)\sec\psi.$$

The arc of the meridian on the earth that is represented by $M'M$ is given by

$$dm=\rho_m d\varphi=\frac{a(1-\epsilon^2)\,d\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}.$$

Hence we have

$$k_m=\frac{(1-\epsilon^2\sin^2\varphi)^{3/2}}{a(1-\epsilon^2)}\left(\frac{ds}{d\varphi}\cos\theta-\frac{d\rho}{d\varphi}\right)\sec\psi.$$

The arc of a parallel on the map between the meridians of longitude λ and $\lambda+d\lambda$ is equal to

$$\rho\left(\frac{\partial\theta}{\partial\lambda}\right) d\lambda, \text{ since } \varphi \text{ is constant.}$$

This arc upon the earth is equal to the expression

$$\rho_n \cos\varphi\, d\lambda = \frac{a\, d\lambda \cos\varphi}{(1-\epsilon^2 \sin^2\varphi)^{1/2}}.$$

Therefore

$$k_p = \frac{\rho\,(1-\epsilon^2\sin^2\varphi)^{1/2}}{a\cos\varphi}\frac{\partial\theta}{\partial\lambda}.$$

The ratio of increase of area, denoted by K, is given by

$$K = k_m k_p \sin\left(\frac{\pi}{2}-\psi\right) = k_m k_p \cos\psi,$$

or

$$K = \frac{\rho(1-\epsilon^2\sin^2\varphi)^2}{a^2\,(1-\epsilon^2)\cos\varphi}\left(\frac{ds}{d\varphi}\cos\theta - \frac{d\rho}{d\varphi}\right)\frac{\partial\theta}{\partial\lambda}.$$

CLASSIFICATION OF POLYCONIC PROJECTIONS.

The general division of polyconic projections is subdivided into the following classes which are not, however, mutually exclusive:

(1) Rectangular polyconic projections.
(2) Stereographic meridian and horizon projections.
(3) Conformal polyconic projections.
(4) Equal area or equivalent polyconic projections.
(5) Conventional polyconic projections.
(6) Ordinary, or American, polyconic projection.

The general differential formulas developed above will now be applied to these classes in the order named.

RECTANGULAR POLYCONIC PROJECTIONS.

The condition that must be fulfilled if the meridians and parallels of the map are to intersect at right angles is expressed analytically by

$$\psi = 0.$$

Since this condition requires, whatever the value of s and ρ, that

$$\tan\psi = 0,$$

we must have

$$\rho\frac{\partial\theta}{\partial\varphi} + \frac{ds}{d\varphi}\sin\theta = 0.$$

Let us introduce as a new variable a function of φ denoted by u and defined by the equation

$$\frac{1}{\rho}\frac{ds}{d\varphi}=\frac{1}{u}\frac{du}{d\varphi}.$$

But

$$\frac{1}{\rho}\frac{ds}{d\varphi}=-\frac{1}{\sin\theta}\frac{\partial\theta}{\partial\varphi}$$

hence

$$\frac{1}{\sin\theta}\frac{\partial\theta}{\partial\varphi}=-\frac{1}{u}\frac{du}{d\varphi}.$$

By integrating this partial differential equation with respect to φ, we obtain the required relation. This integration may be carried through in the following manner.

$$\int\frac{1}{\sin\theta}\frac{\partial\theta}{\partial\varphi}d\varphi=-\int\frac{du}{u}$$

$$\int\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\sin\theta}\frac{\partial\theta}{\partial\varphi}d\varphi=-\int\frac{du}{u}$$

$$\int\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\frac{\partial\theta}{\partial\varphi}d\varphi=-\int\frac{du}{u}$$

$$\int\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\frac{\partial\theta}{\partial\varphi}\frac{d\varphi}{2}+\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\frac{\partial\theta}{\partial\varphi}\frac{d\varphi}{2}=-\int\frac{du}{u}$$

$$\log\sin\frac{\theta}{2}-\log\cos\frac{\theta}{2}=-\log u+\log\Gamma(\lambda).*$$

Log $\Gamma(\lambda)$ is a function of λ that is added since the integration is partial with respect to φ. The function $\Gamma(\lambda)$ is as yet undetermined.

$$\log\tan\frac{\theta}{2}=\log\frac{\Gamma(\lambda)}{u}$$

or

$$\tan\frac{\theta}{2}=\frac{\Gamma(\lambda)}{u}.$$

*This function has no connection with the gamma function defined by the second Eulerian integral.

Since for $\lambda=0$, θ must also be zero, the function $\Gamma(\lambda)$ must vanish with λ. This is the only condition that is required to give a rectangular polyconic projection.

If we choose an arbitrary function for $\Gamma(\lambda)$ that vanishes with λ and another arbitrary function of φ for u and set

$$\tan\frac{\theta}{2}=\frac{\Gamma(\lambda)}{u},$$

then the net will always be rectangular provided that

$$\rho=u\frac{\frac{ds}{d\varphi}}{\frac{du}{d\varphi}},$$

in which s is also an arbitrary function of φ, or provided that

$$s=\int\frac{\rho}{u}\frac{du}{d\varphi}\,d\varphi$$

with ρ arbitrary.

Since in this case of the rectangular polyconic projection $\psi=0$ and $\sec\psi=1$, we have

$$k_m=\frac{(1-\epsilon^2\sin^2\varphi)^{3/2}}{a(1-\epsilon^2)}\left(\frac{ds}{d\varphi}\cos\theta-\frac{d\rho}{d\varphi}\right)$$

$$k_p=\frac{\rho(1-\epsilon^2\sin^2\varphi)^{1/2}}{a\cos\varphi}\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\sin\theta,$$

since

$$\frac{\partial\theta}{\partial\lambda}=\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\sin\theta.$$

If we wish the parallel of latitude φ to lie on the developed base of the cone tangent to the earth at latitude φ, we must have

$$\rho=\frac{a\cot\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}.$$

If, besides, the parallels are to be spaced along the central meridian in proportion to their true distances, we must also take

$$s=\int_0^{\varphi}\frac{a\,(1-\epsilon^2)\,d\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}+\frac{a\cot\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}.$$

With these values we obtain

$$\frac{ds}{d\varphi}=\frac{a(1-\epsilon^2)}{(1-\epsilon^2\sin^2\varphi)^{3/2}}-\frac{a\operatorname{cosec}^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}+\frac{a\,\epsilon^2\cos^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$=\frac{a(1-\operatorname{cosec}^2\varphi)}{(1-\epsilon^2\sin^2\varphi)^{1/2}}=-\frac{a\cot^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}},$$

hence

$$\frac{1}{\rho}\frac{ds}{d\varphi}=-\cot\varphi.$$

Therefore

$$\frac{1}{u}\frac{du}{d\varphi}=-\cot\varphi;$$

by integration, we obtain

$$\log u=-\log\sin\varphi=\log\operatorname{cosec}\varphi,$$

or, passing to exponentials,

$$u=\operatorname{cosec}\varphi.$$

But

$$\tan\frac{\theta}{2}=\frac{\Gamma(\lambda)}{u}=\Gamma(\lambda)\sin\varphi.$$

The length of an arc of the developed parallel is given by

$$\rho\theta=\frac{2a\cot\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}\tan\frac{\theta}{2}\,\frac{\frac{\theta}{2}}{\tan\frac{\theta}{2}}=\frac{2a\cos\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}\,\Gamma(\lambda)\,\frac{\frac{\theta}{2}}{\tan\frac{\theta}{2}}.$$

On the equator, since $\varphi=0$ and $\theta=0$, we obtain for an arc from $\lambda=0$ to λ the value

$$\text{equatorial arc}=2a\,\Gamma(\lambda).$$

If we now add the condition that the equatorial arcs are to be preserved in their true length, we have

$$2a\,\Gamma(\lambda)=a\lambda$$

or

$$\Gamma(\lambda)=\frac{\lambda}{2}.$$

This value gives

$$\tan\frac{\theta}{2}=\frac{\lambda}{2}\sin\varphi.$$

This gives the full determination of the projection. With these values we shall now determine the magnification along the meridians and parallels.

$$\Gamma'(\lambda)=\frac{1}{2}$$

$$\frac{d\rho}{d\varphi}=-\frac{a\ \mathrm{cosec}^2\ \varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}+\frac{a\epsilon^2\cos^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$=\frac{-a\ \mathrm{cosec}^2\ \varphi+a\epsilon^2+a\epsilon^2\cos^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

and

$$\frac{ds}{d\varphi}=-\frac{a\cot^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}.$$

Substituting these values in the differential formulas on pages 12 and 13, we obtain

$$k_{\mathrm{m}}=\frac{\mathrm{cosec}^2\ \varphi}{1-\epsilon^2}-\frac{\epsilon^2(1+\cos^2\varphi)}{1-\epsilon^2}-\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}\cot^2\varphi\cos\theta$$

$$k_{\mathrm{p}}=\frac{\sin\theta}{\lambda\sin\varphi}.$$

The formula for k_{m} shows that the value of k_{m} along the central meridian is equal to unity; that is, the scale is maintained constant along this meridian as was provided by the choice of the value for s. This means that the parallels are spaced along the central meridian in proportion to their distances apart upon the earth. Since this is true, with the known radii we can construct the parallel arcs either by drafting or by plotting by means of computed coordinates. The only things remaining to be determined are the points of intersection of the meridians with these parallels.

In order to determine these points, we have first

$$\rho\tan\frac{\theta}{2}=\frac{a\lambda\cos\varphi}{2(1-\epsilon^2\sin^2\varphi)^{1/2}}.$$

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But the right-hand member of this equation is equal to one-half the arc of the parallel of latitude φ from $\lambda=0$ to the value λ. If then in figure 3 we lay off the distance MN on the tangent to the parallel drawn from the point where it crosses the central meridian and take it equal in length to one-half the arc of this parallel up to the given longitude λ, the angle MCN will be equal to one-half of θ. To determine the point of intersection, from N as center with a radius NM construct an arc intersecting the parallel at M_1. The point M_1 is then the intersection of the meridian λ with the parallel φ.

This projection has been much used by the English War Office for the construction of maps.

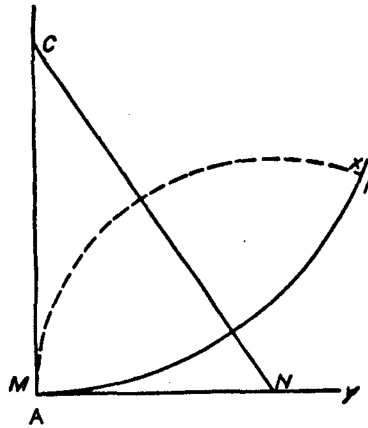

FIG. 3.—Construction of arc of parallel on rectangular polyconic projection.

We can easily determine the radius of curvature of the meridians in this projection. In figure 2

$$M'M=(ds\cos\theta-d\rho),$$

since in this case $\cos\psi=1$.

$$\cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{1-\frac{\lambda^2}{4}\sin^2\varphi}{1+\frac{\lambda^2}{4}\sin^2\varphi}.$$

The angle between two successive radii of curvature is the angle between the tangents to the parallels of φ and $\varphi+d\varphi$

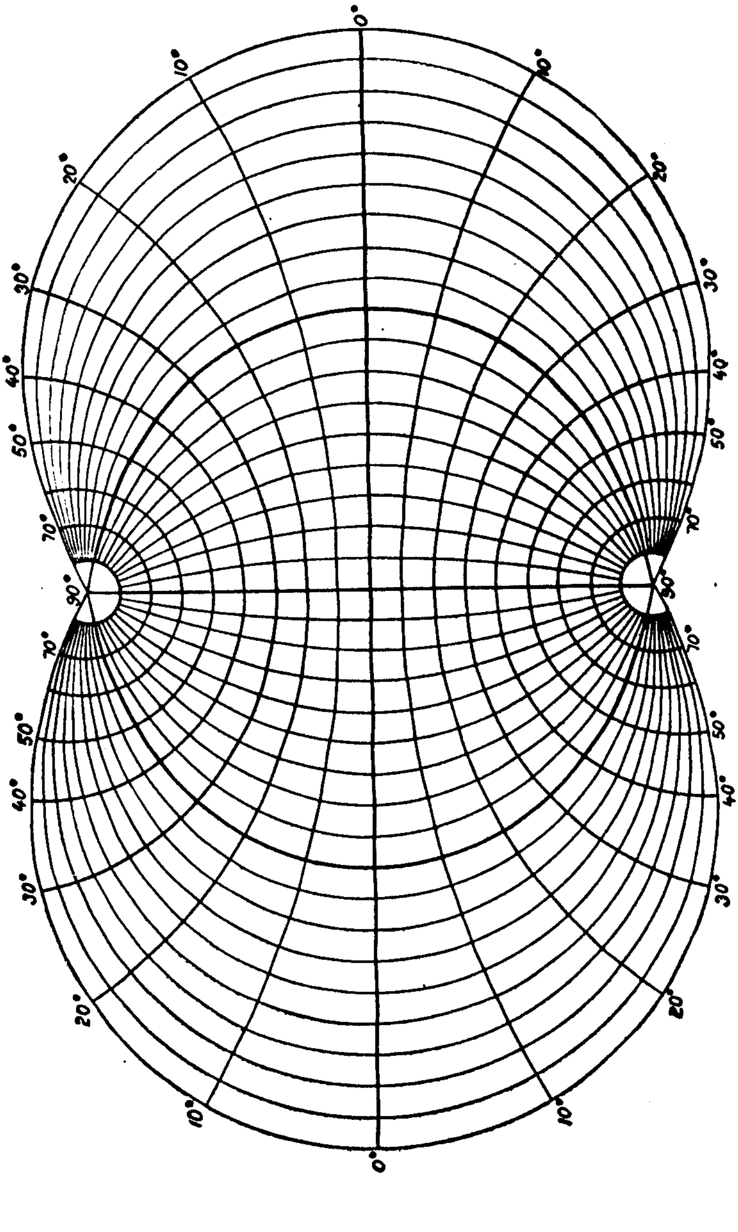

FIG. 4.—Entire surface of the sphere on rectangular polyconic projection.

at the points M and M', respectively, since the projection is rectangular. This angle is evidently equal to $d\theta$.

By differentiation we obtain

$$\sec^2\frac{\theta}{2}\,\frac{d\theta}{2}=\frac{\lambda}{2}\cos\varphi\, d\varphi,$$

since λ is a constant for a given meridian.

Hence

$$d\theta=\frac{\lambda\cos\varphi\, d\varphi}{1+\frac{\lambda^2}{4}\sin^2\varphi}.$$

The radius of curvature of the meridian, denoted by ρ_s is given in the form

$$\rho_s=\frac{M'M}{d\theta}=\frac{\left(\frac{ds}{d\varphi}\cos\theta-\frac{d\rho}{d\varphi}\right)\left(1+\frac{\lambda^2}{4}\sin^2\varphi\right)}{\lambda\cos\varphi}.$$

By substituting the values of $\frac{ds}{d\varphi}$, $\frac{d\rho}{d\varphi}$, and $\cos\theta$ and reducing, we find

$$\rho_s=\frac{a\left[1-\epsilon^2+(1-\epsilon^2)\frac{\lambda^2}{4}\sin^2\varphi+\frac{\lambda^2}{2}\cos^2\varphi\,(1-\epsilon^2\sin^2\varphi)\right]}{\lambda\cos\varphi\,(1-\epsilon^2\sin^2\varphi)^{3/2}}.$$

The magnification of area becomes

$$K=\left(\frac{\operatorname{cosec}^2\varphi}{1-\epsilon^2}-\frac{\epsilon^2\left[1+\cos^2\varphi\right]}{1-\epsilon^2}-\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}\cot^2\varphi\cos\theta\right)\frac{\sin\theta}{\lambda\sin\varphi}.$$

But

$$\cos\theta=\frac{1-\frac{\lambda^2}{4}\sin^2\varphi}{1+\frac{\lambda^2}{4}\sin^2\varphi}$$

and

$$\sin\theta=\frac{\lambda\sin\varphi}{1+\frac{\lambda^2}{4}\sin^2\varphi}.$$

By substituting these values we obtain

$$K=\left[\left(\frac{\operatorname{cosec}^2\varphi}{1-\epsilon^2}\right)\left(1+\frac{\lambda^2}{4}\sin^2\varphi\right)-\frac{\epsilon^2(1+\cos^2\varphi)}{1-\epsilon^2}\left(1+\frac{\lambda^2}{4}\sin^2\varphi\right)\right.$$
$$\left.-\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}\cot^2\varphi\left(1-\frac{\lambda^2}{4}\sin^2\varphi\right)\right]\div\left(1+\frac{\lambda^2}{4}\sin^2\varphi\right)^2$$

or, on reduction,

$$K=\frac{1+\frac{\lambda^2}{4}\sin^2\varphi+\frac{\lambda^2}{2}\cos^2\varphi\,\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}}{\left(1+\frac{\lambda^2}{4}\sin^2\varphi\right)^2}.$$

If we equate this to unity, we shall find the equation of a curve along which there is no exaggeration of area. On reduction this equation becomes

$$\lambda^4\sin^4\varphi+4\lambda^2\sin^2\varphi-8\lambda^2\cos^2\varphi\left(\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}\right)=0,$$

which is satisfied by $\lambda=0$, or by the equation

$$\lambda^2\sin^4\varphi+4\sin^2\varphi-8\cos^2\varphi\left(\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}\right)=0.$$

The areas of all sections north of this curve are diminished and those lying south of it are increased in their representation on the map.

If we confine ourselves to the consideration of the sphere K may be expressed in the form

$$K=\frac{1+\frac{\lambda^2}{4}+\frac{\lambda^2}{4}\cos^2\varphi}{\left(1+\frac{\lambda^2}{4}\sin^2\varphi\right)^2}.$$

The differential element of area of the representation is given in the form

$$dS=a^2\,\frac{1+\frac{\lambda^2}{4}+\frac{\lambda^2}{4}\cos^2\varphi}{\left(1+\frac{\lambda^2}{4}\sin^2\varphi\right)^2}\cos\varphi\,d\varphi\,d\lambda.$$

If the whole area of the sphere is represented on one continuous map, one-fourth of the area of the representation will be given by integration of this expression from $\lambda=0$ to $\lambda=\pi$ and from $\varphi=0$ to $\varphi=\frac{\pi}{2}$.

To obviate the use of the fractions, it is better to let $\lambda=2y$; y will then range from 0 to $\frac{\pi}{2}$ and $d\lambda=2\ dy$.

The total area S is given by

$$S=8a^2\int_0^{\frac{\pi}{2}}\cos\varphi\ d\varphi\int_0^{\frac{\pi}{2}}\frac{1+y^2+y^2\cos^2\varphi}{(1+y^2\sin^2\varphi)^2}\,dy$$

$$\int_0^{\frac{\pi}{2}}\frac{1+y^2+y^2\cos^2\varphi}{(1+y^2\sin^2\varphi)^2}\,dy=-\frac{\pi\cot^2\varphi}{2\left(1+\frac{\pi^2}{4}\sin^2\varphi\right)}$$

$$+\operatorname{cosec}^3\varphi\tan^{-1}\left(\frac{\pi}{2}\sin\varphi\right).$$

$$S=4a^2\int_0^{\frac{\pi}{2}}\left[-\frac{\pi\cot^2\varphi\cos\varphi}{\left(1+\frac{\pi^2}{4}\sin^2\varphi\right)}+2\operatorname{cosec}^2\varphi\cot\varphi\right.$$

$$\left.\tan^{-1}\left(\frac{\pi}{2}\sin\varphi\right)\right]d\varphi.$$

$$S=4a^2\left\{\left[\frac{\pi}{2}\operatorname{cosec}\varphi-\operatorname{cosec}^2\varphi\tan^{-1}\left(\frac{\pi}{2}\sin\varphi\right)\right]_0^{\frac{\pi}{2}}\right.$$

$$\left.+\left(\frac{\pi^2}{4}+2\right)\tan^{-1}\frac{\pi}{2}\right\}.$$

The quantity in brackets has to be evaluated for the lower limit, since it takes the form $\infty-\infty$ at this point. Let us write it in the form

$$\frac{\frac{\pi}{2}\sin\varphi-\tan^{-1}\left(\frac{\pi}{2}\sin\varphi\right)}{\sin^2\varphi},$$

which takes the form $\frac{0}{0}$ at the lower limit.

$$\lim_{\varphi\doteq 0}\left[\frac{\frac{\pi}{2}\sin\varphi-\tan^{-1}\left(\frac{\pi}{2}\sin\varphi\right)}{\sin^2\varphi}\right]$$

$$=\lim_{\varphi \doteq 0}\left[\frac{\frac{\pi}{2}\cos\varphi-\dfrac{\frac{\pi}{2}\cos\varphi}{1+\frac{\pi^2}{4}\sin^2\varphi}}{2\sin\varphi\cos\varphi}\right]=\lim_{\varphi \doteq 0}\left[\frac{\frac{\pi^3}{16}\sin\varphi}{1+\frac{\pi^2}{4}\sin^2\varphi}\right]=0.$$

Therefore,

$$S=a^2[(4+\pi^2)\tan^{-1}\frac{\pi}{2}+2\pi].$$

This value is greater than the surface of the sphere in the approximate ratio of 8 : 5.

The length of the outer meridian for the representation of the sphere is given by four times the integral of $a\,k_m\,d\varphi$ from $\varphi=0$ to $\varphi=\frac{\pi}{2}$ with $\lambda=\pi$ in the value of θ.

For the sphere $k_m=\operatorname{cosec}^2\varphi-\cot^2\varphi\cos\theta$, and for the outer meridian

$$k_m=\frac{1+\frac{\pi^2}{4}(1+\cos^2\varphi)}{1+\frac{\pi^2}{4}\sin^2\varphi}.$$

The length of the meridian is, therefore, given by

$$l=4a\int_0^{\frac{\pi}{2}}\frac{1+\frac{\pi^2}{4}(1+\cos^2\varphi)}{1+\frac{\pi^2}{4}\sin^2\varphi}d\varphi.$$

By means of a table of integrals we find that the value of this integral is given in the form

$$l=2a\pi[(4+\pi^2)^{1/2}-1].$$

The length of a great circle at the outer limit of the map is increased in the ratio

$$(4+\pi^2)^{1/2}-1:1 \text{ or about } 2.72:1.$$

STEREOGRAPHIC MERIDIAN PROJECTION.

In the discussion of the stereographic meridian and horizon projection, it is probably best to consider first the sphere and later to indicate the manner in which the ellipsoidal shape can be taken into account. To employ the differential formulas given before, we need only to set ϵ equal to zero.

Any stereographic projection is a perspective projection of the sphere, either upon a tangent plane or upon a diametral plane, with the center of the projection lying upon the surface of the sphere in such a way that the diameter through the point of projection is perpendicular to the plane upon which the projection is made. We shall make use of the diametral plane since there is only a difference of scale between that and the tangent plane.

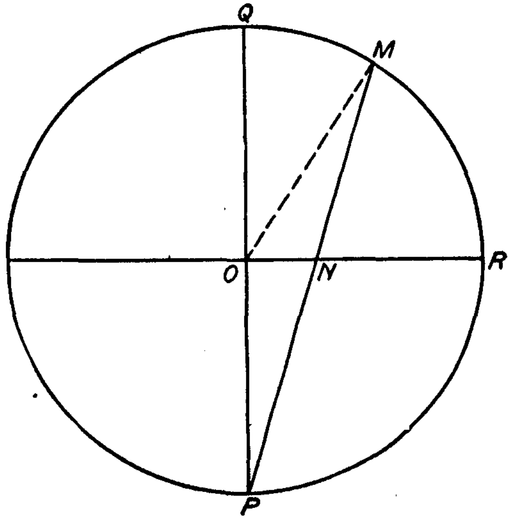

FIG. 5.—Radius from center on stereographic projection.

In figure 5 let the circle $QMRP$ be a plane section of the sphere determined by the diameter PQ and the projecting line PM. P is the point of projection, OR is the trace of the diametral plane upon which the map is to be constructed, and the point Q projected into O forms the center of the map. Let the angle QOM be denoted by p; then the arc QM is the measure of p. All points of the sphere at the arc distance p from Q will lie upon a circle the plane of which is parallel to the plane OR. The lines that project the points of this circle will all lie upon a right circular cone that will cut the plane OR in a circle the radius of which will be equal to ON. OP is equal to a, and the angle OPN is equal to $\frac{p}{2}$.

Hence

$$ON = \rho = a \tan \frac{p}{2}.$$

If we denote the angle between ρ and the X axis in the mapping plane by ω, we have

$$x = \rho \cos \omega = a \tan \frac{p}{2} \cos \omega = \frac{a \sin p \cos \omega}{1 + \cos p}$$

$$y = \rho \sin \omega = a \tan \frac{p}{2} \sin \omega = \frac{a \sin p \sin \omega}{1 + \cos p}.$$

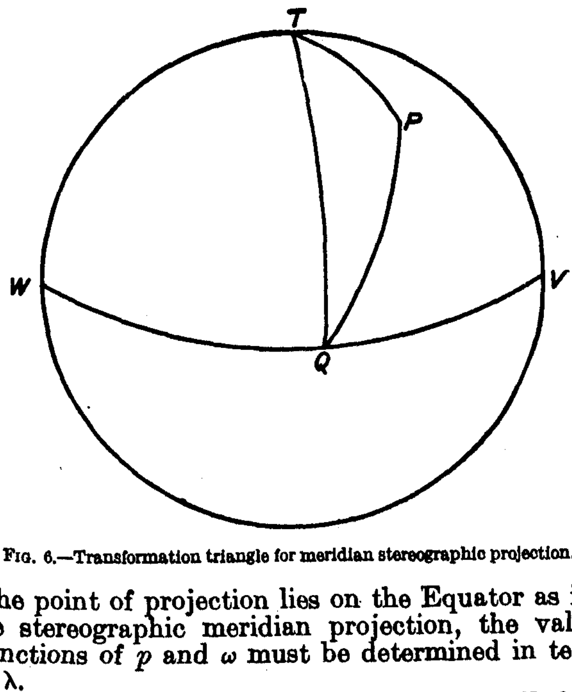

FIG. 6.—Transformation triangle for meridian stereographic projection.

If the point of projection lies on the Equator as it does in the stereographic meridian projection, the values of the functions of p and ω must be determined in terms of φ and λ.

In figure 6, let WQV be the Equator and T the pole and let TQ project into the central meridan of the map.

P is the point that we were considering in the previous figure.

$$PQ=p$$

$$TQ=\frac{\pi}{2}$$

$$TP=\frac{\pi}{2}-\varphi$$

$$\angle PTQ=\lambda$$

$$\angle PQT=\frac{\pi}{2}-\omega.$$

From the trigonometry of the spherical triangle we have the relations

$$\cos p=\cos\lambda\cos\varphi$$

$$\sin p\sin\omega=\sin\varphi$$

$$\sin p\cos\omega=\sin\lambda\cos\varphi.$$

If these values are substituted in the equations for x and y, we obtain

$$x=\frac{a\sin\lambda\cos\varphi}{1+\cos\lambda\cos\varphi}$$

$$y=\frac{a\sin\varphi}{1+\cos\lambda\cos\varphi}.$$

From these equations, by solving for $\sin\lambda$ and $\cos\lambda$, there result

$$\sin\lambda=\frac{x}{y}\tan\varphi$$

$$\cos\lambda=\frac{a\sin\varphi-y}{y\cos\varphi}.$$

Hence

$$\frac{x^2}{y^2}\tan^2\varphi+\frac{(a\sin\varphi-y)^2}{y^2\cos^2\varphi}=1,$$

or, by reduction,

$$x^2+y^2-2ay\operatorname{cosec}\varphi=-a^2$$

or, as usually written,

$$x^2+(y-a\operatorname{cosec}\varphi)^2=a^2\cot^2\varphi.$$

This equation shows that the parallels are circles, and that the parallel of latitude φ has the radius a cot φ, and that the center lies at the point $x=0$, $y=a$ cosec φ. The parallels are therefore circles, nonconcentric, but having their centers on the line $x=0$. The projection is thus seen to be a polyconic projection in the sense of Tissot's definition.

By solving the original equations for sin φ and cos φ we find

$$\sin\varphi=\frac{y\sin\lambda}{a\sin\lambda-x\cos\lambda}$$

$$\cos\varphi=\frac{x}{a\sin\lambda-x\cos\lambda}.$$

By squaring and adding, the equation of the meridians is obtained.

$$\frac{y^2\sin^2\lambda}{(a\sin\lambda-x\cos\lambda)^2}+\frac{x^2}{(a\sin\lambda-x\cos\lambda)^2}=1,$$

or, on reduction,

$$x^2+y^2+2ax\cot\lambda=a^2$$

or, as usually written,

$$(x+a\cot\lambda)^2+y^2=a^2\operatorname{cosec}^2\lambda.$$

The meridians are thus seen to be circles also; the circle for the longitude λ has the radius a cosec λ, and the center lies at the point $x=a\cot\lambda$, $y=0$.

In this projection we have, therefore,

$$\rho=a\cot\varphi$$

$$s=a\operatorname{cosec}\varphi$$

$$\sin\theta=\frac{x}{\rho}=\frac{\sin\lambda\sin\varphi}{1+\cos\lambda\cos\varphi}$$

$$\frac{\partial\theta}{\partial\varphi}=\frac{\sin\lambda}{1+\cos\lambda\cos\varphi}$$

$$\frac{ds}{d\varphi}=-a\cot\varphi\operatorname{cosec}\varphi$$

$$\rho\frac{\partial\theta}{\partial\varphi}+\frac{ds}{d\varphi}\sin\theta=\frac{a\sin\lambda\cot\varphi}{1+\cos\lambda\cos\varphi}-\frac{a\sin\lambda\cot\varphi}{1+\cos\lambda\cos\varphi}=0.$$

Therefore

$\tan\psi=0$, or $\psi=0$, and the projection belongs in the class of the rectangular polyconic projections.

The equations for the magnification along the parallels and along the meridians, respectively, are for the sphere

$$k_m=\frac{\left(\dfrac{ds}{d\varphi}\cos\theta-\dfrac{d\rho}{d\varphi}\right)}{a\cos\psi}$$

$$k_p=\frac{\rho}{a\cos\varphi}\,\frac{\partial\theta}{\partial\lambda}.$$

But

$$\frac{d\rho}{d\varphi}=-a\,\text{cosec}^2\varphi$$

$$\cos\theta=\frac{\cos\lambda+\cos\varphi}{1+\cos\lambda\cos\varphi}$$

and

$$\frac{\partial\theta}{\partial\lambda}=\frac{\sin\varphi}{1+\cos\lambda\cos\varphi}.$$

By substituting these values in the formulas for k_m and k_p we obtain

$$k_m=\frac{\dfrac{-a\cot\varphi\,\text{cosec}\,\varphi\,(\cos\lambda+\cos\varphi)}{1+\cos\lambda\cos\varphi}+a\,\text{cosec}^2\varphi}{a}$$

$$=\frac{1}{1+\cos\lambda\cos\varphi}$$

$$k_p=\frac{a\cot\varphi}{a\cos\varphi}\cdot\frac{\sin\varphi}{1+\cos\lambda\cos\varphi}=\frac{1}{1+\cos\lambda\cos\varphi}.$$

The projection is therefore conformal, since the meridians and parallels form an orthogonal net and the magnification along the meridians and along the parallels is the same.

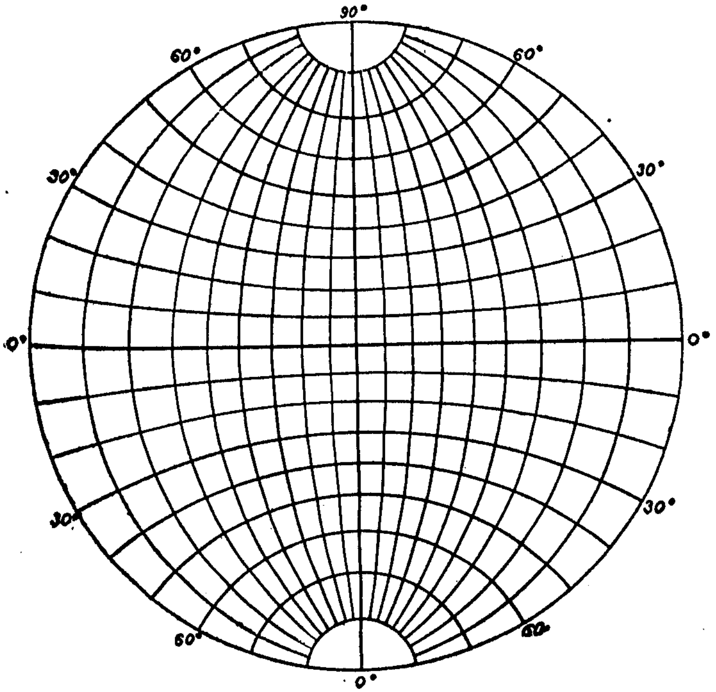

FIG. 7.—Stereographic meridian projection of a hemisphere.

DERIVATION OF STEREOGRAPHIC MERIDIAN PROJECTION BY FUNCTIONS OF A COMPLEX VARIABLE.[a]

The element of length upon the sphere is given in the form

$$dS^2 = a^2 (d\varphi^2 + d\lambda^2 \cos^2\varphi) = a^2 \cos^2\varphi \left(\frac{d\varphi^2}{\cos^2\varphi} + d\lambda^2\right)$$

If we set

$$d\sigma = \frac{d\varphi}{\cos\varphi},$$

dS becomes

$$dS^2 = a^2 \cos^2\varphi\ (d\sigma^2 + d\lambda^2).$$

Any conformal projection may then be expressed as a function either of $\sigma + i\lambda$ or of $\sigma - i\lambda$, in which i denotes as usual $\sqrt{-1}$.

$$\sigma = \int \frac{d\varphi}{\cos\varphi} = \int \frac{d\varphi}{\sin\left(\frac{\pi}{2} + \varphi\right)}$$

$$= \int \frac{\cos^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + \sin^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{2\sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}\, d\varphi.$$

$$\sigma = \int \frac{\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)} \frac{d\varphi}{2} + \int \frac{\sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)} \frac{d\varphi}{2}$$

$$\sigma = +\ \log_e \sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) - \log_e \cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$$

$$\sigma = \log_e \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right),$$

---

[a] See General Theory of the Lambert Conformal Conic Projection, Special Publication No. 53, U. S. Coast and Geodetic Survey.

or, on passing to exponentials,

$$e^{\sigma}=\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

$$e^{\sigma}+e^{-\sigma}=\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+\cot\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

$$=\frac{\sin^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+\cos^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}$$

$$=\frac{2}{2\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}=\frac{2}{\sin\left(\frac{\pi}{2}+\varphi\right)}=\frac{2}{\cos\varphi};$$

or

$$\frac{e^{+\sigma}+e^{-\sigma}}{2}=\sec\varphi$$

$$\cosh\sigma=\sec\varphi$$

$$\frac{e^{+\sigma}-e^{-\sigma}}{2}=\sinh\sigma$$

$$\sinh\sigma=\sqrt{\cosh^2\sigma-1}$$

$$\sinh\sigma=\sqrt{\sec^2\varphi-1}=\tan\varphi.$$

$$\sinh i\lambda=i\sin\lambda.$$

$$\cosh i\lambda=\cos\lambda.$$

If we take

$$x+iy=\frac{ai\left[e^{+\frac{1}{2}(\sigma-i\lambda)}-e^{-\frac{1}{2}(\sigma-i\lambda)}\right]}{e^{+\frac{1}{2}(\sigma-i\lambda)}+e^{-\frac{1}{2}(\sigma-i\lambda)}}$$

we obtain the stereographic meridian projection.

This can also be written in the form

$$x+iy=ai\tanh\left(\frac{\sigma-i\lambda}{2}\right)$$

$$x+iy=\frac{ai\sinh\left(\frac{\sigma-i\lambda}{2}\right)}{\cosh\left(\frac{\sigma-i\lambda}{2}\right)}$$

$$=\frac{ai\sinh\left(\frac{\sigma-i\lambda}{2}\right)\cosh\left(\frac{\sigma+i\lambda}{2}\right)}{\cosh\left(\frac{\sigma-i\lambda}{2}\right)\cosh\left(\frac{\sigma+i\lambda}{2}\right)}$$

$$=\frac{ai\,(\sinh\sigma-\sinh i\lambda)}{\cosh\sigma+\cosh i\lambda}$$

$$=\frac{ai\,(\sinh\sigma-i\sin\lambda)}{\cosh\sigma+\cos\lambda}$$

$$=\frac{a\sin\lambda+ai\sinh\sigma}{\cosh\sigma+\cos\lambda}$$

$$=\frac{a\sin\lambda+ai\tan\varphi}{\sec\varphi+\cos\lambda}$$

$$=\frac{a\sin\lambda\cos\varphi+ai\sin\varphi}{1+\cos\lambda\cos\varphi}.$$

By equating the real parts and the imaginary parts this becomes

$$x=\frac{a\sin\lambda\cos\varphi}{1+\cos\lambda\cos\varphi}$$

$$y=\frac{a\sin\varphi}{1+\cos\lambda\cos\varphi}.$$

We thus by this method arrive at the same values that were obtained before by expressing analytically the results of the direct projection. The fact that the projection can be derived by the use of functions of a complex variable establishes the conformality of the projection.*

*See Coast and Geodetic Survey Special Publication No. 53, The General Theory of the Lambert Conformal Conic Projection.

In order to take into consideration the ellipsoidal shape of the earth, we proceed in the following way. If we denote the element of length upon the ellipsoid by $d\Sigma$, we have

$$d\Sigma^2 = a^2 \left[\frac{(1-\epsilon^2)^2\, d\varphi^2}{(1-\epsilon^2 \sin^2\varphi)^3} + \frac{\cos^2\varphi\, d\lambda^2}{1-\epsilon^2 \sin^2\varphi}\right]$$

$$d\Sigma^2 = \frac{a^2 \cos^2\varphi}{1-\epsilon^2 \sin^2\varphi} \left[\frac{(1-\epsilon^2)^2\, d\varphi^2}{\cos^2\varphi\, (1-\epsilon^2 \sin^2\varphi)^2} + d\lambda^2\right].$$

In this case

$$d\sigma = \frac{(1-\epsilon^2)\, d\varphi}{\cos\varphi\, (1-\epsilon^2 \sin^2\varphi)}$$

$$= \frac{(1-\epsilon^2 \sin^2\varphi - \epsilon^2 \cos^2\varphi)\, d\varphi}{\cos\varphi\, (1-\epsilon^2 \sin^2\varphi)}$$

$$= \frac{d\varphi}{\cos\varphi} - \frac{\epsilon^2 \cos\varphi\, d\varphi}{1-\epsilon^2 \sin^2\varphi}$$

$$= \frac{d\varphi}{\sin\left(\frac{\pi}{2}+\varphi\right)} - \frac{\epsilon}{2}\left(\frac{\epsilon \cos\varphi\, d\varphi}{1-\epsilon \sin\varphi} + \frac{\epsilon \cos\varphi\, d\varphi}{1+\epsilon \sin\varphi}\right)$$

$$= \frac{\left[\cos^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) + \sin^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right] d\varphi}{2 \sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) \cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)} - \frac{\epsilon}{2}\left(\frac{\epsilon \cos\varphi\, d\varphi}{1-\epsilon \sin\varphi} + \frac{\epsilon \cos\varphi\, d\varphi}{1+\epsilon \sin\varphi}\right)$$

$$\sigma = \int \frac{\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)} \frac{d\varphi}{2} + \int \frac{\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)} \frac{d\varphi}{2} - \frac{\epsilon}{2}\int \frac{\epsilon \cos\varphi\, d\varphi}{1-\epsilon \sin\varphi} - \frac{\epsilon}{2}\int \frac{\epsilon \cos\varphi\, d\varphi}{1+\epsilon \sin\varphi}$$

$$\sigma = \log_e \sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) - \log_e \cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) + \frac{\epsilon}{2} \log_e (1-\epsilon \sin\varphi) - \frac{\epsilon}{2} \log_e (1+\epsilon \sin\varphi)$$

$$\sigma = \log_e \left[\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) \cdot \left(\frac{1-\epsilon \sin\varphi}{1+\epsilon \sin\varphi}\right)^{\frac{\epsilon}{2}}\right]$$

$$e^\sigma = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) \cdot \left(\frac{1-\epsilon \sin\varphi}{1+\epsilon \sin\varphi}\right)^{\frac{\epsilon}{2}}.$$

We can now map the ellipsoid conformally upon the sphere by the relations

$$\lambda' = \lambda$$

and

$$\tan\left(\frac{\pi}{4}+\frac{\varphi'}{2}\right)=\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\left(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi}\right)^{\frac{\epsilon}{2}}.$$

The latitudes φ' are computed for the parallels that we may wish to map; that is, for 10°, 20°, etc., or for whatever interval we may choose. This sphere may then be conformally mapped upon the plane, the values of φ' being employed in the computation. Each step is conformal; hence the plane map is a conformal representation of the ellipsoid.

The magnification upon the sphere is given by

$$\frac{dS}{d\Sigma}=\frac{a\cos\varphi'\left(\frac{d\varphi'}{\cos^2\varphi'}+d\lambda^2\right)^{\frac{1}{2}}}{\frac{a\cos\varphi}{(1-\epsilon^2\sin^2\varphi)^{\frac{1}{2}}}\left[\frac{(1-\epsilon^2)^2\,d\varphi^2}{\cos^2\varphi\,(1-\epsilon^2\sin^2\varphi)^2}+d\lambda^2\right]^{\frac{1}{2}}}$$

$$=\frac{\cos\varphi'\,(1-\epsilon^2\sin^2\varphi)^{\frac{1}{2}}}{\cos\varphi}.$$

The total magnification is equal to the product of the values obtained for the ellipsoid upon the sphere and for the sphere upon the plane. The total magnification, which we shall denote by k without subscript, since it is the same at any point in all directions, is given in the form

$$k=\frac{\cos\varphi'\,(1-\epsilon^2\sin^2\varphi)^{\frac{1}{2}}}{\cos\varphi\,(1+\cos\lambda\cos\varphi')}.$$

CONSTRUCTION OF STEREOGRAPHIC MERIDIAN PROJECTION.

It is a very easy matter to construct a stereographic meridian projection graphically. Divide the meridian circle into equal arcs at whatever interval it is desired to construct the meridians and parallels. In figure 8 the divisions are made at 30° intervals. $QR'=30°$; the tangent at R' gives the radius $S'R'$ and the center S' for the parallel of 30°; a similar arc with center distance to the south equal to OS' and with radius equal to $S'R'$ gives the projection of the parallel of 30° S. The tangent at R or SR gives the radius for 60° of latitude, and the same arc transferred to the south gives the projection for 60° S. The center distance $OT = SR$ with radius $TP' = TP$ gives the projection of meridian 60° west and OT' gives 60° east; also the center distance $OU = S'R'$ permits the construction of 30° W. and $OU' = S'R'$ gives the meridian of 30° E.

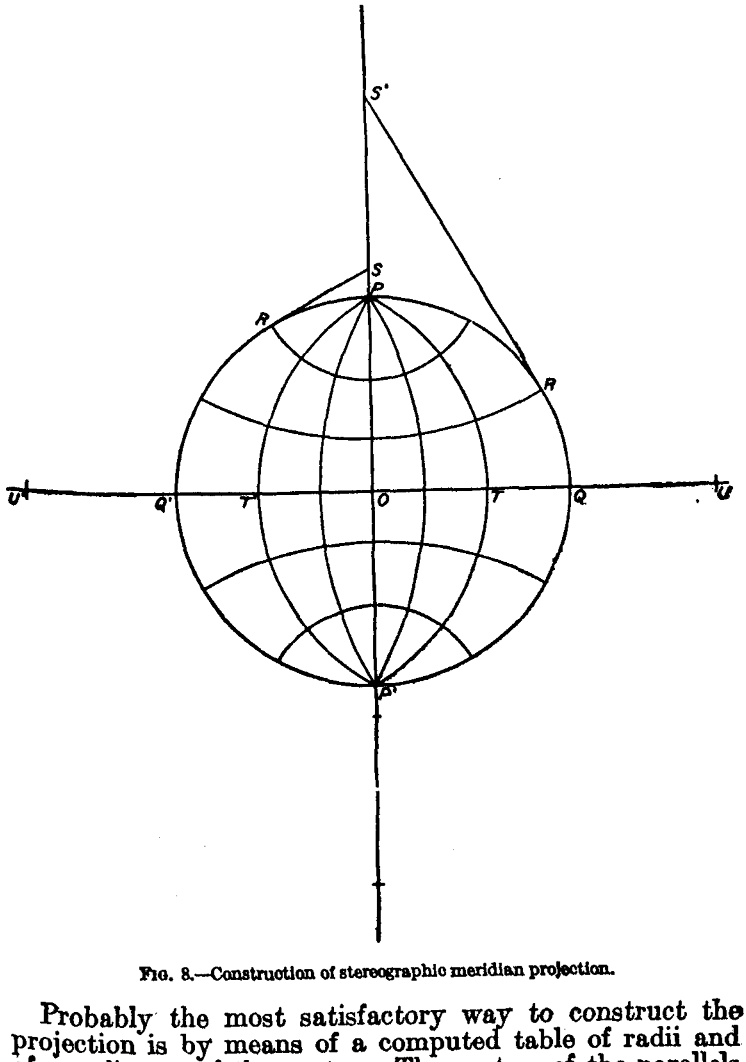

FIG. 8.—Construction of stereographic meridian projection.

Probably the most satisfactory way to construct the projection is by means of a computed table of radii and of coordinates of the center. The centers of the parallels all lie on the Y axis and those of the meridians lie on the X axis. The radii and the distances of the centers of the parallels become, respectively, the distances of the centers and the radii of the meridians. In the table ρ_m and ρ_p denote, respectively, the radii of the meridians and of the parallels; β_m and α_p, the distances of the centers; δ_m and δ_p, the distances of the intersections of the meridians with the Equator and of the parallels with the central meridian. The table, of course, applies to the sphere and not to the ellipsoid. The values are given in terms of the earth's radius, or they are the values for a sphere of unit radius.

TABLE FOR THE STEREOGRAPHIC MERIDIAN PROJECTION.

[In units of the earth's radius.]

φ or λ	ρ_m or α_p	ρ_p or β_m	δ_m or δ_p	φ or λ
Degrees.				*Degrees.*
0	∞	∞	0.00000	0
5	11.47371	11.43005	.04366	5
10	5.75877	5.67128	.08749	10
15	3.86370	3.73205	.13165	15
20	2.92380	2.74748	.17633	20
23° 27′ 30″	2.51204	2.30442	.20762	23° 27′ 30″
25	2.36620	2.14451	.22169	25
30	2.00000	1.73205	.26795	30
35	1.74345	1.42815	.31530	35
40	1.55572	1.19175	.36397	40
45	1.41421	1.00000	.41421	45
50	1.30541	.83911	.46631	50
55	1.22077	.70021	.52057	55
60	1.15470	.57735	.57735	60
65	1.10338	.46631	.63707	65
66° 32′ 30″	1.09009	.43395	.65616	66° 32′ 30″
70	1.06418	.36397	.70021	70
75	1.03528	.26795	.76733	75
80	1.01543	.17633	.83910	80
85	1.00382	.08749	.91633	85
90	1.00000	.00000	1.00000	90

STEREOGRAPHIC HORIZON PROJECTION.

In a stereographic projection the center of the map may lie at any point upon the earth's surface. We have just treated the case in which the center lay upon the equator. If the center is to be in latitude α, we start with the same equation in terms of the arc distance from the center and the azimuth reckoned from the great circle perpendicular to the meridian through the center.

$$x=\frac{a \sin p \cos \omega}{1+\cos p}$$

$$y=\frac{a \sin p \sin \omega}{1+\cos p}.$$

In figure 9 let T be the pole, Q the center of the projection, and let P be any given point.

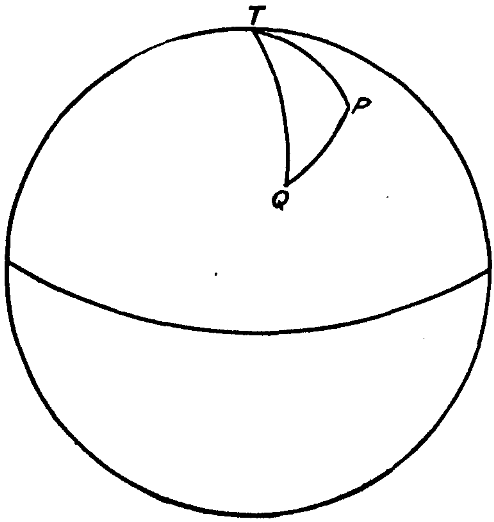

FIG. 9.—Transformation triangle for stereographic horizon projection.

$$TQ=\frac{\pi}{2}-\alpha$$

$$TP=\frac{\pi}{2}-\varphi$$

$$QP=p$$

$$\angle QTP=\lambda$$

$$\angle TQP=\frac{\pi}{2}-\omega.$$

From the trigonometry of the spherical triangle we have

$$\cos p=\sin\alpha\sin\varphi+\cos\alpha\cos\lambda\cos\varphi$$

$$\frac{\sin p}{\cos\varphi}=\frac{\sin\lambda}{\cos\omega}, \text{ or } \sin p\cos\omega=\sin\lambda\cos\varphi,$$

and

$$\sin p\sin\omega=\cos\alpha\sin\varphi-\sin\alpha\cos\lambda\cos\varphi.$$

On the substitution of these values we obtain as definitions of the coordinates of the projection

$$x=\frac{a \sin \lambda \cos \varphi}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}$$

$$y=\frac{a(\cos \alpha \sin \varphi-\sin \alpha \cos \lambda \cos \varphi)}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}.$$

From these equations, by solving for sin φ and cos φ, we find

$$\sin \varphi=\frac{x \sin \alpha \cos \lambda+y \sin \lambda}{a \cos \alpha \sin \lambda-x \cos \lambda-y \sin \alpha \sin \lambda}$$

$$\cos \varphi=\frac{x \cos \alpha}{a \cos \alpha \sin \lambda-x \cos \lambda-y \sin \alpha \sin \lambda}.$$

By squaring and adding there results

$$(x \sin \alpha \cos \lambda+y \sin \lambda)^2+x^2 \cos^2 \alpha$$
$$=(a \cos \alpha \sin \lambda-x \cos \lambda-y \sin \alpha \sin \lambda)^2.$$

By performing the operations and collecting, we obtain finally

$$x^2+y^2+2ax \sec \alpha \cot \lambda+2ay \tan \alpha=a^2,$$

which may also be written

$$(x+a \sec \alpha \cot \lambda)^2+(y+a \tan \alpha)^2=a^2 \sec^2 \alpha \operatorname{cosec}^2 \lambda.$$

This is the equation of the meridians, and they are thus seen to be circles. The meridian of longitude λ has the radius

$\rho_m=a \sec \alpha \operatorname{cosec} \lambda$, with its center at the point,

$x=-a \sec \alpha \cot \lambda,$

$y=-a \tan \alpha.$

The centers, therefore, all lie on the line

$$y=-a \tan \alpha.$$

By solving the original equations for sin λ and cos λ we get

$$\sin\lambda=\frac{x(\sin\alpha+\sin\varphi)}{a\sin\alpha\cos\varphi+y\cos\alpha\cos\varphi}$$

$$\cos\lambda=\frac{a\cos\alpha\sin\varphi-y-y\sin\alpha\sin\varphi}{a\sin\alpha\cos\varphi+y\cos\alpha\cos\varphi}.$$

By squaring and adding we obtain

$$x^2(\sin\alpha+\sin\varphi)^2+(a\cos\alpha\sin\varphi-y-y\sin\alpha\sin\varphi)^2=\cos^2\varphi(a\sin\alpha+y\cos\alpha)^2,$$

or, on developing and arranging,

$$x^2(\sin\alpha+\sin\varphi)^2+y^2(\sin\alpha+\sin\varphi)^2-2ay\cos\alpha(\sin\alpha+\sin\varphi)=a^2(\sin^2\alpha\cos^2\varphi-\cos^2\alpha\sin^2\varphi)$$

or, finally,

$$x^2+\left(y-\frac{a\cos\alpha}{\sin\alpha+\sin\varphi}\right)^2=\frac{a^2\cos^2\varphi}{(\sin\alpha+\sin\varphi)^2}.$$

The parallels are, therefore, circles with their centers all lying on the Y axis. The parallel of latitude φ has the radius

$$\rho_p=\frac{a\cos\varphi}{\sin\alpha+\sin\varphi},$$

with its center at the point

$$x=0,$$

$$y=\frac{a\cos\alpha}{\sin\alpha+\sin\varphi}.$$

The parallel of latitude $-\alpha$ is evidently a straight line, since the radius becomes infinite for this value, as does also the distance of the center from the center of the projection.

The projection is seen to be a polyconic projection in accordance with the definition of Tissot.

For the parallels we have

$$\rho=\frac{a \cos \varphi}{\sin \alpha+\sin \varphi}$$

$$s=\frac{a \cos \alpha}{\sin \alpha+\sin \varphi}$$

$$\sin \theta=\frac{x}{\rho}=\frac{\sin \lambda(\sin \alpha+\sin \varphi)}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}$$

$$\cos \theta=\frac{s-y}{\rho}=\frac{\cos \lambda+\cos \alpha \cos \varphi+\sin \alpha \cos \lambda \sin \varphi}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}.$$

s in this case is not reckoned from the Equator; but, since we need only the derivative of s with respect to φ, it will answer the purpose to leave it as it is. In fact, s could be reckoned from any fixed point in the line of centers and in this case it is reckoned from the origin which lies at latitude α.

$$\frac{\partial \theta}{\partial \varphi}=\frac{\cos \alpha \sin \lambda}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}$$

$$\frac{\partial \theta}{\partial \lambda}=\frac{\sin \alpha+\sin \varphi}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}$$

$$\frac{ds}{d\varphi}=-\frac{a \cos \alpha \cos \varphi}{(\sin \alpha+\sin \varphi)^2}$$

$$\frac{d\rho}{d\varphi}=-\frac{a(1+\sin \alpha \sin \varphi)}{(\sin \alpha+\sin \varphi)^2}.$$

These values may now be substituted in the general differential formulas and by that means we obtain the following results:

$$\rho\frac{\partial \theta}{\partial \varphi}+\frac{ds}{d\varphi}\sin \theta=\frac{a \cos \alpha \sin \lambda \cos \varphi}{(\sin \alpha+\sin \varphi)(1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi)}$$

$$-\frac{a \cos \alpha \sin \lambda \cos \varphi}{(\sin \alpha+\sin \varphi)(1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi)}=0.$$

Therefore

$$\tan \psi=0$$

or

$$\psi=0.$$

The parallels and meridians form, then, an orthogonal net of circles.

$$k_m = \frac{\left(\frac{ds}{d\varphi}\cos\theta - \frac{d\rho}{d\varphi}\right)}{a\cos\psi} = -\frac{\cos\alpha\cos\varphi}{(\sin\alpha+\sin\varphi)^2}\times$$

$$\frac{\cos\lambda+\cos\alpha\cos\varphi+\sin\alpha\cos\lambda\sin\varphi}{1+\sin\alpha\sin\varphi+\cos\alpha\cos\lambda\cos\varphi}+\frac{1+\sin\alpha\sin\varphi}{(\sin\alpha+\sin\varphi)^2}$$

$$=\frac{1}{1+\sin\alpha\sin\varphi+\cos\alpha\cos\lambda\cos\varphi}$$

$$k_p = \frac{\rho}{a\cos\varphi}\cdot\frac{\partial\theta}{\partial\lambda}$$

$$=\frac{1}{\sin\alpha+\sin\varphi}\cdot\frac{\sin\alpha+\sin\varphi}{1+\sin\alpha\sin\varphi+\cos\alpha\cos\lambda\cos\varphi}$$

$$=\frac{1}{1+\sin\alpha\sin\varphi+\cos\alpha\cos\lambda\cos\varphi}.$$

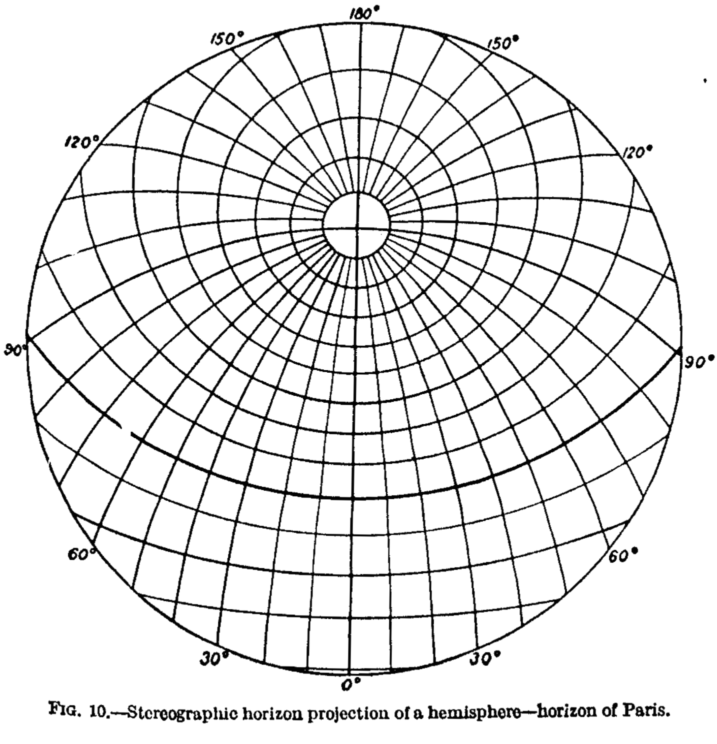

FIG. 10.—Stereographic horizon projection of a hemisphere—horizon of Paris.

The projection is thus shown to be conformal, since the meridians and parallels are orthogonal and the magnification along both is the same. We might have taken this for granted since we found that the stereographic meridian projection was conformal and the nature of the projection is not changed by moving the point of projection to a different point upon the sphere.

In taking account of the spheroid we proceed as in the case of the stereographic meridian projection. The magnification at a point (the same in all directions) would then be

$$k=\frac{\cos\varphi'\,(1-\epsilon^2\sin^2\varphi)^{1/2}}{\cos\varphi\,(1+\sin\alpha'\sin\varphi'+\cos\alpha'\cos\lambda\cos\varphi')}.$$

DERIVATION OF STEREOGRAPHIC HORIZON PROJECTION BY FUNCTIONS OF A COMPLEX VARIABLE.

The projection, being a conformal projection, can be expressed in terms of a function of a complex variable either of $\sigma+i\lambda$ or of $\sigma-i\lambda$. Let us take

$$x+iy=\frac{ai\sinh\left(\frac{\sigma-i\lambda-\beta}{2}\right)}{\cosh\left(\frac{\sigma-i\lambda+\beta}{2}\right)}$$

$$=\frac{ai\sinh\left(\frac{\sigma-i\lambda-\beta}{2}\right)\cosh\left(\frac{\sigma+i\lambda+\beta}{2}\right)}{\cosh\left(\frac{\sigma-i\lambda+\beta}{2}\right)\cosh\left(\frac{\sigma+i\lambda+\beta}{2}\right)}$$

$$=\frac{ai[\sinh\sigma-\sinh(i\lambda+\beta)]}{\cosh(\sigma+\beta)+\cosh i\lambda}$$

$$=\frac{ai[\sinh\sigma-\sinh i\lambda\cosh\beta-\cosh i\lambda\sinh\beta]}{\cosh\sigma\cosh\beta+\sinh\sigma\sinh\beta+\cosh i\lambda}.$$

But

$$\cosh\sigma=\sec\varphi$$

$$\sinh\sigma=\tan\varphi$$

$$\sinh i\lambda=i\sin\lambda$$

$$\cosh i\lambda=\cos\lambda.$$

By substituting these values we obtain

$$x+iy=\frac{ai\,(\tan\varphi-i\sin\lambda\cosh\beta-\cos\lambda\sinh\beta)}{\sec\varphi\cosh\beta+\tan\varphi\sinh\beta+\cos\lambda}$$

$$=\frac{a\sin\lambda\cosh\beta+ai\,(\tan\varphi-\cos\lambda\sinh\beta)}{\sec\varphi\cosh\beta+\tan\varphi\sinh\beta+\cos\lambda}.$$

By equating the real parts and the imaginary parts, we get

$$x=\frac{a\sin\lambda\cosh\beta}{\sec\varphi\cosh\beta+\tan\varphi\sinh\beta+\cos\lambda}$$

$$y=\frac{a\,(\tan\varphi-\cos\lambda\sinh\beta)}{\sec\varphi\cosh\beta+\tan\varphi\sinh\beta+\cos\lambda}.$$

Let

$$\cosh\beta=\sec\alpha,$$

then

$$\sinh\beta=\tan\alpha.$$

Substituting these values we obtain

$$x=\frac{a\sec\alpha\sin\lambda}{\sec\alpha\sec\varphi+\tan\alpha\tan\varphi+\cos\lambda}$$

$$y=\frac{a(\tan\varphi-\tan\alpha\cos\lambda)}{\sec\alpha\sec\varphi+\tan\alpha\tan\varphi+\cos\lambda}.$$

On multiplying both numerator and denominator by $\cos\alpha$ $\cos\varphi$, we derive

$$x=\frac{a\sin\lambda\cos\varphi}{1+\sin\alpha\sin\varphi+\cos\alpha\cos\lambda\cos\varphi}$$

$$y=\frac{a(\cos\alpha\sin\varphi-\sin\alpha\cos\lambda\cos\varphi)}{1+\sin\alpha\sin\varphi+\cos\alpha\cos\lambda\cos\varphi}.$$

We thus arrive at the same equations that were obtained before.

PROOF THAT CIRCLES PROJECT INTO CIRCLES IN STEREOGRAPHIC PROJECTIONS.

It can be proved in a general way that, in any stereographic projection, any circle upon the sphere is projected into a circle upon the plane of the map. Straight lines must, of course, be considered as circles of infinite radii, with centers at infinity. Any circle either great or small which passes through the point of projection will be projected into a straight line, since all of the projecting lines will lie in the plane of the circle and will cut the mapping plane in a straight line, which is formed by the intersection of the plane of the circle with the mapping plane.

Let us now take any other circle upon the sphere. Make a great-circle section of the sphere containing the point of projection and the pole of the given circle. This great circle necessarily will also pass through the point that projects into the center of the map, i. e., the point antipodal to

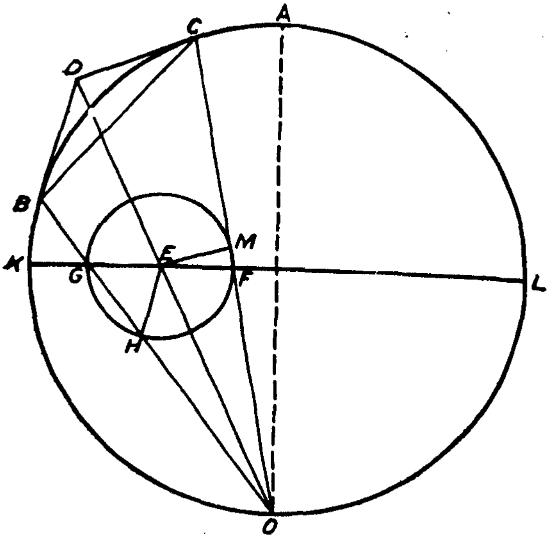

FIG. 11.—Proof that circles project into circles on stereographic projections.

the point of projection. After this is done turn the great circle section into the plane of the page. The plane of this section will evidently be perpendicular to the plane of the given circle, since the plane of any great circle containing the pole of the given circle would partake of this property.

In figure 11 let O be the point of projection, KL the trace of the mapping plane, BC the trace of the plane of the circle, and let A be the point that projects into the center of the map. The lines that project the circle under consideration will evidently form an oblique cone that has the given circle as a circular section. Any plane parallel to the plane of this circle will also cut the cone in a circle.

We shall now prove analytically that any such oblique cone that has one system of circular sections has also another system of circular sections. If we have a cone passing through the circle $z=0$, $x^2+y^2=a^2$, it will be a perfectly general one if we take the apex at the point $x=f$, $y=0$, $z=h$ in the plane $y=0$. A line through this point is given by the equations

$$x-f=\alpha(z-h)$$

$$y=\beta(z-h).$$

This line intersects the plane $z=0$ in the point the coordinates of which are

$$x_1=f-\alpha h$$

$$y_1=-\beta h.$$

Since this point is to lie on the circle, we have

$$(f-\alpha h)^2+\beta^2h^2=a^2.$$

But

$$\alpha=\frac{x-f}{z-h}$$

$$\beta=\frac{y}{z-h}.$$

By substituting these values we obtain

$$(fz-hx)^2+h^2y^2=a^2(z-h)^2.$$

This is the equation of a cone bearing the same relation to the plane $y=0$ that the projecting cone bears to the plane of the great circle. This equation may be written in the form

$$h^2\ (x^2+y^2+z^2-a^2)=z[2fhx+(a^2-f^2+h^2)z-2ha^2].$$

Hence, if the conical surface is cut by either of the planes,

$$z=\gamma$$

or

$$2fhx+(a^2-f^2+h^2)z-2ha^2=\delta,$$

the points of intersection will satisfy an equation of the form

$$x^2+y^2+z^2+2Ax+2Bz+D=0$$

for all values of γ and δ, and the sections will therefore be plane sections of a sphere. Therefore, there are two series of circular sections made by two systems of parallel planes, and both systems are parallel to the plane $y=0$.

The trace of the cone upon the plane $y=0$ has for its equation:

$$(fz-hx)^2-a^2(z-h)^2=0.$$

This is, therefore, the equation of the two generating lines which lie in that plane. The equation of the two planes in opposite systems giving the circular sections is

$$(z-\gamma)\ [2fhx+(a^2-f^2+h^2)\ z-2ha^2-\delta]=0.$$

By adding these two equations we get an equation of the form

$$x^2+z^2+A'x+B'y+C'=0.$$

This shows that the four points in which the two generating lines in the plane $y=0$ meet the planes forming the circular sections lie upon a circle. Hence the first system of planes makes the same angle with the one of the generating lines that the second system makes with the other. We will now show that the mapping plane fulfills the conditions for the second system of circular sections. The mapping plane is evidently perpendicular to the plane of the great circle *ALOK*, and it thus fulfills the first condition. The further condition is that it must make the same angle with one of the elements of the cone lying in the plane of the great circle that the plane of the circle on the sphere makes with the other element in this plane. In figure 11

$$\angle CBO=\frac{1}{2}\text{ arc } OLAC=\frac{1}{2}(\text{arc } OLA+\text{arc } AC)=\frac{\pi}{2}+\frac{1}{2}\text{ arc } AC$$

$$\angle KFO=\frac{1}{2}\ (\text{arc } OK+\text{arc } LAC)=\frac{\pi}{2}+\frac{1}{2}\text{ arc } AC,$$

Therefore

$$\angle CBO=\angle KFO$$

and

$$\angle BCO=\angle FGO.$$

It is thus seen that the points *B*, *C*, *F*, and *G* lie upon a circle and all the conditions are fulfilled for a circular section.

Construct the tangents *BD* and *CD*, draw *EM* parallel to *CD*, and draw *EH* parallel to *BD*.

Then

$$DC : EM = DO : EO = DB : EH,$$

but

$$DC = DB.$$

Therefore

$$EM = EH,$$

$$\angle EGH = \frac{1}{2} (\text{arc } OL + \text{arc } KB) = \frac{\pi}{4} + \frac{1}{2} \text{ arc } KB$$

$$\angle EHG = \pi - \angle EHO = \pi - \angle DBO = \pi - \frac{1}{2} \text{arc } OLACB$$

$$= \pi - \frac{1}{2} (\text{arc } OLACBK - \text{arc } BK)$$

$$= \pi - \frac{3}{4}\pi + \frac{1}{2} \text{ arc } BK = \frac{\pi}{4} + \frac{1}{2} \text{ arc } BK.$$

Therefore

$$\angle EGH = \angle EHG$$

and

$$EH = EG.$$

In a similar way it can be proved that

$$EM = EF.$$

But, since

$$EH = EM,$$

$$EG = EF,$$

therefore the projection of D is the center of the circle that maps the given circle. D is, of course, the apex of the cone tangent to the sphere along the given circle.

The stereographic horizon projection can be constructed either by computation of the radii and centers or directly by graphic construction. The formulas for computation are for the meridians

$$\rho_m = a \sec \alpha \operatorname{cosec} \lambda$$

$$x_m = -a \sec \alpha \cot \lambda$$

$$y_m = -a \tan \alpha$$

and for the parallels

$$\rho_p = \frac{a \cos \varphi}{\sin \alpha + \sin \varphi} = \frac{a \cos \varphi}{2 \sin \left(\frac{\alpha+\varphi}{2}\right) \cos \left(\frac{\alpha-\varphi}{2}\right)}$$

$$x_p = 0$$

$$y_p = \frac{a \cos \alpha}{\sin \alpha + \sin \varphi} = \frac{a \cos \alpha}{2 \sin \left(\frac{\alpha+\varphi}{2}\right) \cos \left(\frac{\alpha-\varphi}{2}\right)}.$$

The forms last given should be used for logarithmic computation.

CONSTRUCTION OF STEREOGRAPHIC HORIZON PROJECTION.

The method of graphical construction for the parallels is as follows: Let us suppose that we wish to construct a projection for $\alpha=30°$. In figure 12 the point of projection is supposed to be in the perpendicular to the plane of the paper at E. Let the plane of the central meridian (that through the point of projection) cut the mapping plane or the plane of the paper in the line YY'. This central meridian section is then turned upon YY' as an axis until it falls in the plane of the paper. The eye will then be at O, and A will be the point that projects into the center of the map. Construct the angle AEQ equal to 30°; then QQ' is the trace of the equitorial plane upon the plane of the central meridian. The diameter PP' perpendicular to QQ' is the axis of the earth turned with the plane of the central meridian. YY' is the projection of the central meridian, since the plane was turned upon this line as an axis; hence, if any point is projected upon this line the corresponding point upon the map will be determined. P and P' are the poles; draw OP and OP'. Then p is the North Pole of the map and p' is the South Pole of the same.

To determine the circle that forms the projection of any parallel, lay off the arc CQ equal to the latitude; in the figure $CQ=45°$. Construct CB perpendicular to PP' and construct tangents at B and C meeting in the axis produced at D. Draw OB, OC, and OD; then B' and c' are points on the circle, and D' is the center of the same. With D' as center and with radius $D'B'$ or $D'c'$ construct the circle, and the circle so drawn in the figure is the projection of the parallel of 45° of latitude. OQ determines the point q on the Equator, and OF drawn parallel to PP' locates the center at F; with the radius Fq draw the arc OqA; this arc is the projection of the Equator.

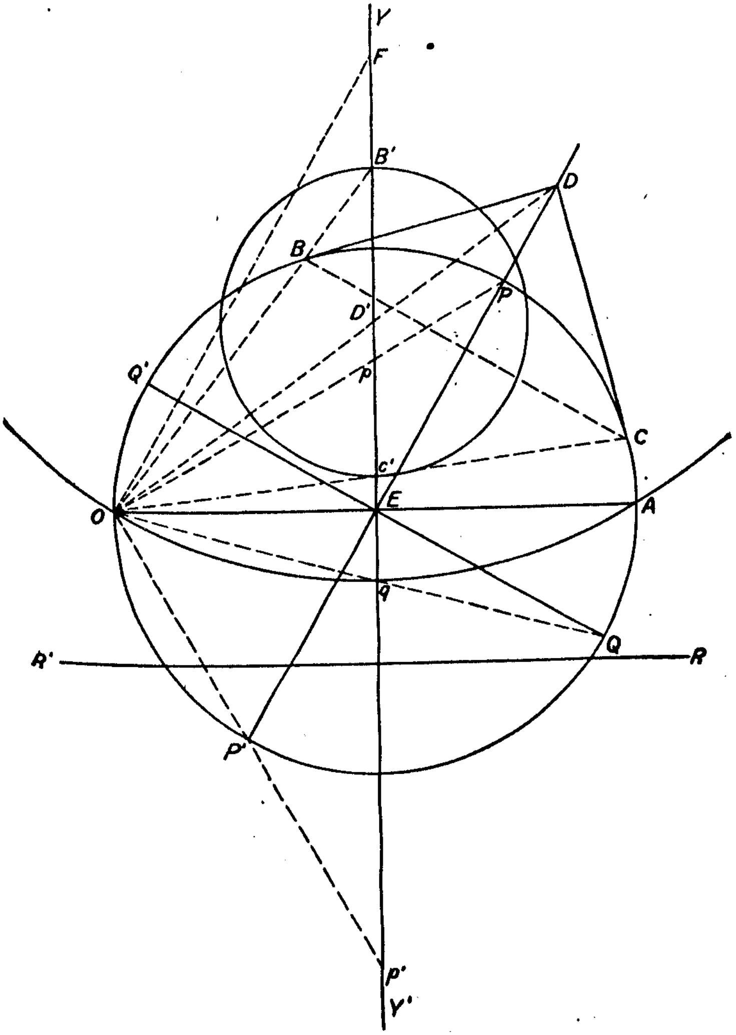

FIG. 12.—Construction of parallels on stereographic horizon projection.

In a similar manner the projections of any desired parallels can be drawn. It is evident that any two of the points B', c', and D' will be sufficient to determine the circle, since we know that the center lies upon YY'. The circle which represents the parallel of latitude $-\alpha$ has an infinite radius with center at infinity on the line YY'; it is therefore a straight line perpendicular to YY'. The lower point at which the parallel crosses the central meridian is given by

$$y_p - \rho_p = \frac{a(\cos\alpha - \cos\varphi)}{\sin\alpha + \sin\varphi}.$$

This takes the form 0/0 for $\varphi = -\alpha$, and the limit must be determined for this point.

$$\lim_{\varphi \doteq -\alpha} \frac{a(\cos\alpha - \cos\varphi)}{\sin\alpha + \sin\varphi} = \lim_{\varphi \doteq -\alpha} \frac{a \sin\varphi}{\cos\varphi} = -a \tan\alpha,$$

or, otherwise,

$$\frac{a(\cos\alpha - \cos\varphi)}{\sin\alpha + \sin\varphi} = a \tan\frac{1}{2}(\varphi - \alpha),$$

which for $\varphi = -\alpha$ becomes $-a \tan\alpha$.

The straight line parallel, therefore, conicides with the line of centers for the meridians; and hence must be the perpendicular bisector of pp'. It is the line RR' drawn in the figure.

In figure 13 the details of the construction of the meridians are given. p and p' are determined in the same way as in figure 12. To determine the coordinates of p and of p', we set $x=0$ in the equation of the meridian and solve for y. We thus find that

$$y = -a \tan\alpha \pm a \sec\alpha;$$

therefore

$$Ep = -a \tan\alpha + a \sec\alpha$$

and

$$Ep' = -a \tan\alpha - a \sec\alpha.$$

The middle point of pp' is given by

$$\frac{1}{2}(Ep + Ep') = -a \tan\alpha.$$

The perpendicular bisector of pp' is, of course, the line of centers of the meridians, since they must all pass through the points p and p' and they thus have pp' as a common chord. This line of centers is the line RR' in the figure.

The length of Fp' is equal to the length of Ep' minus the length of EF; hence the length of $Fp' = a \sec \alpha$. The center for the arc that is the projection of the meridian of longitude λ lies on the line RR' at the point, $x_m = -a \sec \alpha \cot \lambda$. With p' as a center and with any convenient radius construct a circle; divide this circumference into

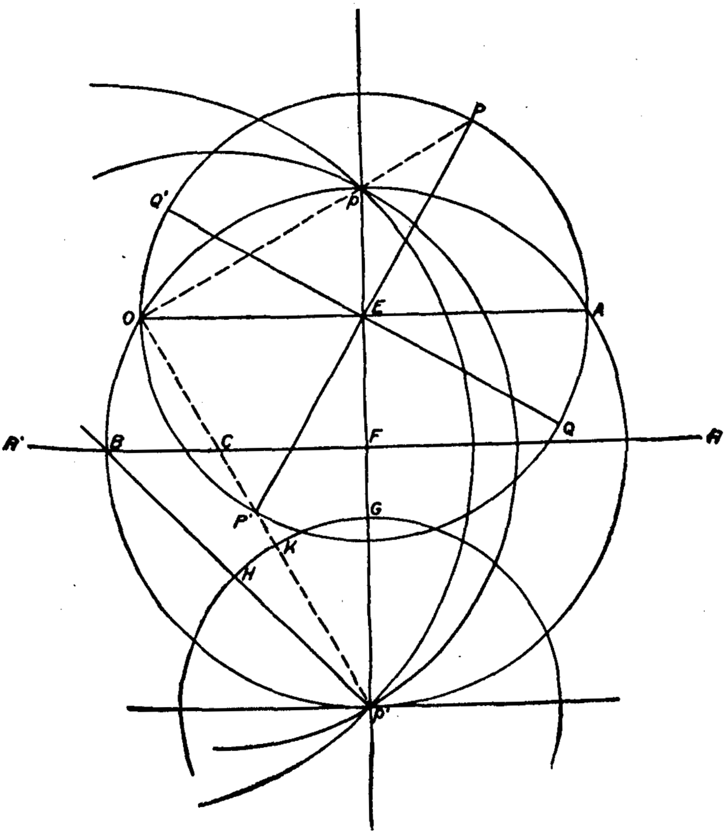

FIG. 13.—Construction of meridians on stereographic horizon projection.

equal arcs for whatever interval it is desired to construct the meridians, the initial point of the subdivision being the point where this circle intersects the central meridian. In the figure we have

$$BF = Fp' \tan \angle Bp'F;$$

but

$$Fp' = a \sec \alpha.$$

If then the angle $Bp'F = \frac{\pi}{2} - \lambda$, we shall have

$$BF = a \sec \alpha \cot \lambda.$$

The arc GH must be taken as the complement of the longitude, for which we wish to construct the meridian. GK is 30°; therefore C is the center of the meridian for $\lambda = 60°$. The meridians all pass through p and p', so that they may be constructed as soon as we have located the centers. F is, of course, the center for the meridian of $\lambda = 90°$.

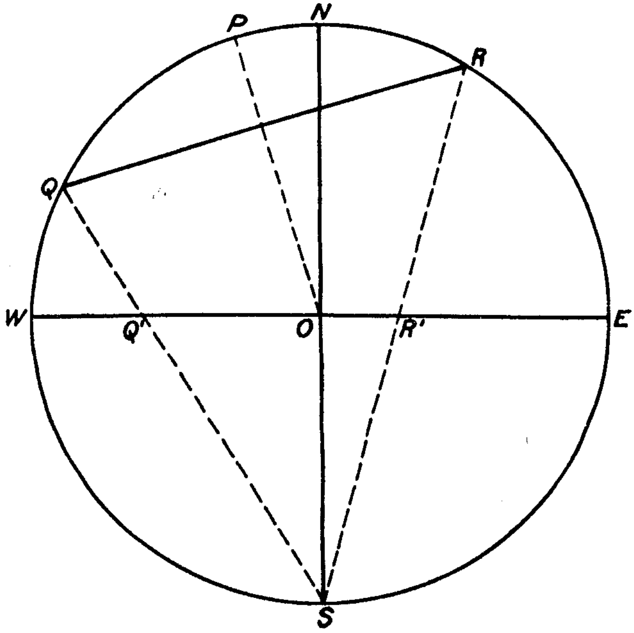

FIG. 14.—Elements of a small circle on stereographic projection.

SOLUTION OF PROBLEMS IN STEREOGRAPHIC PROJECTIONS.

We shall now give the demonstration of the solutions of a few problems connected with stereographic projections. The plane of the projection is called the primitive plane, and the circle formed by the intersection of the primitive plane with the sphere is called the primitive circle. The polar distance of a point on the sphere is the angular distance on the sphere from one of the poles of the primitive circle. The polar distance of a circle is the angular distance of any point of its circumference from either of its own poles. The inclination of a circle is the angle between its plane and the primitive plane. It is measured by the arc distance between the pole of the given circle and the pole of the primitive circle, since this measures the angle between the perpendiculars to the planes of the two circles.

In figure 14 let *NESW* be the primitive circle and let *QR* be the trace of the plane of a small circle, with *P* as its pole; then $PR = PQ$ is its polar distance and *PN* is its inclination. The diameter *WE* is called the line of measures of the circle *QR*; *NS* is perpendicular to *WE* at the center

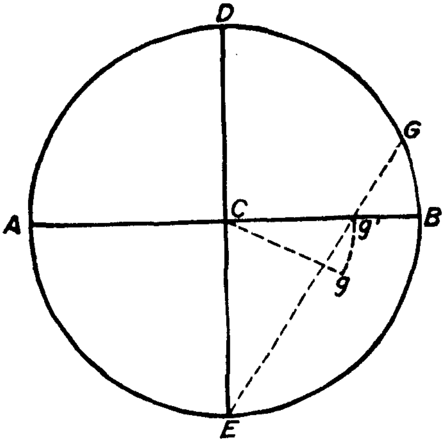

FIG. 15.—Determination of the arc distance from the center on stereographic projection

of the primitive circle. *S* is the point of projection and *Q'* and *R'* are the projections of the extreme or principal elements of the oblique circular cone *SQR* which is formed by the projecting lines of the points of the circle *QR*. Denoting the polar distance of the circle by κ and the inclination by ξ, we have

$$OR' = a \tan \frac{1}{2}(\kappa - \xi)$$

$$OQ' = a \tan \frac{1}{2}(\kappa + \xi)$$

Problem 1.—To determine the shortest distance between the center of the map and another point the projection of which is given; that is, to determine the arc of a great circle between them:

In figure 15 let *DBEA* be the primitive circle and let *AB* be the line of measures; *g* is the given point. Construct *Cg′* equal to *Cg* and draw *Eg′* from the point of sight *E* and prolong it to meet the primitive circle at *G;* then *DG* is the arc distance, since all points of polar distance *DG* are projected into the circle of which the arc *gg′* forms a part. Therefore, the great circle distance of *Cg* and *Cg′* are equal; *DG* is evidently the polar distance of *g′*, and hence also of *g*. If the given point lies on the line of measures the construction is the same as that given for the determination of the great circle distance of *g′*.

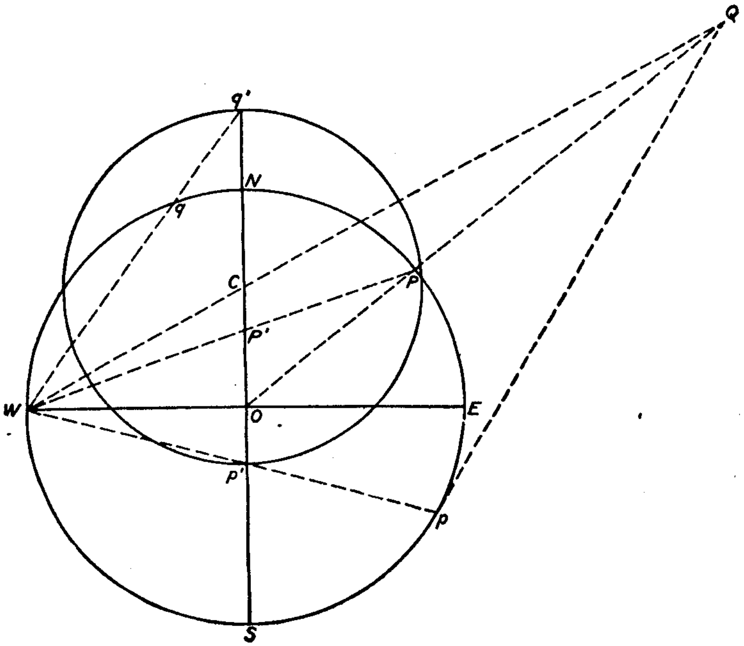

FIG. 16.—Projection of a circle with given projection of pole and given polar distance on stereographic projection.

Problem 2.—To construct the projection of a given circle, its polar distance and the projection of its pole being given:

In figure 16 let *P′* be the projection of the pole. *NESW* is the primitive circle with *NS* passing through *P′* and with *WE* perpendicular to *NS; NS* is then the line of measures, with *W* as the point of projection. Draw *WP′P* and from *P* lay off the arcs *Pp* and *Pq* equal to the given polar distance. Draw *Wp* and *Wq*, thus locating

p' and q' in the line of measures. A circle constructed on $p'q'$ as diameter is the required projection, since $p'q'$ is the projection of the diameter of the circle on the line of measures. This circle can be determined in another way by locating p and p' as before; then at p draw the

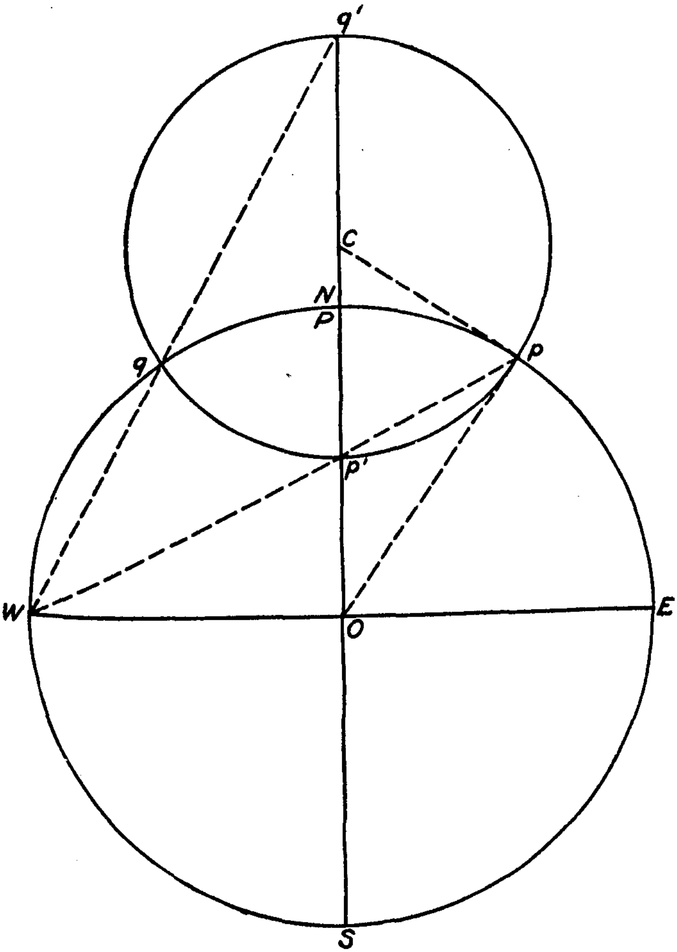

FIG. 17.—Projection of circle whose pole projection lies on the primitive circle on stereographic projection.

tangent pQ meeting OP produced at Q; then WQ locates C the center of the required circle. With C as center and with Cp' as the radius, we can construct the circle. If P' lies on the primitive circle, P and P' will coincide, and the construction is evident from figure 17.

Problem 3.—To project a great circle, the projection of the pole being given:

In this case the polar distance is $\frac{\pi}{2}$ and $Pp = Pq = \frac{\pi}{2}$ in figure 18. The circle passes through W and E; hence it is sufficient to locate either p' or q'; WC is parallel to OP,

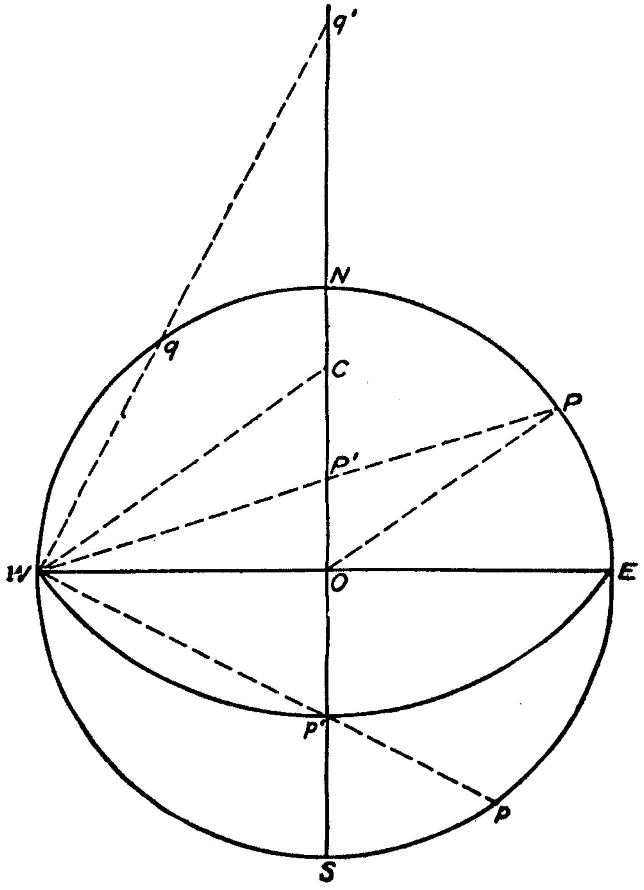

FIG. 18.—Projection of a great circle with given pole projection on stereographic projection.

and in this manner C can be located; with C as center, with CE as radius, the circle can be constructed.

Problem 4.—To find the locus of centers of all great circles passing through a given point:

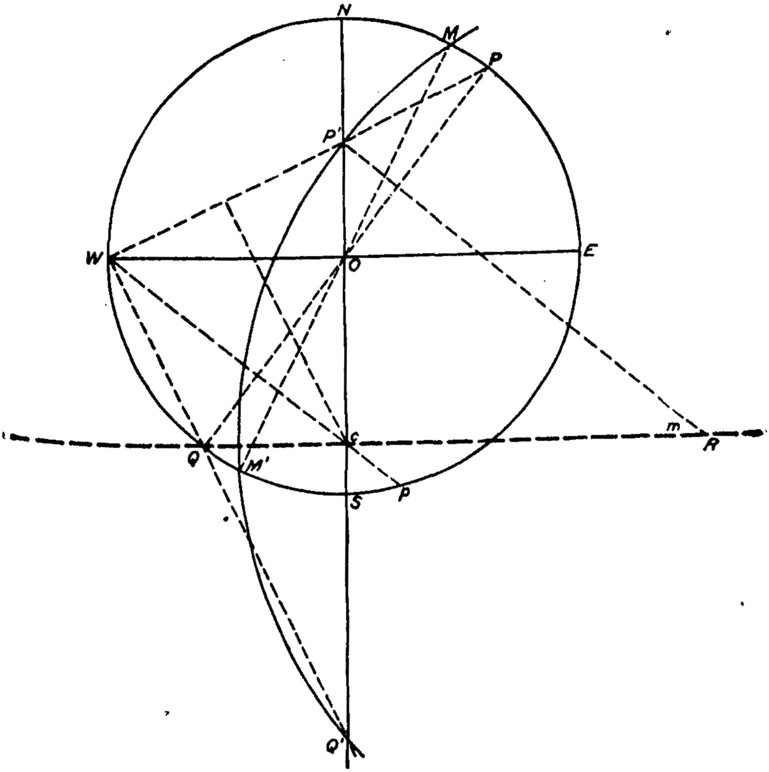

FIG. 19.—Locus of centers of great circles through a given point on stereographic projection.

In figure 19 let P' be the projection of the given point through which the great circles are to pass; draw the diameter $NP'S$ and the perpendicular diameter WE. The projections of all great circles through P' must also pass through a point at the distance of π from P'; accordingly draw the diameter PQ and draw WQ, cutting NS the line of measures in Q'; then Q' is the projection of the antipode of P. Since all the required circles pass through P' and Q', their centers must lie on the straight line perpendicular to $P'Q'$ at its middle point c; this line is called the line of centers.

Since a great circle may always be drawn through the points W, P', and E, the point c may be found by drawing a perpendicular bisector to WP' intersecting NS in c.

The triangle $WP'c$ is isosceles, and the angle $P'Wp$ equals the angle $WP'S$, which is measured by $\frac{1}{2}\left(\frac{\pi}{2}+\text{arc } PN\right)$ $=\frac{1}{2}$ arc PNW; that is, the arc $PEp=$ arc PNW. Hence lay off the arc $PEp=$ arc PNW and draw Wcp. This is the same as laying off a polar distance PNW from P; thus the line of centers is the projection of a small circle passing through the line of sight and having the polar distance $PNW=\pi-\xi$, where ξ denotes the inclination of the circle.

From figure 19 $WQ=PE$; $QSp=\pi-(pE+WQ)=\pi-PEp=\pi-PNW=WQ$; hence lay off $WQp=2PE$, and draw Wp, thus locating c. Wp is evidently perpendicular to PQ, so that c can be located in that way.

$\angle WEp=\angle POE=\angle WOQ$; hence a line joining E and p is parallel to PQ; this gives another method for locating c.

Problem 5.—To draw a great circle through P, making a given angle with NS:

In figure 19 the tangent to the required circle at P makes the given angle (m) with $P'OS$; the perpendicular to the tangent makes with $P'OS$ the angle $\frac{\pi}{2}-m$. Hence construct $SP'R=\frac{\pi}{2}-m$ with $P'R$ intersecting the line of centers at R, the center of the required circle.

The projection of a great circle always meets the primitive circle at the extremities of a diameter as MM' in figure 19.

Problem 6.—To find the projection of a pole of a given circle:

In figure 18 let $Wp'E$ be a great circle; draw the perpendicular diameters WE and NS, and draw $Wp'p$; lay off pP equal to $\frac{\pi}{2}$ and draw WP, thus locating P', the required pole.

In figure 16 let $p'q'$ be a given small circle; through its center c draw NS and draw WE at right angles; draw Wp' to locate p and Wq' to locate q; bisect the arc $qNEp$, locating P, and draw WP, thus locating P', the projection of the required pole.

Problem 7.—To construct the projection of a great circle passing through the projections of two given points:

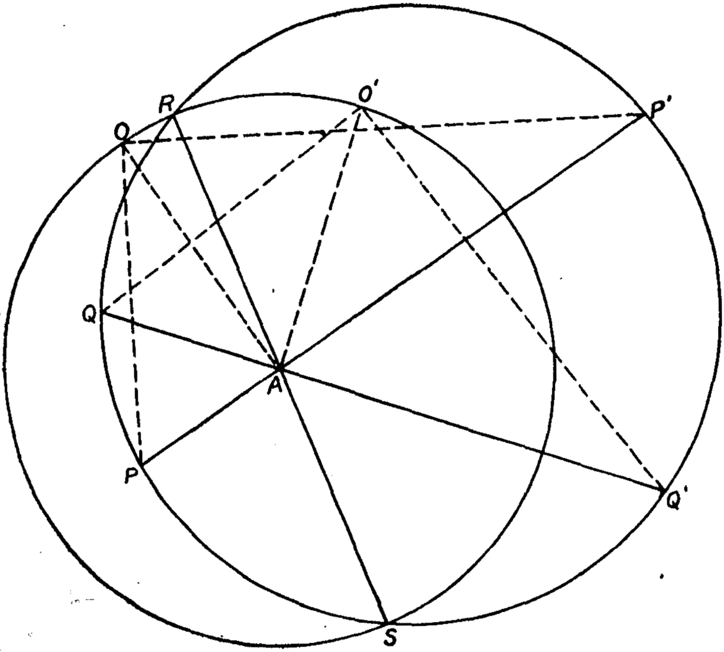

FIG. 20.—Projection of a great circle through the projections of two given points on stereographic projection.

In figure 20 let $ORO'S$ be the primitive circle and let P and Q be the projections of the two given points, and let A be the center of the projection. The lines that project any two antipodal points are perpendicular to each other; we can then easily determine the projections of the points antipodal to P and Q through which the projected circle must necessarily pass. Draw PA and prolong it beyond A; at A erect the perpendicular AO, intersecting the primitive circle at O; draw OP and erect upon it the perpendicular OP' intersecting PA produced in P'; P' is then the projection of the point antipodal to P. The triangle OPP' is the projecting triangle turned on the projected line PP' as an axis into the plane of the paper. In a similar way Q' can be determined, but a circle passed through P, Q, and P' is the required projection. It may be seen that the construction is correct from the consideration that AP' must be a third proportional to AP and AO. If the point of which P is the projection has the polar distance p, then $AP = a \tan \frac{p}{2}$ and $AP' = a \tan \frac{1}{2}(\pi - p)$ $= a \cot \frac{p}{2}$; but $OA = a$, and so we have

$$OP : OA = OA. : AP'.$$

This establishes the validity of the construction.

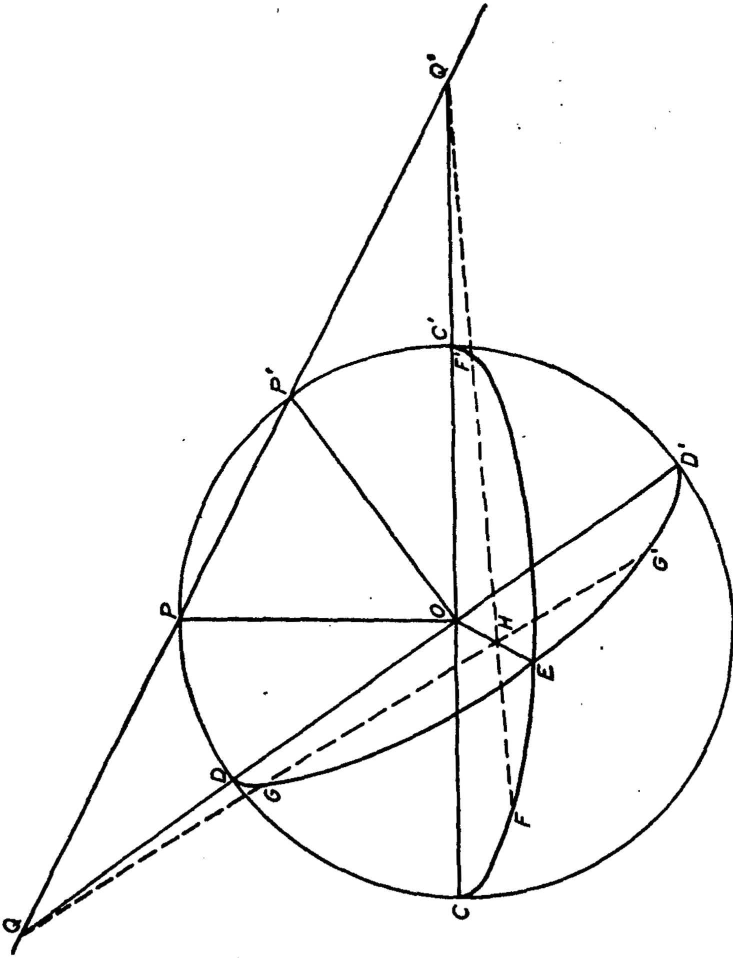

Fig. 21.—Plane through the poles of two great circles.

As a basis for the next problem we shall prove that if a plane passes through the poles of two great circles it cuts off equal arcs on the two circles.

In figure 21 let P be the pole of the great circle CEC' and let P' be the pole of DED' with the center of the sphere at O. The triangle OPP' is isosceles; therefore, the line PP' is equally inclined to the planes of the great circles, since it is equally inclined to their perpendiculars OP and OP'. Produce PP' in both directions to intersect the planes of the circles, the one at Q and the other at Q'. The triangle OPQ = the triangle $OP'Q'$, since $OP = OP'$, $\angle OPQ = \angle OP'Q'$, and $\angle POQ = \angle P'OQ'$. Therefore, $QO = Q'O$ and $QD = Q'C'$. Pass a plane through PP' and let $QGHG'$ be its trace on the plane of DED' and let $Q'F'HF$ be the trace on the plane of CEC'. Then $\angle OQH = \angle OQ'H$, since the corresponding right triangles are equal. The arc DG will therefore equal the arc $C'F'$, and the arc $G'D'$ will equal the arc CF, since Q and Q' are the same distance from their respective great circles. But the arc $GEG' = \pi - (DG + D'G')$ and the arc $FEF' = \pi - (F'C' + CF)$. Therefore, the arc GEG' is equal to the arc FEF', and the proposition is proved.

Problem 8.—To determine the shortest distance between two points whose projections P and Q are given; that is, to determine the arc of a great circle between them:

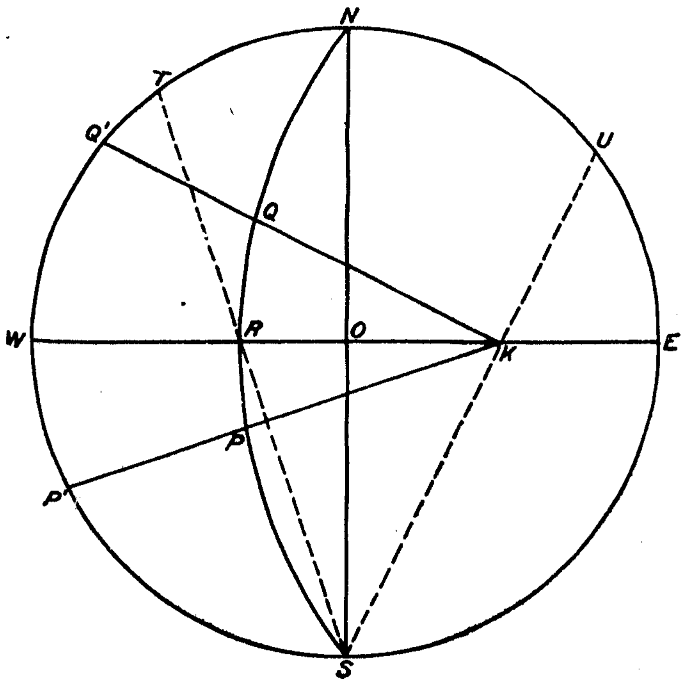

FIG. 22.—Great circle arc between two points on stereographic projection.

In figure 22 construct the projection of the great circle passing through P and Q, the projections of the two given points, by the method of problem 7. Draw NS the diameter determined by the intersections of this great circle projection with the primitive circle and draw the perpendicular diameter WE. This diameter is then the line of measures. Locate the projection of the pole of SRN by drawing SRT and by laying off $TU=\frac{\pi}{2}$, and by then drawing SU, thus locating K, the projection of the pole. Draw KP and KQ and prolong them to intersect the primitive circle in P' and Q', respectively; then $P'WQ'$ is the great circle arc, between the given points of which P and Q are the projections. KP' and KQ' are the projections of circles passing through the point of projection and through the pole of the great circle of which $SPQN$ is the projection. But the point of projection is the pole of the primitive circle; hence the planes that determine the projections KP' and KQ' cut off equal arcs on the great circle, whose projection is $SPQN$ and the primitive circle. Therefore, the arc $P'Q'$ is equal to the arc of which PRQ is the projection.

This problem can be solved, together with that of determining the projection of the great circle passing through the projections of the two given points in the following manner:

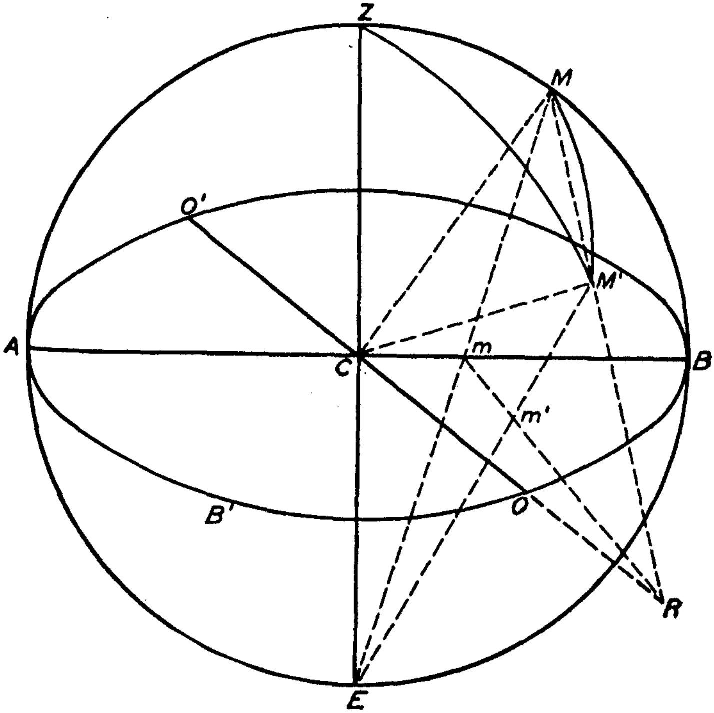

FIG. 23.—Sphere showing intersection of given lines.

In figure 23 let Z be the zenith and C the center of the sphere and let MM' be the arc of a great circle joining the points M and M'. If E is the point of projection, m and m' are evidently the projections of M and M'. Produce the chord MM' until it meets mm' produced in R; then RO is evidently in the plane of the great circle MM', and also in the primitive plane. Therefore, the points O and O' lie on the projection of the great circle and the projection is fully determined, since it is a circle passing through m, m', O, and O'. If MM' is parallel to mm', then evidently OO' is also parallel to each of these lines.

Now, in figure 24 let *NESW* be the primitive circle and let *WE* be the line of measures; also let *m* and *m'* be the projections of the given points. Take $On' = Om'$ and $On = Om$; draw *Sn'* to intersect the primitive circle in *p'* and *Sn* to intersect it in *p*. On *mm'* construct the triangle *Dmm'*, having $mD = Sn$ and $m'D = Sn'$; prolong *Dm'* to *q'*, making $m'q' = n'p'$, and prolong *Dm* to *q*, making $mq = np$. Then *qq'* is the chord distance between the given points, and this chord being laid off anywhere on

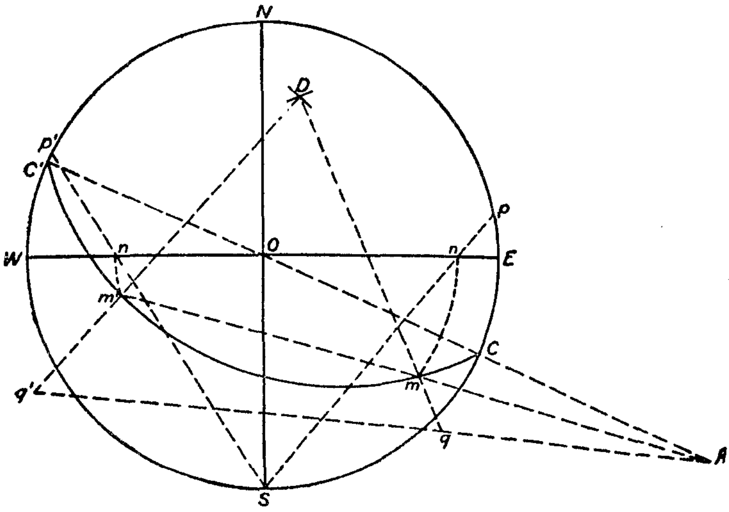

FIG. 24.—Projection of great circle through two points and length of arc between them on stereographic projection.

the primitive circle will give the great-circle-arc distance. The triangle *Dqq'* is evidently the triangle *EMM'* of figure 23 turned on *mm'* as an axis into the plane of the projection or into the primitive plane. Prolong *mm'* and *qq'* until they intersect at *R*, and draw *RO* intersecting the primitive circle in *C* and *C'*. A circle made to pass through *C*, *m*, *m'*, and *C'*, is the required projection of the great circle through the points *M* and *M'* of the sphere.

This same problem can be solved by the method of descriptive geometry in the following way:

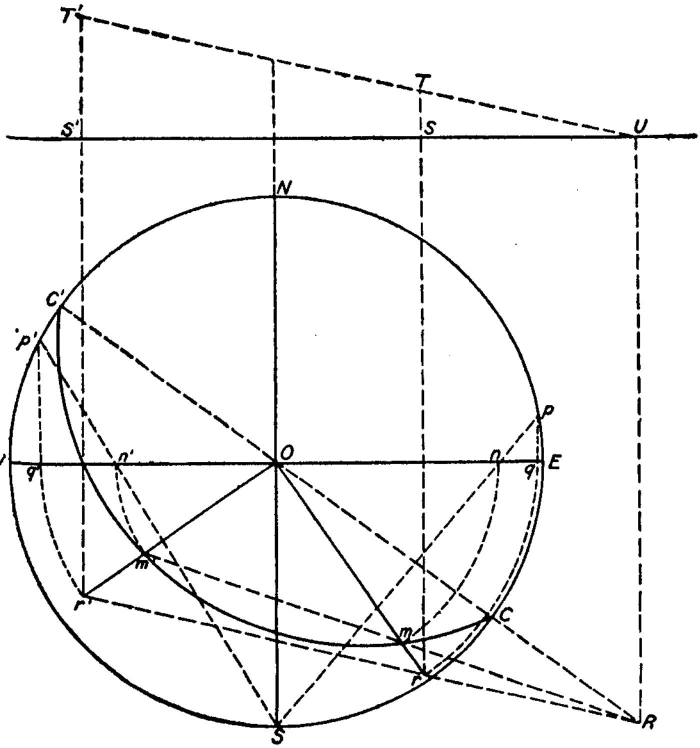

FIG. 25.—Projection of great circle through two points on stereographic projection, second method.

In figure 25 RO is the trace of the great circle plane on the horizontal plane; we need to determine, then, this trace of the plane of M, M' and the center of the sphere. n and n', p and p' are determined as before; from p let fall the perpendicular pq upon WE and from p', the perpendicular $p'q'$; prolong Om to r, making $Or=Oq$, and prolong Om' to r', making $Or'=Oq'$. r and r' are then the orthographic horizontal projections of the given points M and M' on the sphere. Draw $S'U$ parallel to WE; let fall the perpendiculars $r's'$ and rs and prolong them, making $S'T'=p'q'$ and $ST=pq$. T and T' are the orthographic vertical projections of M and M', and TT' is the vertical projection of the line MM' and rr' is the horizontal projection of the same line. Prolong TT' until it intersects the line $S'S$ at U and erect the perpendicular UR intersecting $r'r$ prolonged in R. R is the trace of the line MM' on the horizontal plane, which is here the primitive plane. RO is then the trace of the great circle plane on the horizontal or primitive plane. This determines the points C and C', through which the projection of the great circle must pass. A circle made to pass through the points C, m, m', and C' is the required projection. Note that $m'm$ produced passes through the point R, as it should.

Problem 9.—To lay off on a great circle an arc of given length from a given point P:

Determine the projection of the pole of the given great circle projection. In figure 22 let K be the projection of the pole of the great circle of which the arc $SPRQN$ is the projection; draw KP intersecting the primitive circle in P'. Lay off the given arc $P'Q'$ on the primitive circle and draw KQ' intersecting the projection of the great circle in Q; then PQ is the projection of the required arc.

Problem 10.—The projection of a great circle and that of a point being given, to construct the projection of the great circle passing through the given point and perpendicular to the given great circle:

Determine the projection of the pole of the given great circle and then construct the projection of the great circle passing through this pole and the given point; this is the required projection.

Problem 11.—To construct the projection of a great circle which passes through a given point and which is inclined at a certain angle z to the primitive plane:

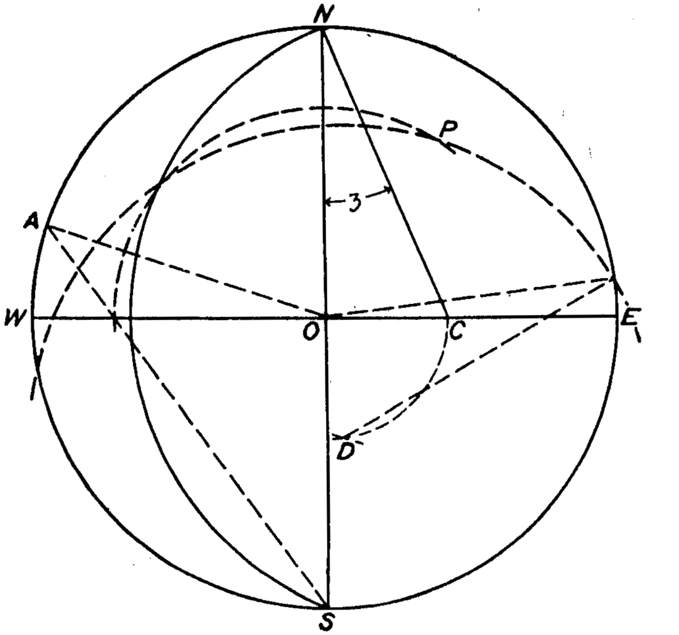

FIG. 26.—Projection of great circle with given inclination to the primitive plane on stereographic projection.

In figure 26 if the given point lies on the primitive circle, as N, draw NS and WE, the line of measures. Construct the angle ONC equal to the given angle z; then C is the center and CN the radius of the required projection. If the projection of the given point is not on the primitive circle, but is at some other point, as P, construct the arc CD with O as a center with OC as a radius. Construct another arc with P as a center and with CN as a radius intersecting the first arc in D; then with D as a center and with DP as a radius construct the required projection. (Remark.—If the given point does not lie on the primitive circle, the construction is not always possible; in fact, the angle z can not be less than the angle WOA.)

Problem 12.—To determine the inclination of two great circles with respect to each other:

This problem is solved by determining the projections of the poles of the given circles, and then by measuring the great-circle-arc distance between them. Apply the method of problem 6 and then that of problem 8. With great circles the inclination of the planes is equal to the angle between the radii of the two circles drawn to the point of intersection, since the inclination is equal to the angle between the given circles. The method of the problem can, however, be applied to any circles, either great or small. Even with small circles we may draw the projections of the parallel great circles and then determine their inclination with respect to each other by the

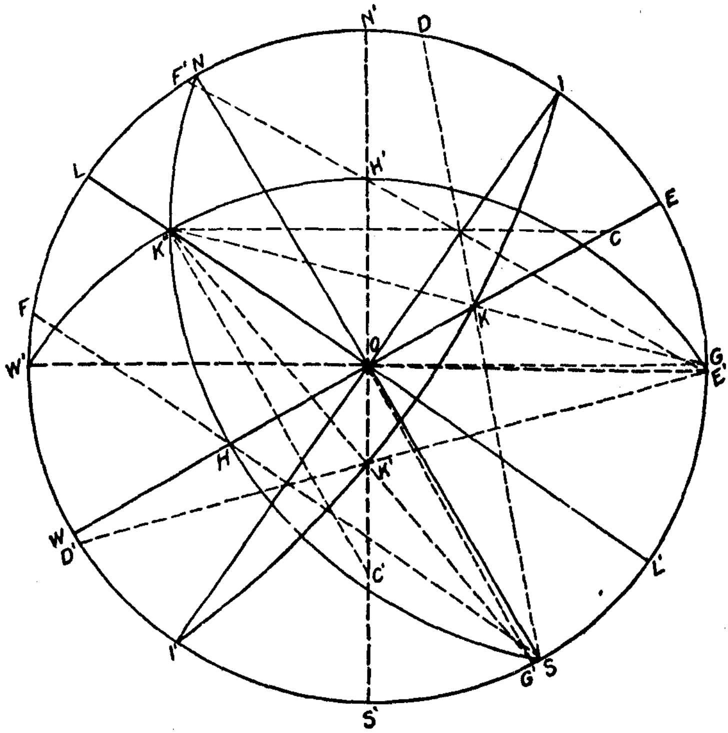

FIG. 27.—Determination of the inclination of the planes of two great circles on stereographic projection.

radii drawn to the point of intersection. In figure 27 let SHN be the projection of a great circle, with C as the center for the arc; also let $E'H'W'$ be the projection of another great circle with C' as the center for the arc. The angle between the arcs is then equal to $CK''C'$, since the angle between the radii is equal to the angle between the tangents, and, the projection being conformal, the angle between the circles is preserved in their representations. Locate the projection of the pole of each of the given great circles; K is the projected pole of the first circle and K' is that of the second circle. A great circle passing through the pole of a given great circle has its plane necessarily perpendicular to that of the given great circle; therefore the great circle which passes through the poles of the two great circles has its plane perpendicular to the plane of each of the given circles. K'' must then be the projection of the pole of this great circle of which $IKK'I'$ is the projected arc. GG' is therefore the great circle arc of which KK' is the projection; or the angle GOG' is the angle that measures the inclination of the planes of the given great circles. The angle GOG' should, therefore, equal the angle $CK''C'$; the impossibility of making a perfect construction may cause some deviation from equality in the constructed figure.

Problem 13.—The projection of a point being given, to construct the meridian and parallel passing through the point:

If the problem is to be determinate, we must have the primitive circle given and the projection of one of the poles.

In figure 28 let $NESW$ be the primitive circle and let P be the projection of the pole; locate the south pole by drawing WP and then WP' perpendicular to WP; RR' is the perpendicular bisector of PP', and is therefore the line of centers for the meridians. Let Q be the projection of the given point; pass a circle through P, Q, and P', and this is the projection of the meridian through the given point. Construct a tangent to PQP' at Q, meeting NS in T; then T is the center of the projection of the parallel and TQ is the radius; this fully determines the projection of the parallel which is the arc QQ'.

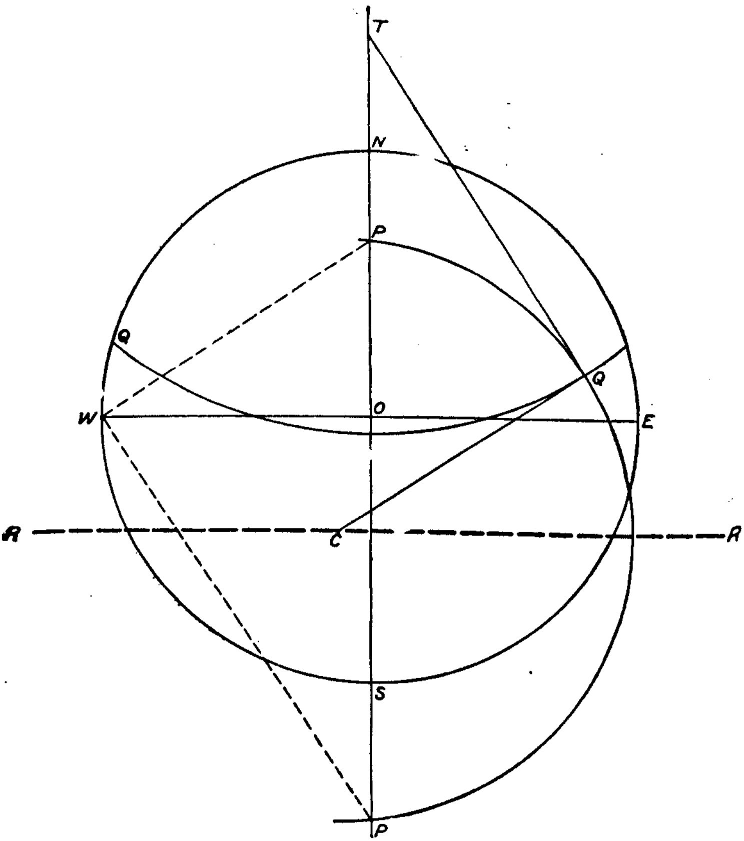

FIG. 28.—Projection of the meridian and parallel through a given point on stereographic projection.

Problem 14.—To construct the projections of the circles parallel to a given circle:

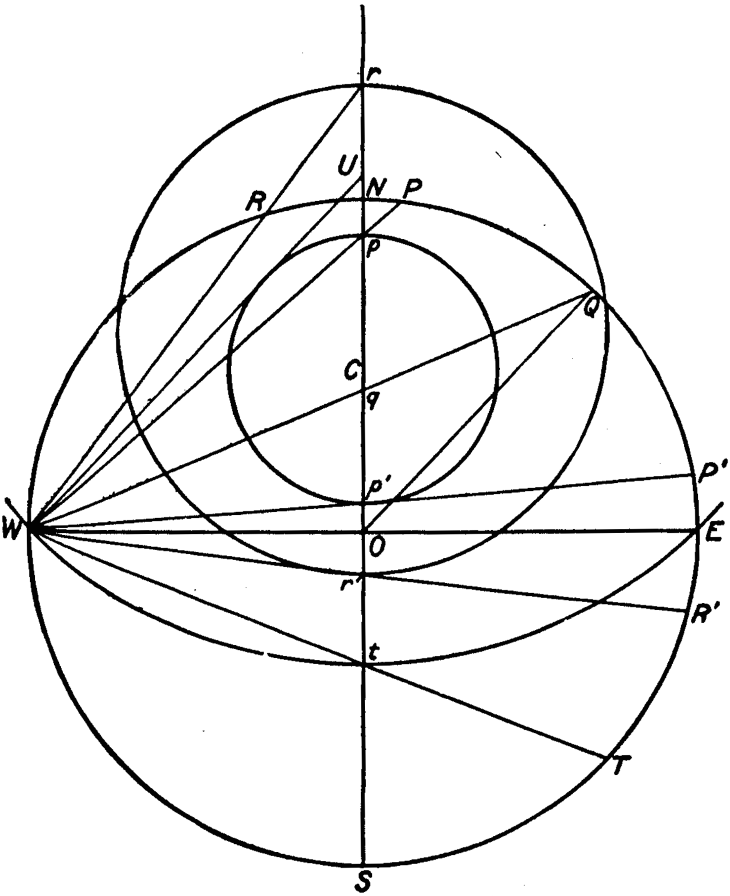

FIG. 29.—Projection of circles parallel to given circle on stereographic projection.

In figure 29 let pp' with center at C be the given circle. Draw NcS and the perpendicular diameter WE; draw $Wp'P'$ and WpP; bisect the arc PP', thus locating Q the pole of the given circle. From Q lay off the polar distance of the required parallel circle. In the figure $QR = QR' = \frac{\pi}{3}$; draw WR and WR', thus locating the extremities of the diameter of the given circle rr'; the center is given by bisecting this line. For the parallel great circle take $QT=\frac{\pi}{2}$; WT locates t and WU parallel to OQ locates U, the center of the required great circle projection.

CONFORMAL POLYCONIC PROJECTIONS.

Since we are to have a conformal projection, it is best to treat the case for a sphere and then to take into account the ellipsoidal shape in the same way that we did in treating the stereographic projections.

In the treatment of the rectangular polyconic projections, we found that

$$\tan\frac{\theta}{2}=\frac{\Gamma(\lambda)^*}{u},$$

and for the sphere that

$$k_m=\frac{1}{a}\left(\frac{ds}{d\varphi}\cos\theta-\frac{d\rho}{d\varphi}\right)$$

$$k_p=\frac{\rho}{a\cos\varphi}\cdot\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\sin\theta;$$

also

$$\frac{1}{\rho}\frac{ds}{d\varphi}=\frac{1}{u}\frac{du}{d\varphi}.$$

If the projection is to be conformal, it must be rectangular, and, in addition, the scale at any given point must be the same along the meridian that it is along the parallel, or $k_m=k_p$.

Hence

$$\left(\frac{ds}{d\varphi}\cos\theta-\frac{d\rho}{d\varphi}\right)=\frac{\rho}{\cos\varphi}\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\sin\theta,$$

or

$$\Gamma'(\lambda)=\frac{\Gamma(\lambda)\cos\varphi}{\rho\sin\theta}\left(\frac{ds}{d\varphi}\cos\theta-\frac{d\rho}{d\varphi}\right).$$

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*See p. 15.

But

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2u\,\Gamma(\lambda)}{u^2+\Gamma^2(\lambda)}$$

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{u^2-\Gamma^2(\lambda)}{u^2+\Gamma^2(\lambda)}.$$

Substituting these values and the value of

$$\frac{ds}{d\varphi} = \frac{\rho}{u}\frac{du}{d\varphi},$$

we obtain

$$\Gamma'(\lambda) = \frac{[u^2+\Gamma^2(\lambda)]\cos\varphi}{2\rho u}\left(\frac{\rho}{u}\frac{du}{d\varphi}\frac{u^2-\Gamma^2(\lambda)}{u^2+\Gamma^2(\lambda)} - \frac{d\rho}{d\varphi}\right)$$

$$= \frac{\cos\varphi}{2u^2}\frac{du}{d\varphi}[u^2-\Gamma^2(\lambda)] - \frac{\cos\varphi}{2\rho u}\frac{d\rho}{d\varphi}[u^2+\Gamma^2(\lambda)]$$

$$= -\Gamma^2(\lambda)\left(\frac{\cos\varphi}{2\rho u}\frac{d\rho}{d\varphi} + \frac{\cos\varphi}{2u^2}\frac{du}{d\varphi}\right) + u^2\left(\frac{\cos\varphi}{2u^2}\frac{du}{d\varphi} - \frac{\cos\varphi}{2\rho u}\frac{d\rho}{d\varphi}\right)$$

$$= -\left(u\frac{d\rho}{d\varphi} + \rho\frac{du}{d\varphi}\right)\frac{\cos\varphi}{2\rho u^2}\left[\Gamma^2(\lambda) + u^2\frac{u\frac{d\rho}{d\varphi} - \rho\frac{du}{d\varphi}}{u\frac{d\rho}{d\varphi} + \rho\frac{du}{d\varphi}}\right].$$

Since $\Gamma(\lambda)$ is independent of φ, $\Gamma'(\lambda)$ is also independent of φ; consequently the two expressions dependent upon φ must reduce to constants. We can set one of them equal to unity, because u can be multiplied by any constant without changing the value of either s or ρ; and if so, $\Gamma(\lambda)$ would be multiplied by the same constant, so that θ would not be changed thereby.

Accordingly let

$$u^2 \frac{u\frac{d\rho}{d\varphi} - \rho\frac{du}{d\varphi}}{u\frac{d\rho}{d\varphi} + \rho\frac{du}{d\varphi}} = 1,$$

or

$$u\frac{d\rho}{d\varphi} - \rho\frac{du}{d\varphi} = \frac{1}{u}\frac{d\rho}{d\varphi} + \frac{\rho}{u^2}\frac{du}{d\varphi}$$

$$\left(u - \frac{1}{u}\right)\frac{d\rho}{d\varphi} = \rho\left(\frac{du}{d\varphi} + \frac{1}{u^2}\frac{du}{d\varphi}\right)$$

$$\left(u - \frac{1}{u}\right)d\rho = \rho d\left(u - \frac{1}{u}\right)$$

$$\frac{d\rho}{\rho} = \frac{d\left(u - \frac{1}{u}\right)}{u - \frac{1}{u}}$$

by integration

$$\log_e \rho = \log_e\left(u - \frac{1}{u}\right) + \log_e \frac{c}{2}$$

in which the constant of integration is taken in the form $\log_e \frac{c}{2}$. It determines the scale of the projection. Passing to exponentials, we obtain

$$\rho = \frac{c}{2}\left(u - \frac{1}{u}\right).$$

But

$$\frac{1}{\rho}\frac{ds}{d\varphi} = \frac{1}{u}\frac{du}{d\varphi}$$

or

$$ds = \frac{\rho}{u}\,du,$$

substituting the value of ρ, we get

$$ds = \frac{c}{2}\left(1 - \frac{1}{u^2}\right)du.$$

Therefore, by integration,

$$s=\frac{c}{2}\left(u+\frac{1}{u}\right),$$

in which the constant of integration may be taken as zero, since the addition of any quantity would only serve to change the point from which s is reckoned.

From these results we obtain

$$s+\rho=cu$$

$$s-\rho=\frac{c}{u}$$

or, by multiplication,

$$s^2-\rho^2=c^2.$$

This equation shows that the circle with the origin as center, constructed with the radius c, cuts all the parallels at right angles. Any circle drawn through the two points of intersection of this circle and the line of centers of the parallels will also cut the parallels orthogonally, for the tangents drawn to it from any point in this line of centers are equal. Therefore, these circles, since they form the orthogonal trajectories of the parallels of the map, are none other than the projections of the meridians. The two common points in the line of centers of the parallels are the poles of the map.

If, then, we take two arbitrary points to represent the two poles, the meridians of the map will be the arcs of circles which pass through these two points and the parallels will be other arcs of circles having their centers at various points of the prolongation of the line of poles and each passing through the point of contact of the tangent drawn from the center to any one of the meridians; for example, to the circumference described upon the line of poles as diameter.

We have yet to find the expressions for u, ρ, and s in terms of φ, and that for $\Gamma(\lambda)$ in terms of λ, by which expressions we may be able to tell, in the first series of arcs, the one that corresponds to a given meridian λ and, in the second series of arcs, the one that corresponds to the parallel of latitude φ.

In the expression for $\Gamma'(\lambda)$ on page 73, if we let $\frac{n}{2}$ represent the second constant, we have

$$\left(u\frac{d\rho}{d\varphi}+\rho\frac{du}{d\varphi}\right)\frac{\cos\varphi}{2\rho u^2}=-\frac{n}{2}$$

or, by substitution in the equation on page 73,

$$\Gamma'(\lambda)=\frac{n}{2}[1+\Gamma^2(\lambda)]$$

$$\frac{\Gamma'(\lambda)\,d\lambda}{1+\Gamma^2(\lambda)}=\frac{n}{2}d\lambda,$$

by integration,

$$\tan^{-1}\Gamma(\lambda)=\frac{n}{2}\lambda+c'$$

or

$$\Gamma(\lambda)=\tan\left(\frac{n}{2}\lambda+c'\right).$$

Hence

$$\tan\frac{\theta}{2}=\frac{1}{u}\tan\left(\frac{n}{2}\lambda+c'\right).$$

Since for $\lambda=0$, we have $\theta=0$; therefore, $c'=0$ and

$$\Gamma(\lambda)=\tan\frac{n}{2}\lambda$$

and

$$\tan\frac{\theta}{2}=\frac{1}{u}\tan\frac{n}{2}\lambda.$$

To determine u, we may write

$$\left(u\frac{d\rho}{d\varphi}+\rho\frac{du}{d\varphi}\right)\frac{\cos\varphi}{2\rho u^2}=-\frac{n}{2}$$

in the form

$$\frac{d(u\rho)}{d\varphi}\cdot\frac{\cos\varphi}{2\rho u^2}=-\frac{n}{2}.$$

But

$$u\rho=\frac{c}{2}\left(u^2-1\right)$$

and

$$\frac{d(u\rho)}{d\varphi}'=cu\frac{du}{d\varphi}.$$

By substituting these values, we obtain

$$\frac{\cos\varphi}{u^2-1}\frac{du}{d\varphi}=-\frac{n}{2}$$

$$\frac{du}{u^2-1}=\frac{-n}{2}\frac{d\varphi}{\cos\varphi}$$

$$\frac{1}{2}\left(\frac{du}{u-1}-\frac{du}{u+1}\right)=-\frac{n}{2}\frac{d\varphi}{\sin\left(\frac{\pi}{2}+\varphi\right)}$$

$$=-\frac{n}{2}\frac{\left[\cos^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+\sin^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right]d\varphi}{2\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}$$

$$\frac{1}{2}\left(\frac{du}{u-1}-\frac{du}{u+1}\right)=-\frac{n}{4}\left[\frac{\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}d\varphi+\frac{\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}d\varphi\right]$$

or

$$\frac{du}{u+1}-\frac{du}{u-1}=n\left[\frac{\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}\frac{d\varphi}{2}+\frac{\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}\frac{d\varphi}{2}\right]$$

By integration

$$\log_e\frac{u+1}{u-1}=n\left[\log_e\sin\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-\log_e\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right]+\log_e k,$$

$\log_e k$ being the constant of integration. Passing to exponentials we obtain

$$\frac{u+1}{u-1}=k\tan^n\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

or

$$u=\frac{k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}{k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}$$

$$s=\frac{c}{2}\left(u+\frac{1}{u}\right)=c\,\frac{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}$$

$$\rho=\frac{c}{2}\left(u-\frac{1}{u}\right)=c\,\frac{2k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}$$

$$\Gamma(\lambda)=\tan\frac{n}{2}\lambda$$

$$\tan\frac{\theta}{2}=\frac{\Gamma(\lambda)}{u}=\frac{k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}{k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}\tan\frac{n}{2}\lambda.$$

The value of s gives the distance of the center for the circle that is to represent the parallel of latitude φ from the intersection of the central meridian with the parallel that is represented by a straight line; ρ is the radius of this parallel; the parallel is therefore fully determined by these two quantities, since the centers of the parallels must lie on the central meridian. In order to construct the meridians, we must determine on the parallel of φ the value of θ, the angle at the center of parallel φ, that corresponds to the meridian of longitude λ; this method of plotting the meridians by coordinates will be unnecessary, however, if we determine the equation of the meridians.

We have

$$x = \rho \sin \theta.$$

$$y = s - \rho \cos \theta.$$

But

$$\tan\frac{\theta}{2}=\frac{\Gamma(\lambda)}{u}$$

or

$$u = \Gamma(\lambda)\cot\frac{\theta}{2} = \tan\frac{n}{2}\lambda\cot\frac{\theta}{2}.$$

Hence

$$\rho = \frac{c}{2}\left(u - \frac{1}{u}\right) = \frac{c}{2}\left(\tan\frac{n}{2}\lambda\cot\frac{\theta}{2} - \cot\frac{n}{2}\lambda\tan\frac{\theta}{2}\right)$$

or

$$\rho = 2c\,\frac{\left(\sin\frac{n}{2}\lambda\cos\frac{\theta}{2}\right)^2 - \left(\cos\frac{n}{2}\lambda\sin\frac{\theta}{2}\right)^2}{\sin n\lambda\sin\theta};$$

also

$$s = 2c\,\frac{\left(\sin\frac{n}{2}\lambda\cos\frac{\theta}{2}\right)^2 + \left(\cos\frac{n}{2}\lambda\sin\frac{\theta}{2}\right)^2}{\sin n\lambda\sin\theta}.$$

$$\rho = 2c\,\frac{\left(\sin\frac{n}{2}\lambda\cos\frac{\theta}{2} - \cos\frac{n}{2}\lambda\sin\frac{\theta}{2}\right)\left(\sin\frac{n}{2}\lambda\cos\frac{\theta}{2} + \cos\frac{n}{2}\lambda\sin\frac{\theta}{2}\right)}{\sin n\lambda\sin\theta}$$

$$= 2c\,\frac{\sin\frac{1}{2}(n\lambda-\theta)\sin\frac{1}{2}(n\lambda+\theta)}{\sin n\lambda\sin\theta} = \frac{c(\cos\theta - \cos n\lambda)}{\sin n\lambda\sin\theta}.$$

$$s - \rho\cos\theta = \frac{2c}{\sin n\lambda\sin\theta}\left[\left(\sin\frac{n}{2}\lambda\cos\frac{\theta}{2}\right)^2 2\sin^2\frac{\theta}{2} + \left(\cos\frac{n}{2}\lambda\sin\frac{\theta}{2}\right)^2 2\cos^2\frac{\theta}{2}\right]$$

$$= \frac{4c\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\left(\sin^2\frac{n}{2}\lambda + \cos^2\frac{n}{2}\lambda\right)}{\sin n\lambda\sin\theta}$$

$$= \frac{c\sin\theta}{\sin n\lambda} = y.$$

$$\rho\sin\theta = \frac{c(\cos\theta - \cos n\lambda)}{\sin n\lambda} = x,$$

or

$$\frac{c\cos\theta}{\sin n\lambda} = x + c\cot n\lambda.$$

Therefore

$$y^2 + (x + c\cot n\lambda)^2 = c^2\operatorname{cosec}^2 n\lambda.$$

Since this equation contains only λ and is independent of φ and θ, it is the equation of the meridians. The meridians are therefore circles with centers upon the X axis (the straight line parallel of the map) lying at the distance $= -c \cot n\lambda$ from the origin and having the radius $= c \operatorname{cosec} n\lambda$.

Since for $x=0$, $y=\pm c$, all of the meridians pass through the two points which are distant $+c$ and $-c$ from the origin; $2c$ is therefore the length of the central meridian included between the poles.

As an aid to construction, we may assume the equation

$$k \tan^n\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right);$$

then

$$s=c \operatorname{cosec} \psi$$

and

$$\rho=c \cot \psi.$$

A special case of this projection is given by the values $k=1$ and $n=1$ · in which case $\psi=\varphi$, and

$$s=c \operatorname{cosec} \varphi$$

$$\rho=c \cot \varphi$$

and the equation of the meridians becomes

$$y^2+(x+c \cot \lambda)^2=c^2 \operatorname{cosec}^2 \lambda.$$

This is evidently the stereographic meridian projection, which has already been discussed under that heading.

DETERMINATION OF THE CONFORMAL PROJECTION IN WHICH THE MERIDIANS AND PARALLELS ARE REPRESENTED BY CIRCULAR ARCS.

This projection is the one devised by Lagrange. His problem was to determine the general conformal projection in which the meridians and parallels were both represented by circular arcs.

Since the projection is to be conformal, we can express it in the form of a function of a complex variable.*

*See The General Theory of the Lambert Conformal Conic Projection, Special Publication No. 53, U. S. Coast and Geodetic Survey.

Let i denote as usual $\sqrt{-1}$ and assume the relations,

$$x-iy=f_1(\sigma+i\lambda)$$
$$x+iy=f_2(\sigma-i\lambda),$$

then f_1 and f_2 are conjugate functions of a complex variable that are only limited to being analytical functions. From these we find at once

$$x=\frac{1}{2}[f_1(\sigma+i\lambda)+f_2(\sigma-i\lambda)]$$

$$y=\frac{i}{2}[f_1(\sigma+i\lambda)-f_2(\sigma-i\lambda)],$$

or, denoting $f_1(\sigma+i\lambda)$ by f_1 and $f_2(\sigma-i\lambda)$ by f_2,

$$x=\frac{1}{2}(f_1+f_2)$$

$$iy=-\frac{1}{2}(f_1-f_2)$$

$$\frac{\partial x}{\partial \sigma}=\frac{1}{2}(f'_1+f'_2)$$

$$\frac{\partial x}{\partial \lambda}=\frac{i}{2}(f'_1-f'_2)$$

$$\frac{\partial y}{\partial \sigma}=+\frac{i}{2}(f'_1-f'_2)$$

$$\frac{\partial y}{\partial \lambda}=-\frac{1}{2}(f'_1+f'_2).$$

From these equations it follows that

$$\frac{\partial x}{\partial \sigma}=-\frac{\partial y}{\partial \lambda} \text{ and } \frac{\partial x}{\partial \lambda}=+\frac{\partial y}{\partial \sigma}.$$

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From these we obtain at once

$$\frac{\partial^2 x}{\partial \sigma^2} = -\frac{\partial^2 y}{\partial \sigma\, \partial \lambda} = -\frac{\partial^2 x}{\partial \lambda^2}$$

$$\frac{\partial^2 y}{\partial \sigma^2} = +\frac{\partial^2 x}{\partial \sigma\, \partial \lambda} = -\frac{\partial^2 y}{\partial \lambda^2}$$

$$W^2 = \left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial y}{\partial \sigma}\right)^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 = \frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \sigma} - \frac{\partial x}{\partial \sigma}\frac{\partial y}{\partial \lambda}$$

$$= \frac{1}{4}[(f'_1 + f'_2)^2 - (f'_1 - f'_2)^2] = f'_1 f'_2.$$

Therefore

$$W = \sqrt{f'_1(\sigma + i\lambda)\, f'_2(\sigma - i\lambda)}.$$

If the coordinates of a plane curve are expressed in terms of an independent variable t in the form

$$x = \varphi(t)$$
$$y = \Psi(t),$$

the expression for the radius of curvature is given in the form

$$\frac{1}{R} = \pm \frac{\frac{dx}{dt}\frac{d^2 y}{dt^2} - \frac{dy}{dt}\frac{d^2 x}{dt^2}}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}.$$

Since in the expressions for x and y in terms of f_1 and f_2, σ is a function of the latitude and λ is merely the longitude, σ is constant along a given parallel and λ is constant along a given meridian; in other words, σ remaining constant, we obtain a parallel by variation of λ, and λ being constant, we get a meridian by variation of σ. Therefore if we neglect the sign

$$\frac{1}{R_m} = \frac{\frac{\partial x}{\partial \sigma}\frac{\partial^2 y}{\partial \sigma^2} - \frac{\partial y}{\partial \sigma}\frac{\partial^2 x}{\partial \sigma^2}}{\left[\left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial y}{\partial \sigma}\right)^2\right]^{3/2}}$$

$$\frac{1}{R_p} = \frac{\frac{\partial x}{\partial \lambda}\frac{\partial^2 y}{\partial \lambda^2} - \frac{\partial y}{\partial \lambda}\frac{\partial^2 x}{\partial \lambda^2}}{\left[\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2\right]^{3/2}}$$

or by substituting the values on page 82

$$\frac{1}{R_m}=\frac{1}{W^3}\left[\frac{\partial x}{\partial \sigma}\frac{\partial^2 x}{\partial \sigma\,\partial \lambda}+\frac{\partial y}{\partial \sigma}\frac{\partial^2 y}{\partial \sigma\,\partial \lambda}\right]=\frac{1}{W^2}\frac{\partial W}{\partial \lambda}$$

$$\frac{1}{R_p}=\frac{1}{W^3}\left[\frac{\partial x}{\partial \lambda}\frac{\partial^2 x}{\partial \lambda\,\partial \sigma}+\frac{\partial y}{\partial \lambda}\frac{\partial^2 y}{\partial \lambda\,\partial \sigma}\right]=\frac{1}{W^2}\frac{\partial W}{\partial \sigma},$$

or, again paying no attention to sign,

$$\frac{1}{R_m}=\frac{\partial}{\partial \lambda}\left(\frac{1}{W}\right)$$

$$\frac{1}{R_p}=\frac{\partial}{\partial \sigma}\left(\frac{1}{W}\right),$$

in which

$$W=\sqrt{f'_1(\sigma+i\lambda)\,f'_2(\sigma-i\lambda)}.$$

If the meridians and parallels are to be circles, R_m must be independent of σ, and R_p must be independent of λ. This fact is analytically expressed by

$$\frac{\partial}{\partial \sigma}\left(\frac{1}{R_m}\right)=0 \quad\text{and}\quad \frac{\partial}{\partial \lambda}\left(\frac{1}{R_p}\right)=0.$$

These two conditions lead to the same condition; that is, to

$$\frac{\partial^2}{\partial \sigma\,\partial \lambda}\left(\frac{1}{W}\right)=0.$$

From this it follows that, if the projection is conformal, the condition that one system of curves forming the net is to be made up of circles, makes it necessary that the other set should also be circular arcs; this includes, of course, straight lines as special cases of circles with infinite radii and with centers at infinity.

If, in order to simplify the analysis, we set

$$\frac{1}{\sqrt{f'_1(\sigma+i\lambda)}}=g_1(\sigma+i\lambda)$$

$$\frac{1}{\sqrt{f'_2(\sigma-i\lambda)}}=g_2(\sigma-i\lambda),$$

then

$$\frac{1}{W}=g_1(\sigma+i\lambda)\ g_2(\sigma-i\lambda)$$

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{W}\right)=g'_1(\sigma+i\lambda)\ g_2(\sigma-i\lambda)+g_1(\sigma+i\lambda)\ g'_2(\sigma-i\lambda)$$

$$\frac{\partial^2}{\partial\sigma\,\partial\lambda}\left(\frac{1}{W}\right)=ig_1''(\sigma+i\lambda)\ g_2(\sigma-i\lambda)-ig_1(\sigma+i\lambda)\ g_2''(\sigma-i\lambda)$$

so that from the required condition we have

$$\frac{g_1''(\sigma+i\lambda)}{g_1(\sigma+i\lambda)}=\frac{g_2''(\sigma-i\lambda)}{g_2(\sigma-i\lambda)}$$

The two members of this equation are conjugate complex functions, and the equality can only exist on condition that the members are each equal to a real constant. Let us use β^2 for this constant and, for the sake of abbreviation, let us denote the variable $\sigma+i\lambda$ by z and $g_1(z)$ by Z. The differential equation then becomes

$$\frac{d^2Z}{dz^2}=\beta^2 Z.$$

Multiply both members by $2\frac{dZ}{dz}$ and we have

$$\frac{2dZ}{dz}\frac{d^2Z}{dz^2}=2\beta^2 Z\frac{dZ}{dz}.$$

By integration,

$$\left(\frac{dZ}{dz}\right)^2=\beta^2Z^2-\gamma^2,$$

$-\gamma^2$ being the constant of integration.

$$\frac{dZ}{\sqrt{\beta^2Z^2-\gamma^2}}=dz$$

or

$$\frac{\beta dZ}{\sqrt{\beta^2Z^2-\gamma^2}}=\beta dz.$$

Integrating again, we obtain

$$\log_e(\beta Z + \sqrt{\beta^2 Z^2 - \gamma^2}) = \beta z + \delta$$

or

$$\beta Z + \sqrt{\beta^2 Z^2 - \gamma^2} = e^{\beta z + \delta}.$$

Taking reciprocals we get

$$\beta Z - \sqrt{\beta^2 Z^2 - \gamma^2} = \gamma^2 e^{-\beta z - \delta}.$$

By addition, we obtain

$$Z = \frac{e^{\delta}}{2\beta} e^{\beta z} + \frac{\gamma^2 e^{-\delta}}{2\beta} e^{-\beta z}.$$

Now, for abbreviation let

$$\frac{e^{\delta}}{2\beta} = A_1 \text{ and } \frac{\gamma^2 e^{-\delta}}{2\beta} = B_1$$

and we have

$$Z = A_1 e^{\beta z} + B_1 e^{-\beta z}$$

or

$$g_1(\sigma + i\lambda) = A_1 e^{\beta(\sigma + i\lambda)} + B_1 e^{-\beta(\sigma + i\lambda)}.$$

But

$$f'_1(\sigma + i\lambda) = \frac{1}{g_1^2(\sigma + i\lambda)}.$$

Hence

$$\frac{d[f_1(z)]}{dz} = \frac{1}{(A_1 e^{\beta z} + B_1 e^{-\beta z})^2}$$

$$= \frac{e^{2\beta z}}{(A_1 e^{2\beta z} + B_1)^2}$$

$$d[f_1(z)] = \frac{1}{2A_1\beta} \frac{d(A_1 e^{2\beta z} + B_1)}{(A_1 e^{2\beta z} + B_1)^2}.$$

By integration

$$f_1(z) = -\frac{1}{2A_1\beta} \frac{1}{A_1 e^{2\beta z} + B_1} + C.$$

If we set $-2A_1^2\beta = M$ and $-2A_1B_1\beta = N$ and restore the value of z, we obtain

$$f_1(\sigma + i\lambda) = C + \frac{1}{Me^{2\beta(\sigma + i\lambda)} + N}.$$

Since $f_1(\sigma+i\lambda)$ is equal to $x-iy$, the constant C tends only to translate the origin. Let us suppose that C is a complex quantity in the form of $a+ib$. If we transpose C to the left-hand member, we have

$$x-a-i(y+b)=\frac{1}{Me^{2\beta(\sigma+i\lambda)}+N}.$$

a and b may be either positive or negative and either or both may be zero. No generality is lost if we set them both equal to zero, since they may be accounted for by a mere translation of axes.

Now, let $M=-Ai$ and $N=-Bi$ and we get

$$x-iy=\frac{ie^{-\beta(\sigma+i\lambda)}}{Ae^{\beta(\sigma+i\lambda)}+Be^{-\beta(\sigma+i\lambda)}}.$$

By multiplying both terms of the fraction by $Ae^{\beta(\sigma-i\lambda)}+Be^{-\beta(\sigma-i\lambda)}$, we get

$$x-iy=\frac{iAe^{-2i\beta\lambda}+iBe^{-2\beta\sigma}}{A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma}}$$

$$=\frac{A\sin 2\beta\lambda+i\,(A\cos 2\beta\lambda+Be^{-2\beta\sigma})}{A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma}}.$$

By equating the real parts and the imaginary parts, we obtain

$$x=\frac{A\sin 2\beta\lambda}{A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma}}$$

$$y=-\frac{A\cos 2\beta\lambda+Be^{-2\beta\sigma}}{A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma}}.$$

On the sphere

$$\sigma=log_e\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

and on the ellipsoid

$$\sigma=log_e\left[\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\left(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi}\right)^{\epsilon/2}\right].$$

That the meridians and parallels are both circles, we already know, since the function f_1 was determined on this condition; but in order to obtain their equations, we must proceed in the usual way. If we eliminate σ, we shall have the equation of the λ meridian and, by the elimination of λ, we may obtain the equation of the parallel of latitude φ.

$$x^2+y^2=\frac{A^2+2ABe^{-2\beta\sigma}\cos 2\beta\lambda+B^2e^{-4\beta\sigma}}{(A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma})^2}$$

$$=\frac{e^{-2\beta\sigma}}{A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma}}.$$

Therefore

$$\frac{y}{x^2+y^2}=-(Ae^{2\beta\sigma}\cos 2\beta\lambda+B)$$

$$\frac{x}{x^2+y^2}=Ae^{2\beta\sigma}\sin 2\beta\lambda.$$

From these, by the elimination of σ, we obtain

$$\frac{y+B\,(x^2+y^2)}{x}=-\cot 2\beta\lambda$$

or

$$x^2+y^2+\frac{1}{B}y+\frac{1}{B}x\cot 2\beta\lambda=0.$$

$$\left(x+\frac{\cot 2\beta\lambda}{2B}\right)^2+\left(y+\frac{1}{2B}\right)^2=\frac{1}{4B^2\sin^2 2\beta\lambda}.$$

This is a circle, the center being at the point

$$x_0=-\frac{\cot 2\beta\lambda}{2B}$$

$$y_0=-\frac{1}{2B}$$

and its radius being

$$\rho_0=\frac{1}{2B\sin 2\beta\lambda}.$$

This equation is identically satisfied by the values $x=0$, $y=0$, and by $x=0$, $y=-\frac{1}{B}$; since all meridians pass through these points, they represent the two poles; the Y axis is the central meridian.

If we eliminate λ, we get

$$\left(\frac{y}{x^2+y^2}+B\right)^2+\frac{y^2}{(x^2+y^2)^2}=A^2e^{4\beta\sigma}.$$

Developing and arranging, we get

$$x^2+y^2+2B\ (x^2+y^2)y+B^2\ (x^2+y^2)^2=A^2e^{4\beta\sigma}\ (x^2+y^2)^2.$$

Dividing by x^2+y^2, since this can only vanish for $x=0$, $y=0$, we get $(A^2e^{4\beta\sigma}-B^2)\ (x^2+y^2)-2By=1$
or

$$x^2+y^2-\frac{2By}{A^2e^{4\beta\sigma}-B^2}=\frac{1}{A^2e^{4\beta\sigma}-B^2}$$

or

$$x^2+\left(y-\frac{B}{A^2e^{4\beta\sigma}-B^2}\right)^2=\frac{A^2e^{4\beta\sigma}}{(A^2e^{4\beta\sigma}-B^2)^2}$$

This is a circle with center at the point

$$x_0=0,\ y_0=\frac{B}{A^2e^{4\beta\sigma}-B^2}$$

and with radius

$$\rho_0=\frac{Ae^{2\beta\sigma}}{A^2e^{4\beta\sigma}-B^2}$$

Since we know that the projection is conformal, it is known that the magnification is the same at any point in all directions. We can determine its value along a parallel and in that way determine its value in all directions.

$$\frac{\partial x}{\partial\lambda}=\frac{2A\beta\cos 2\beta\lambda\ (A^2e^{2\beta\sigma}+B^2e^{-2\beta\sigma})+4A^2B\beta}{(A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma})^2}$$

$$\frac{\partial y}{\partial\lambda}=\frac{2A\beta\sin 2\beta\lambda\ (A^2e^{2\beta\sigma}+B^2e^{-2\beta\sigma})-4AB^2\beta e^{-2\beta\sigma}\sin 2\beta\lambda}{(A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma})^2}$$

$$\left(\frac{dS}{d\lambda}\right)^2=\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2=\frac{4A^2\beta^2}{(A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma})^2}.$$

But on the earth

$$\left(\frac{dS_1}{d\lambda}\right)^2=\frac{a^2\cos^2\varphi}{1-\epsilon^2\sin^2\varphi},$$

from which it follows that

$$k=\frac{dS_1}{dS}=\frac{2A\beta\sqrt{1-\epsilon^2\sin^2\varphi}}{a\cos\varphi\,(A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma})}.$$

In order to derive the equations in their usual form, we shall move the origin down to the point $-\frac{1}{2B}$. The value of x will remain the same, but the new value of y will equal the old value of y increased by $\frac{1}{2B}$ or $y'=y+\frac{1}{2B}$. The equations are, therefore,

$$x=\frac{A\sin 2\beta\lambda}{A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma}}$$

$$y=\frac{A^2e^{2\beta\sigma}-B^2e^{-2\beta\sigma}}{2B\,(A^2e^{2\beta\sigma}+2AB\cos 2\beta\lambda+B^2e^{-2\beta\sigma})}.$$

The equation of the meridians now becomes

$$\left(x+\frac{\cot 2\beta\lambda}{2B}\right)^2+y^2=\frac{1}{4B^2\sin^2 2\beta\lambda}$$

and that of the parallels

$$x^2+\left[y-\frac{A^2e^{4\beta\sigma}+B^2}{2B(A^2e^{4\beta\sigma}-B^2)}\right]^2=\frac{A^2e^{4\beta\sigma}}{(A^2e^{4\beta\sigma}-B^2)^2}.$$

To identify this projection with the one formerly obtained, let

$$\frac{1}{2B}=c,\ 2\beta=n,\ \text{and}\ \frac{A}{B}=k.$$

Then

$$x=\frac{2ck\sin n\lambda}{k^2e^{n\sigma}+2k\cos n\lambda+e^{-n\sigma}}$$

$$y=\frac{c(k^2e^{n\sigma}-e^{-n\sigma})}{k^2e^{n\sigma}+2k\cos n\lambda+e^{-n\sigma}}$$

$$(x+c\cot n\lambda)^2+y^2=c^2\operatorname{cosec}^2 n\lambda$$

$$x^2+\left[y-\frac{c(k^2e^{2n\sigma}+1)}{k^2e^{2n\sigma}-1}\right]^2=\frac{4c^2k^2e^{2n\sigma}}{(k^2e^{2n\sigma}-1)^2}.$$

But for the sphere

$$e^{n\sigma}=\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

or for the spheroid

$$e^{n\sigma}=\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\left(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi}\right)^{\frac{n\epsilon}{2}}.$$

Therefore, for the sphere

$$x=\frac{2ck\sin n\lambda\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+2k\cos n\lambda\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}$$

$$y=\frac{c\left[k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1\right]}{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+2k\cos n\lambda\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}$$

$$x^{2}+\left\{y-\frac{c\left[k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1\right]}{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}\right\}^{2}=\frac{4c^{2}k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{\left[k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1\right]^{2}}$$

We thus see that

$$s=c\frac{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}$$

$$\rho=\frac{2ck\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{k^{2}\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}.$$

If we denote that intersection which lies nearest the origin by y_0 (that is to say the y value for $\lambda=0$), we have

$$\tan\frac{\theta}{2}=\frac{y-y_{0}}{x}=\frac{y-s+\rho}{x}.$$

By performing the indicated operations, we obtain

$$\tan\frac{\theta}{2}=\frac{k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}{k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)+1}\tan\frac{n\lambda}{2}.$$

The projection is thus found to be identical with the one previously obtained by a different procedure.

With these values the magnification (denoted by k' for distinction) for the ellipsoid becomes

$$k'=\frac{2ckn\sqrt{1-\epsilon^{2}\sin^{2}\varphi}}{a\cos\varphi\,(k^{2}e^{n\sigma}+2k\cos n\lambda+e^{-n\sigma})},$$

in which

$$e^{n\sigma}=\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\left(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi}\right)^{\frac{n\epsilon}{2}}.$$

If the parallel, the latitude of which is $-\alpha$, is to be represented by the circle of infinite radius or by the straight line, among the circles of parallels, which forms the perpendicular bisector of the line joining the poles of the projection, then the radius of this parallel and the distance of its center from the origin must become infinite. This will be the case if

$$\frac{c}{k^{2}\tan^{2n}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-1}=\infty;$$

hence

$$k^{2}\tan^{2n}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-1=0$$

or

$$k=\cot^{n}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\tan^{n}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

If, for the sake of abbreviation, we set

$$k\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\tan^{n}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=m,$$

the expression for the center of the parallel becomes

$x_0=0$, $y_0=\dfrac{c(m^2+1)}{m^2-1}$, and the radius becomes $\rho_0=\dfrac{2cm}{m^2-1}$.

The equation for the parallel becomes

$$x^2+\left[y-\frac{c(m^2-1)}{m^2-1}\right]^2=\frac{4c^2\,m^2}{(m^2-1)^2}.$$

The equation of the meridians remains as before

$$(x+c\cot n\lambda)^2+y^2=c^2\operatorname{cosec}^2 n\lambda.$$

The coordinates expressed in terms of m become

$$x=\frac{2cm\sin n\lambda}{1+2m\cos n\lambda+m^2},$$

$$y=\frac{c\,(m^2-1)}{1+2m\cos n\lambda+m^2},$$

and the magnification for the sphere becomes

$$k=\frac{2cmn}{a\cos\varphi\,(1+2m\cos n\lambda+m^2)},$$

and for the spheroid

$$k'=\frac{2cmn\sqrt{1-\epsilon^2\sin^2\varphi}}{a\cos\varphi\,(1+2m\cos n\lambda+m^2)}$$

with the value for m in the last form

$$m=k\tan^n\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\left(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi}\right)^{\frac{n\epsilon}{2}}.$$

Since both φ and α must be less than $\frac{\pi}{2}$, if φ is greater than $-\alpha$, then

$$\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)>\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

or

$$\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)>1$$

and

$$m>1.$$

In a similar way it may be shown that when $\varphi<-\alpha$, then $m<1$.

The parallel circles whose latitudes are greater than $-\alpha$ lie on the positive side of y; those with latitudes less than $-\alpha$ lie on the negative side.

In the expressions for the projection to which we have arrived, c, α, and n are constants that we can determine to fit such conditions as we may require the projection to fulfill, these being limited, of course, to the conditions that are possible in a conformal map.

c determines the scale of the projection and it may be any real constant, so that it only remains to determine α and n. If $\alpha=0$, then the straight line parallel represents the equator and m becomes

$$m=\tan^{n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right),$$

so that $k=1$.

SPECIAL CASES OF THE PROJECTION.

If n converges to zero, and at the same time c converges to ∞ in such a way that $cn=2a$, we obtain a projection in which the parallels are represented by straight lines perpendicular to the Y axis since their centers lie at infinity on the Y axis. In the same way the meridians have infinite radii with centers at infinity on the X axis; consequently they are perpendicular to this axis.

To determine the values we have

$$x=\lim_{\substack{n\doteq 0\\ cn\doteq 2a\\ m\doteq 1}}\left[\frac{2cm\sin n\lambda}{1+2m\cos n\lambda+m^2}\right]$$

$$x=\lim_{\substack{n\doteq 0\\ cn\doteq 2a\\ m\doteq 1}}\left[\frac{2m\,cn\left(\lambda-\frac{n^2\lambda^3}{6}+\ldots\right)}{1+2m\cos n\lambda+m^2}\right]$$

The limiting value of this is seen to be

$$x = a\lambda.$$

$$y = \lim_{\substack{n \doteq 0 \\ cn \doteq 2a \\ m \doteq 1}} \left[\frac{c(m^2-1)}{1+2m\cos n\lambda + m^2}\right]$$

$$= \lim_{\substack{n \doteq 0 \\ cn \doteq 2a \\ m \doteq 1}} \frac{[c(m^2-1)]}{4}$$

$$= \frac{1}{4} \lim_{\substack{n \doteq 0 \\ cn \doteq 2a}} \left[\frac{cn\left\{\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1\right\}}{n}\right]$$

$$= \frac{a}{2} \lim_{n \doteq 0} \left[\frac{\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)-1}{n}\right]$$

$$= \frac{a}{2} \lim_{n \doteq 0} \left[\frac{2\tan^{2n}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\log_e \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{1}\right]^*$$

The value of this expression at the limit is

$$y = a \log_e \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right).$$

We have thus arrived at the Mercator projection as a special case of Lagrange's projection. Although it is not a polyconic projection in the accepted sense, yet it appears as a special case of one of the important projections of the polyconic class. Lambert s conformal conic projection can also be obtained as a special case by letting B become equal to zero in the equations containing the A and B constants.

* $\left[\text{Since } \frac{d}{dx} a^x = a^x \log_e a\right]$

If n becomes equal to unity, we obtain the stereographic projection and the equations take the form

$$x=\frac{2cm \sin \lambda}{1+2m \cos \lambda+m^2}$$

$$y=\frac{c(m^2-1)}{1+2m \cos \lambda+m^2}$$

with $m=\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$

Substituting this value of m and reducing, we obtain

$$x=\frac{c \cos \alpha \sin \lambda \cos \varphi}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}$$

$$y=\frac{c\,(\sin \alpha+\sin \varphi)}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}.$$

If we now let $y'=y-\sin \alpha$, which merely moves the origin and does not change the nature of the projection, we obtain after dropping the primes

$$x=\frac{c \cos \alpha \sin \lambda \cos \varphi}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}$$

$$y=\frac{c \cos \alpha(\cos \alpha \cos \varphi-\sin \alpha \cos \lambda \cos \varphi)}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}.$$

Now by replacing $c \cos \alpha$ by a, we arrive at the values previously obtained

$$x=\frac{a \sin \lambda \cos \varphi}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}$$

$$y=\frac{a(\cos \alpha \cos \varphi-\sin \alpha \cos \lambda \cos \varphi)}{1+\sin \alpha \sin \varphi+\cos \alpha \cos \lambda \cos \varphi}.$$

GENERAL STUDY OF DOUBLE CIRCULAR PROJECTIONS.

In order to enter upon some points not yet discussed, we shall study in general those projections in which the meridians are represented by a system of circles passing through two common points which form the poles of the projection and in which the parallels are represented by a system of curves orthogonal to the meridians. The centers of the circles forming the meridians will all lie upon the perpendicular bisector of the common chord which forms the line joining the poles of the projection. The tangents drawn to the various circumferences from any point of the prolongation of the common chord are equal, since they are in each case a mean proportional between the same secant and the external segment of the same. If from this point as center, with a radius equal to one of these tangents, we describe a circle, it will intersect all the circular arcs representing the meridians at right angles. We thus see that the orthogonal trajectories of the meridians of the map—that is, the parallels—are also circumferences, so that they belong to the polyconic projections. The locus of centers of the parallels is a straight line passing through the projections of the two poles and perpendicular to the locus of centers of the meridians.

Every point of either prolongation of the line ot poles of the map can be considered as the center of the projection of one of the parallels, and the radius of this projection is then equal to the tangent drawn through the point in question to one of the meridians of the map; for example, to the circumference described upon the line of poles as diameter. Reciprocally, if in a projection with orthogonal curves the parallels are circumferences having their centers upon the prolongations of one of the diameters of a given circumference and as radii the tangents drawn from the various centers to this circumference, the meridians will also be circumferences which pass through the two extremities of the given diameter. This will not be true if the radii of the parallels are determined by any other condition than the one mentioned. The rectangular polyconic projection of the English War Office, already discussed, furnishes an example of an othogonal projection in which the parallels, but not the meridians, are circumferences.

The properties which we have just pointed out are not the only ones which we can extend from the stereographic projection to all conformal projections with circular meridians and from these to projections with circular meridians and orthogonal parallels. In figure 30 let P and P' be the projections of the poles, O the middle point of the line PP', $APA'P'$ the circumference described upon PP' as a diameter, AA' the diameter perpendicular to PP'; in addition, let S be the center of the projection of any parallel, U and U', D and D', F and F' the points where this projection intersects, respectively, the circumference

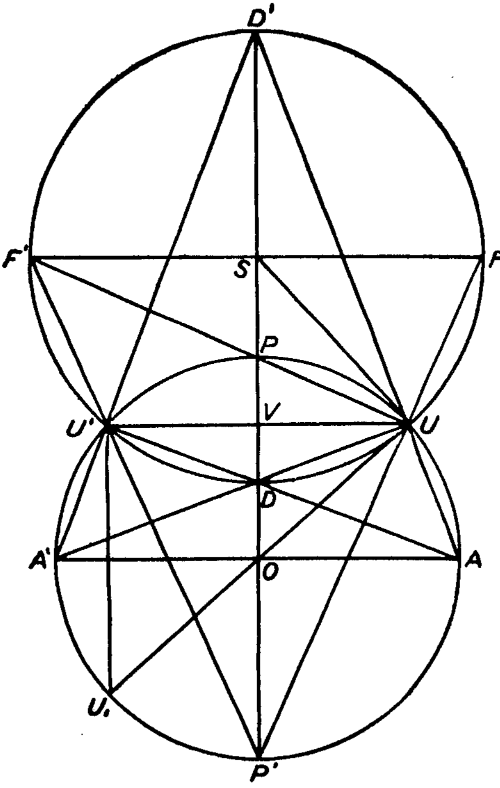

FIG. 30.—Geometrical relations between orthogonal circular meridians and parallels, first figure.

$APA'P'$, the line PP', and the perpendicular erected at S upon this line; finally, let V be the intersection of PP' with UU', and let U_1 be symmetrical to U with respect to O, so that $U'U_1$ is parallel to PP'

The point D being the bisector of the arc UDU', UD will bisect the angle formed by the chord UU' and the tangent OU; the point A' being the bisector of the arc

$U'A'U_1$, UA' also bisects the angle $U'UU_1$; therefore, the three points U, D, A' lie on a straight line which makes it possible to construct the point D without describing the circumference S when U is given. Since the angles AUA', DUD', each being inscribed in a semicircle, are right angles, the three points A, U, D' also lie on a straight line, which is the bisector of the angle formed by one of the sides of the triangle $U'UU_1$ with the prolongation of the other.

The angle PUA', which subtends, upon the circumference O, an arc equal to a quarter of the circumference, is equal to the half of a right angle; the same is true of the angle DUF'', which subtends upon the circumference S an arc equal to a quadrant; the two angles are, therefore, equal, and, as two of their sides UA' and UD coincide, the two others, UP and UF'', also coincide; that is to say, that the points U, P, F'' are in a straight line. Since UP' is perpendicular to UP and UF to UF'', the points P', U, F are also in a straight line. It follows from this that UD is the bisector of the right angle PUP' and UD' of the adjacent angle PUF; therefore, $DP : DP' = D'P : D'P' = UP : UP'$. The projection of each parallel is the locus of the points the distances of which to the projections of the two poles have a given fixed ratio. The lines UP and UP' are in their turn bisectors of the right angles DUD' and DUA; therefore, the ratio of the distances of any point of the circumference O to the two points D and D' is constant.

In figure 31 the letters already appearing in figure 30 are employed with the same signification. The semicircumference PAP' is the projection of a particular meridian. Let us now consider the projection $PMGP'$ of any meridian. Let T be the center, G and M its intersections with AA' and the circumference S, respectively, and, finally, let G' and M' be the points of intersection of the arc which completes the circumference T with the same two lines, respectively. With regard to the two circumferences S and T, we should have to point out the same properties that were pointed out as obtaining between the two circumferences S and O. It will be sufficient to indicate the following facts: Since M lies on the parallel circle which is the locus of points with distances from P and P' in the ratio DP to DP', the ratio of MP to MP' is the same as that of DP to DP'; therefore, the line MD is the bisector of the angle PMP', and it should pass through the mid-point G' of the arc $PG'P'$; then the three points M, D, G' are in a straight line; the same is true of the three points D', M, G, as also of G, D, M' and of G', M', D'. The three points D', G, G' are thus the vertices of a triangle the altitudes of which intersect in D and the feet of these perpendiculars are at O, M', and M.

Let us construct the angle POI equal to that which the meridian PMP' makes with the straight line meridian PP'; the three points P', G, I will be in a straight line,

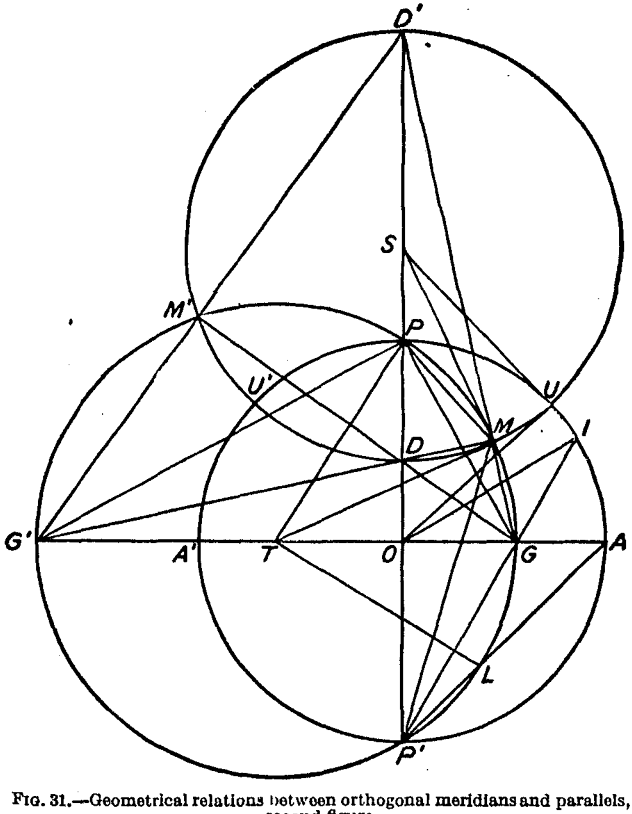

FIG. 31.—Geometrical relations between orthogonal meridians and parallels, second figure.

because the angle $OP'G$ which subtends the arc PMG upon the circumference T is equal to half the angle formed by the chord PP' with the tangent at P'; that is, to half the angle POI; hence upon the circumference O it ought to subtend an arc equal to PI; that is to say, that the prolongation of $P'G$ ought to pass through I. We have, then, to determine directly the point G, a process analogous to that which may be made use of in the stereographic projection upon a meridian.

Let us construct TL perpendicular to TP and intersecting in L the projection PMP' of the meridian; the three points P', L, A are in a straight line, for the angle $PP'L$, which has its vertex upon the circumference T and intercepts the same arc as the angle at the center PTL, is equal to half this angle or to half a right angle; therefore, the prolongation of $P'L$ ought to pass through the point A.

The radius OP or OA of the circumference described upon the line of poles as diameter being taken as unity, we define the modified latitude of a parallel as the arc AU of this circumference comprised between the straight line parallel AA' of the map and the projection UDU' of the parallel in question. This arc which we denote by φ' is also the half of the angle at which, from the center of the projection of the parallel, one would see the circumference described upon the line of poles as diameter; this arc varies with φ from 0 to $\frac{\pi}{2}$ and from 0 to $-\frac{\pi}{2}$. For the abbreviation of the formulas we shall often use in them in place of the arc that has just been defined the modified colatitude p', which is the complement of φ' and which represents the arc PU comprised between the projection of the pole and that of the parallel; p' can then vary from 0 to π with the colatitude p.

Every circumference described from a point S of the prolongation of PP' as center, with the tangent SU for radius, is, in any system of projection with orthogonal intersections and with circular meridians, the projection of a parallel; that which varies from one system to another is the position of this parallel upon the globe, or, inversely, it is the expression of φ' or of p' as a function of φ or p, respectively. Whatever this expression may be, if we call r the radius SD or SU or SM of the projection of the parallel and s the distance OS from its center to the center of the map, we shall have from the right angled-triangle OSU

$$r = \cot \varphi'$$

$$s = \operatorname{cosec} \varphi'$$

$$s^2 - r^2 = 1.$$

Since the three points A, D, U' are in a straight line, the angle at A of the triangle OAD is equal to $\frac{\varphi'}{2}$, and it results, in this triangle and the triangle OAD', that $OD = \tan\frac{\varphi'}{2}$, and $OD' = \cot\frac{\varphi'}{2}$. We thus have $OD \times OD' = 1$, as it ought to be, since the tangent OU is the mean proportional between OD and OD'.

The constant ratio of the distances of any point of the projection of a parallel to the projections P and P' of the two poles will be

$$\frac{UP}{UP'} = \tan PP'U = \tan\frac{p'}{2}.$$

Let us now consider the meridians. The longitude will be reckoned as starting from that meridian the projection of which is the straight line PP', and we shall define the modified longitude of a meridian the angle at which its projection intersects the projection of the central meridian, an angle which we shall denote by λ'; this angle is also half the angle at which, from the center of the projection of the meridian, we should see the line of poles of the map. Therefore, for the meridian projected into PGP', λ' will be the angle which PP' makes with the tangent at P to the arc PGP', or, what amounts to the same thing, to the angle OTP. The projection can vary without the arc PGP' ceasing to be the projection of a meridian; that which will vary will be the position of this meridian upon the earth or, inversely, the expression of λ' as a function of λ. Whatever this expression may be, if we call R the radius TG or TP or TM of the projection of the meridian, and S the distance OT of its center from the center of the map, the right-angled triangle OTP will give

$$R = \operatorname{cosec}\lambda'$$

$$S = \cot\lambda'$$

$$R^2 - S^2 = 1,$$

and the triangles OPG and OPG' will give

$$OG = \tan\frac{\lambda'}{2},\ OG' = \cot\frac{\lambda'}{2}.$$

We thus have $OG \times OG' = 1$, which ought to be so, since OP is a mean proportional between OG and OG'.

The coordinates φ' and λ' or p' and λ' determine the position of any point of the map; however, we shall make use also of a third variable depending upon the first two. This will be the angle OSM formed by the radius SM of the projection of the parallel with the straight line meridian or, what amounts to the same thing, the angle OTM formed by the radius TM of the projection of the meridian with the straight line parallel. We denote this angle by θ; it is the angle at which one would see, either from the center of the projection of a parallel or from the center of the projection of the meridian, the distance of any point M to the center of the map.

Half of θ is equal to the inscribed angle $OG'M$, which subtends upon the circumference T the same arc as the angle at the center OTM, or to the angle $OG'D$, since the three points G', D, M are in a straight line; but the tangent of this angle is given by the ratio of OD to OG'. We have, then,

$$\tan \frac{\theta}{2} = \tan \frac{\lambda'}{2} \tan \frac{\varphi'}{2}.$$

From this equation we deduce

$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{\sin \lambda' \sin \varphi'}{1 + \cos \lambda' \cos \varphi'}$$

$$\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{\cos \lambda' + \cos \varphi'}{1 + \cos \lambda' \cos \varphi'}.$$

The coordinates of M with respect to the axes OA and OP are

$$x = r \sin \theta = \frac{\sin \lambda' \cos \varphi'}{1 + \cos \lambda' \cos \varphi'}$$

$$y = R \sin \theta = \frac{\sin \varphi'}{1 + \cos \lambda' \cos \varphi'}.$$

We have for the square of the distance OM to the origin

$$x^2+y^2=\frac{1-\cos\lambda'\cos\varphi}{1+\cos\lambda'\cos\varphi'}.$$

We should note that the general equation of the circles traced upon the sphere and that of circles traced upon the map have exactly the same form when we take for coordinates φ and λ on the sphere and φ' and λ' upon the plane. On the unit sphere we have

$$x=\cos\lambda\cos\varphi$$

$$y=\sin\lambda\cos\varphi$$

$$z=\sin\varphi.$$

If we substitute these values in the equation of a plane

$$Ax+By+Cz+D=0,$$

we obtain

$$(A\cos\lambda+B\sin\lambda)\cos\varphi+C\sin\varphi+D=0.$$

This is the equation of a circle determined by the intersection of the plane with the sphere.

The general equation of a circle in the plane is given by

$$(x-a)^2+(y-b)^2=c^2,$$

or on substitution of the values of x and y in terms of φ' and λ' we obtain

$$\left(\frac{\sin\lambda'\cos\varphi'}{1+\cos\lambda'\cos\varphi'}-a\right)^2+\left(\frac{\sin\varphi'}{1+\cos\lambda'\cos\varphi'}-b\right)^2=c^2,$$

or on development

$$\frac{1-\cos\lambda'\cos\varphi'}{1+\cos\lambda'\cos\varphi'}-\frac{2a\sin\lambda'\cos\varphi'}{1+\cos\lambda'\cos\varphi'}-\frac{2b\sin\varphi'}{1+\cos\lambda'\cos\varphi'}=c^2-a^2-b^2$$

$$1-\cos\lambda'\cos\varphi'-2a\sin\lambda'\cos\varphi'-2b\sin\varphi'=c^2-a^2-b^2$$
$$+(c^2-a^2-b^2)\cos\lambda'\cos\varphi'$$

$$(a^2+b^2-c^2-1)\cos\lambda'\cos\varphi'-2a\sin\lambda'\cos\varphi'-2b\sin\varphi'$$
$$+a^2+b^2-c^2+1=0$$

or

$$(A'\cos\lambda'+B'\sin\lambda')\cos\varphi'+C'\sin\varphi'+D'=0,$$

A', B', C', and D' being constants depending upon the position of the center and the radius of the circle. In the meridian stereographic projection we have $\varphi' = \varphi$ and $\lambda' = \lambda$, so that it is only necessary to take A', B', C', and D' proportional to A, B, C, and D, respectively, in order that the two circles may correspond to each other. Therefore, in the stereographic projection on a meridian, and as a consequence also upon the horizon of any place, every circle is projected into a circle. This fact has already been proved in another place by the use of analytic geometry.*

Let us now determine the expressions for the scale along the meridian and for that along the parallels. When the point M is displaced infinitesimally upon the projection of the meridian, the arc described is equal to $R\left(\frac{\partial\theta}{\partial\varphi'}\right) d\varphi'$. and when displaced upon the parallel the arc described is equal to $r\left(\frac{\partial\theta}{\partial\lambda'}\right) d\lambda'$; therefore, we have

$$k_m = R\left(\frac{\partial\theta}{\partial\varphi'}\right)\frac{d\varphi'}{d\varphi}$$

$$k_p = \frac{r}{\cos\varphi}\left(\frac{\partial\theta}{\partial\lambda'}\right)\frac{d\lambda'}{d\lambda}.$$

Now, if we take the logarithms of the two members of the formula which gives the value of $\tan\frac{\theta}{2}$ and then differentiate, we obtain

$$\frac{d\theta}{\sin\theta} = \frac{d\lambda'}{\sin\lambda'} + \frac{d\varphi'}{\sin\varphi'},$$

which gives for the partial derivative values the following expressions:

$$\frac{\partial\theta}{\partial\varphi'} = \frac{\sin\theta}{\sin\varphi'}, \text{ and } \frac{\partial\theta}{\partial\lambda'} = \frac{\sin\theta}{\sin\lambda'}.$$

On substituting these values and the values of r and R we obtain

$$k_m = \frac{\sin\theta}{\sin\lambda'\sin\varphi'}\frac{d\varphi'}{d\varphi}$$

$$k_p = \frac{\sin\theta}{\cos\varphi\tan\varphi'\sin\lambda'}\frac{d\lambda'}{d\lambda},$$

*See p. 43.

or, on substituting the value of sin θ,

$$k_m = \frac{1}{1+\cos\lambda'\cos\varphi'}\frac{d\varphi'}{d\varphi}$$

$$k_p = \frac{1}{1+\cos\lambda'\cos\varphi'}\frac{\cos\varphi'}{\cos\varphi}\frac{d\lambda'}{d\lambda}.$$

CONFORMAL DOUBLE CIRCULAR PROJECTIONS.

In the conformal polyconic projection the condition $k_m = k_p$ gives in the case of the double circular orthogonal net

$$\frac{\sec\varphi'}{\sec\varphi}\frac{d\varphi'}{d\varphi} = \frac{d\lambda'}{d\lambda}.$$

The left-hand member of this equation is a function of φ alone and the right-hand member a function of λ alone; it is therefore necessary that they should be equal to the same constant n; hence

$$d\lambda' = n\,d\lambda$$

and

$$\frac{d\varphi'}{\cos\varphi'} = n\frac{d\varphi}{\cos\varphi}.$$

By integrating the first equation we get

$$\lambda' = n\lambda,$$

no constant of integration being introduced, since λ' vanishes with λ. In the second equation let $\varphi' = \frac{\pi}{2} - p'$ and let $\varphi = \frac{\pi}{2} - p$ and we obtain

$$\frac{dp'}{\sin p'} = n\frac{dp}{\sin p}.$$

Let us write this in the form

$$\cot\frac{p'}{2}\frac{dp'}{2} + \tan\frac{p'}{2}\frac{dp'}{2} = n\cot\frac{p}{2}\frac{dp}{2} + n\tan\frac{p}{2}\frac{dp}{2};$$

on integration this becomes

$$\log_e \sin\frac{p'}{2} - \log_e \cos\frac{p'}{2} = n \log_e \sin\frac{p}{2} - n \log_e \cos\frac{p}{2}$$

$$- n \log_e \sin\frac{p_0}{2} + n \log_e \cos\frac{p_0}{2},$$

or

$$\log_e \tan\frac{p'}{2} = n \log_e \tan\frac{p}{2} - n \log_e \tan\frac{p_0}{2},$$

or, on passing to exponentials,

$$\tan\frac{p'}{2} = \left(\frac{\tan\frac{p}{2}}{\tan\frac{p_0}{2}}\right)^n.$$

The constant which enters into the expression for tan $\frac{p'}{2}$, denoted by tan $\frac{p_0}{2}$, is determined by the fact that the straight line parallel is to have the colatitude p_0. When p is equal to p_0, p' becomes equal to $\frac{\pi}{2}$ and $r = \infty$. In the further discussion we shall consider $p_0 > \frac{\pi}{2}$ and reckon p and p' from the North Pole. That will throw the straight-line parallel into the Southern Hemisphere.

The angles are everywhere preserved except at the poles; in order that they may be preserved also at these two points, it is necessary that we should have n equal to unity, and then we have the stereographic projection upon the horizon of the place of the central meridian which has the latitude $\varphi_0 = p_0 - \frac{\pi}{2}$.

CAYLEY'S PRINCIPLE.

This puts us in position to explain what is sometimes called Cayley's principle.* Since in the stereographic projection n must equal unity, the meridians in the horizon projection are simply the same arcs as those of the

* See Cayley's Collected Mathematical Papers, Vol. VII, p. 397. Also mentioned in the ninth edition of the Encyclopædia Britannica, Vol. X, p. 203, in which place some a[illegible]ishing mathematical analysis is given in explanation of the principle.

stereographic meridian projection. The parallels are determined by the equation

$$\tan\frac{p'}{2}=\frac{\tan\frac{p}{2}}{\tan\frac{p_0}{2}}.$$

Parallels constructed for p' on the meridian projection are the parallels for p on the horizon projection. The circle constructed with its diameter consisting of the chord for $\varphi_0=p_0-\frac{\pi}{2}$ in the meridian projection becomes the projection of the horizon circle in the horizon projection. In figure 32, $pMp'N$ is the meridian circle of the original meridian projection and $PQP'Q'$ is the horizon circle for $p_0=\frac{2\pi}{3}$ constructed on the chord of the meridian circle for $\varphi_0=\frac{\pi}{6}\cdot$ Tangents to the computed p' points of the meridian circle would determine the centers and radii of the arcs for the horizon projection; or the radii and center distances can be computed from the expressions for r and s in terms of $\varphi'=\frac{\pi}{2}-p'$.

If we let p_0 become $\frac{\pi}{2}$ and then let n converge to zero while leaving constant the product of n by the length OP in figure 31, which we have chosen as unity in the former analysis, we obtain again Mercator's projection. If we maintain this product equal to two, we shall have constantly

$$OG=\lambda\,\frac{\tan\frac{\lambda'}{2}}{\frac{\lambda'}{2}},\ \text{and}\ OD=\frac{2}{n}\,\frac{1-\left(\tan\frac{p}{2}\right)^n}{1+\left(\tan\frac{p}{2}\right)^n}.$$

The limiting values of these expressions as $n\doteq 0$ are given in the form

$$OG=\lambda,\ \text{and}\ OD=\log_e\cot\frac{p}{2}.*$$

* For the derivation of these limits see p. 94.

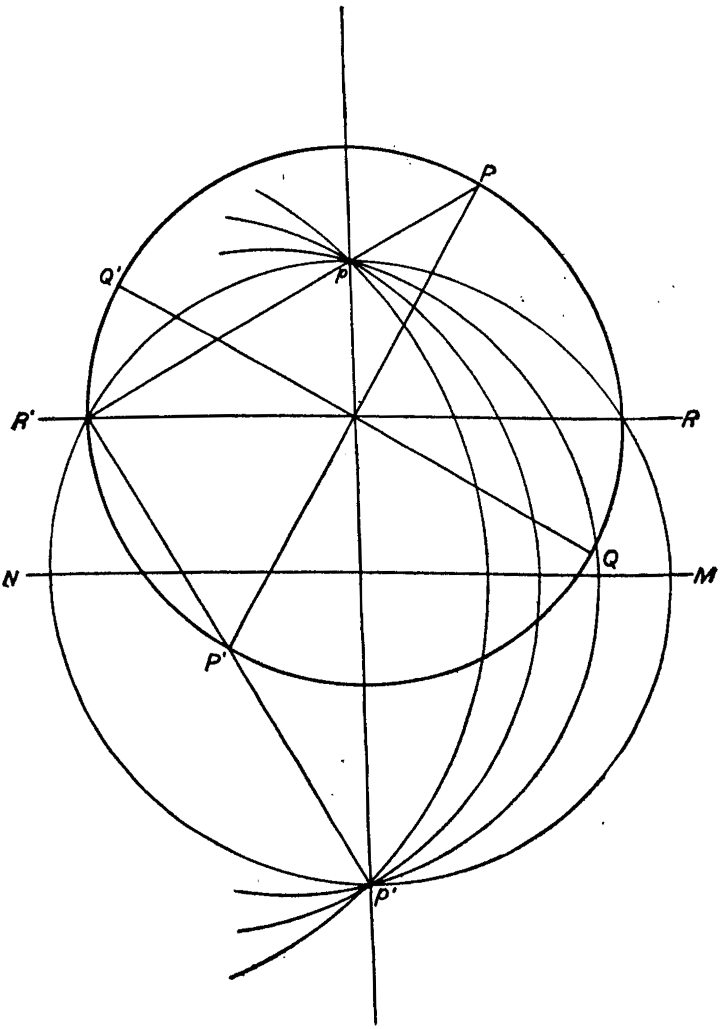

FIG. 32.—Cayley's principle.

DISCUSSION OF THE MAGNIFICATION ON THE CONFORMAL DOUBLE CIRCULAR PROJECTION.

The values which we have found for k_m and k_p in any system of rectangular projections with circular meridians and parallels have now become equal to each other and we have for the ratio of the lengths at each point of a conformal projection

$$k=\frac{n \sin \theta}{\cos \varphi \tan \varphi' \sin \lambda'}.$$

It results from this equation that, upon any given parallel, k increases or diminishes at the same time as λ. When the value of $\sin \theta$ is substituted, we obtain

$$k=\frac{n \sec \varphi}{\sec \varphi' + \cos \lambda'}=\frac{n \sin p'}{\sin p\ (1+\cos \lambda' \sin p')}.$$

A point of discontinuity is found when $\cos \lambda' \sin p' = -1$. Within the limits of the map this can happen only when $p'=\frac{\pi}{2}$ and $\lambda'=\pm\pi$. In the stereographic projection this point is the antipode of the center of the map. If n is less than unity it would fall outside of the map of the whole surface; but if n is greater than unity it would fall inside of the map of the earth's surface, since we should have $n\lambda=\pm\pi$.

For convenience we will write the above expression in the form

$$\frac{n}{k}= \sin p\left[\frac{1}{2}\left(\tan \frac{p'}{2}+ \cot \frac{p'}{2}\right)+\cos \lambda'\right].$$

In this expression we need only to replace λ' by $n\lambda$ and $\tan \frac{p'}{2}$ by $\left(\cot \frac{p_0}{2} \tan \frac{p}{2}\right)^n$ to obtain k directly as a function of p and λ. In order to see immediately what happens to k at the poles, we shall make this substitution and express the result in the form

$$\frac{n}{k}=\left(\cot \frac{p_0}{2}\right)^n\left(\sin \frac{p}{2}\right)^{1+n}\left(\cos \frac{p}{2}\right)^{1-n}$$
$$+\left(\tan \frac{p_0}{2}\right)^n\left(\sin \frac{p}{2}\right)^{1-n}\left(\cos \frac{p}{2}\right)^{1+n}+\sin p \cos n\lambda.$$

We shall need the derivatives of k with respect to p of the first two orders; we have

$$\frac{\sin p}{k}\frac{\partial k}{\partial p}=\frac{n\cos p'}{1+\sin p'\cos\lambda'}-\cos p$$

or

$$\frac{\partial}{\partial p}\left(\frac{1}{k}\right)=-\cot p'+\frac{1}{n}\,(\operatorname{cosec} p'+\cos\lambda')\cos p$$

$$n\sin p\sin p'\,\frac{\partial^2}{\partial p^2}\left(\frac{1}{k}\right)=n^2-n\cos p\cos p'$$
$$-\sin^2 p\,(1+\cos\lambda'\sin p'),$$

or

$$n\sin p\sin p'\left[\frac{1}{k^2}\frac{\partial^2 k}{\partial p^2}-\frac{2}{k^3}\left(\frac{\partial k}{\partial p}\right)^2\right]=\sin^2 p\,(1+\cos\lambda'\sin p')$$
$$+n\cos p\cos p'-n^2.$$

Let us first suppose $n<1$. Then at the two poles, that is, for $p=0$ and for $p=\pi$, we should have $k=\infty$; within the interval k would pass upon each meridian through a minimum. Denoting by a subscript m the value which applies for k a minimum, we should have, by equating to zero the first derivative of k with respect to p,

$$\frac{\cos p'_m}{1+\cos\lambda'\sin p'_m}=\frac{\cos p_m}{n}$$

$$k_m=\frac{\tan p'_m}{\tan p_m}$$

$$\sin p_m\sin p'_m\left[\frac{1}{k^2}\frac{\partial^2 k}{\partial p^2}\right]=\frac{\cos p'_m}{\cos p_m}-n.$$

The corresponding point is situated in the Northern Hemisphere.

The values which the above expression for $\frac{\sin p}{k}\frac{\partial k}{\partial p}$ assumes for $p=0$ and for $p=\pi$ are, respectively, $n-1$ and $1-n$, so that the first is negative and the second is positive. But for $p'=\frac{\pi}{2}$, $p(=p_0)>\frac{\pi}{2}$; hence the expression is positive for $p'=\frac{\pi}{2}$, and, in fact, it is positive for $p=\frac{\pi}{2}$. The point at which the minimum is found lies, therefore, in the Northern Hemisphere.

The values of p_m and p'_m for a given value of n on any given meridian would have to be determined by successive approximations until the equation containing p_m, p'_m, λ', and n would be satisfied by the value obtained. For particular meridians the equation becomes much simpler. Thus for the central meridian it becomes

$$\tan\frac{\varphi'_m}{2}=\frac{\sin\varphi_m}{n}.$$

When this value is substituted in the equation for the second derivative, we obtain

$$\sin p_m \sin p'_m\left[\frac{1}{k^2}\frac{\partial^2 k}{\partial p^2}\right]_m = n\left[\frac{1+\cos\phi_m - n^2}{n^2+\sin^2\varphi_m}\right].$$

It is upon this meridian that we obtain the smallest of all the minima.

Let us now suppose $n>1$. The conditions are now changed, since $k=0$ at the poles. The value of k upon each meridian passes through a maximum instead of a minimum; this maximum is found in the Southern Hemisphere and lies between the colatitude p_0 and the South Pole. This is shown by the fact that $\frac{\sin p}{k}\frac{\partial k}{\partial p}$ is equal to $n-1$ for $p=0$, a positive result; for $p=p_0$, $p'=\frac{\pi}{2}$, and the value is $-\cos p_0$, still positive, since $p_0>\frac{\pi}{2}$; for $p=\pi$ the value becomes $1-n$, a negative result. Hence the maximum lies between the straight line parallel and the South Pole.

When n is slightly greater than unity, it may happen that, starting at zero, the value of k would pass through a maximum in the Northern Hemisphere; then it would fall to a minimum in the same hemisphere, and finally pass through a maximum in the Southern Hemisphere to return to zero at the South Pole. This depends upon whether $\frac{\cos p'_m}{\cos p_m}$ becomes greater than n; this may well happen if n is but slightly greater than unity.

Lagrange proposed to profit by the fact that n and p_0 were arbitrary parameters to so determine them that k would vary as slowly as possible at a given point upon the meridian and upon the parallel in the vicinity of the principal place of the country the map of which he wished to construct. One part of the condition is fulfilled by making the meridian of the place become the central or straight line meridian, for in that case the derivative of k with

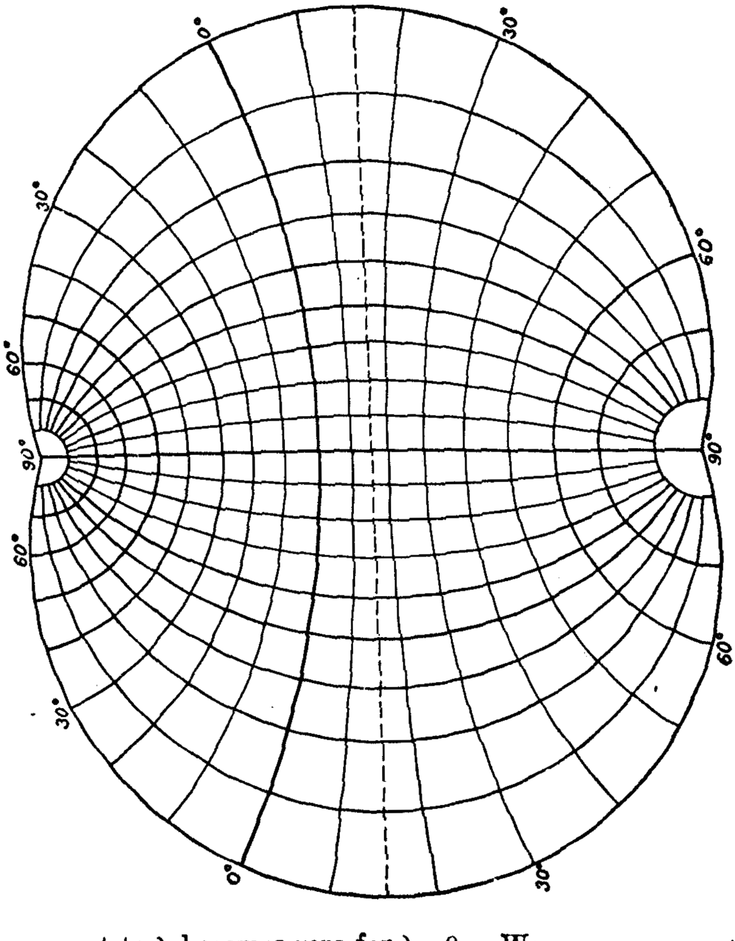

Fig. 33.—Lagrange's projection with Paris as center of least alteration.

respect to λ becomes zero for $\lambda = 0$. We can now equate to zero the first derivative of k with respect to p upon this meridian; it would merely be necessary to consider φ_m as the latitude of the given place. The second derivative will also become equal to zero if we take

$$n=\sqrt{1+\cos^2\varphi_m}.$$

Having thus found n, we would calculate φ'_m by means of the formula

$$\tan\frac{\varphi'_m}{2}=\frac{\sin\varphi_m}{n}.$$

Then we should have for the determination of p_o

$$\tan\frac{p_o}{2}=\tan\frac{p_m}{2}\left(\cot\frac{p'_m}{2}\right)^{\frac{1}{n}}.$$

For example, if the principal place was found on the Equator, we should have

$$\varphi_m=0,\ n=\sqrt{2},\ \varphi'_m=0,\ \text{and}\ p_o=\frac{\pi}{2}.$$

The Equator would then be represented by a straight line and the system of projection would be defined by the equations

$$\lambda'=\lambda\sqrt{2}$$

$$\tan\frac{p'}{2}=\left(\tan\frac{p}{2}\right)^{\sqrt{2}}.$$

A special case considered by Lagrange is given by the values of definition

$$s=\cot\frac{\varphi}{2}$$

$$S=\cot\frac{\lambda}{2}.$$

Hence

$$\operatorname{cosec}\varphi'=\cot\frac{\varphi}{2}$$

$$\cot\lambda'=\cot\frac{\lambda}{2}$$

or

$$\lambda'=\frac{\lambda}{2}$$

$$n=\frac{1}{2}$$

$$\tan\frac{p'}{2}=\left(\tan\frac{p}{2}\right)^{\frac{1}{2}}.$$

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Hence $p_0 = \frac{\pi}{2}$ and the Equator is represented by a straight line. The whole surface of the earth may be represented on a unit circle with the projection as defined, and the projection is so given in figure 34.

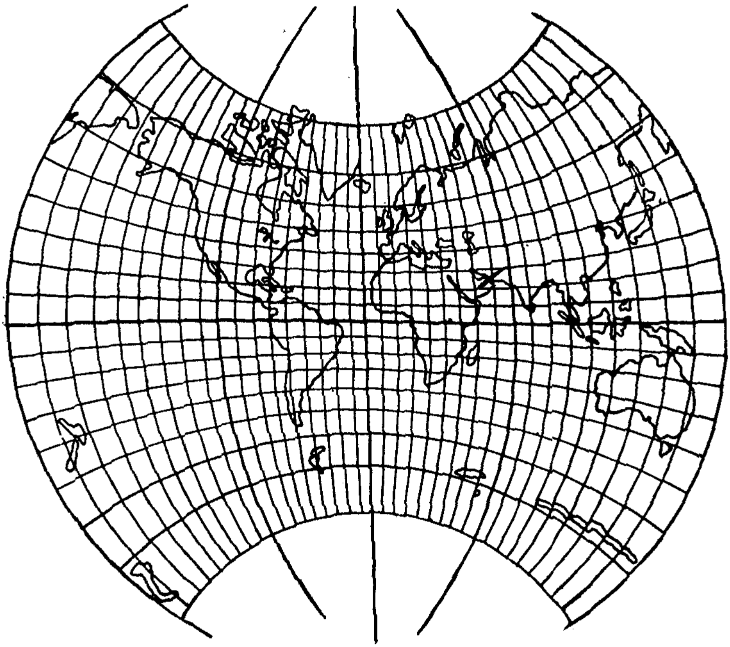

FIG. 34.—Lagrange's projection, earth's surface in a circle.

EQUIVALENT OR EQUAL-AREA POLYCONIC PROJECTIONS.

An equivalent or equal-area projection is one in which the proportion of areas is preserved constant; that is to say, that any portion of the map bears the same ratio to the region it represents that any other portion does to the region which it represents, or the ratio of area of any part is equal to the ratio of area of the whole representation. This is expressed analytically by the equation

$$k_m k_p \cos \psi = 1.$$

In the polyconic projection this becomes for the sphere

$$\frac{\rho}{a^2 \cos \varphi}\left(\frac{ds}{d\varphi}\cos\theta - \frac{d\rho}{d\varphi}\right)\frac{\partial\theta}{\partial\lambda} = 1.$$

Integrating partially with respect to λ and θ with φ remaining constant, we obtain

$$\frac{\rho}{a^2 \cos \varphi}\left(\frac{ds}{d\varphi} \sin \theta - \frac{d\rho}{d\varphi}\theta\right) = \lambda,$$

no constant being added, since θ and λ vanish together. In this expression s and ρ are any function of φ that we may choose. θ would then be determined by the above equation. Inversely, if we give the relation which should obtain between θ, φ, and λ subject to the condition that λ should be a linear function of θ and $\sin \theta$, there would be an infinity of equal-area polyconic projections which would satisfy this relation. In fact, u and v being given functions of φ, the assigned relation would be

$$u \sin \theta - v\,\theta = \lambda,$$

in which

$$u = \frac{\rho}{a^2 \cos \varphi}\frac{ds}{d\varphi}$$

$$v = \frac{\rho}{a^2 \cos \varphi}\frac{d\rho}{d\varphi},$$

or

$$\rho^2 = \rho_0{}^2 + 2a^2 \int_0^{\varphi} v \cos \varphi \, d\varphi.$$

$$s = s_0 + a^2 \int_0^{\varphi} \frac{u}{\rho} \cos \varphi \, d\varphi$$

ρ_0 and s_0 denoting the two constants of integration.

There is no equivalent polyconic projection that is at the same time rectangular. In a rectangular polyconic projection we have

$$\frac{ds}{d\varphi} = \frac{\rho}{u}\frac{du}{d\varphi}$$

and

$$\tan \frac{\theta}{2} = \frac{\Gamma(\lambda)}{u}$$

$$\frac{\partial \theta}{\partial \lambda} = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)} \sin \theta.$$

By substituting these values we obtain

$$\frac{\rho \sin \theta}{a^2 \cos \varphi}\left(\frac{\rho}{u}\frac{du}{d\varphi}\cos \theta - \frac{d\rho}{d\varphi}\right)\frac{\Gamma'(\lambda)}{\Gamma(\lambda)} = 1;$$

but

$$\sin \theta = \frac{2u\, \Gamma(\lambda)}{u^2 + \Gamma^2(\lambda)}$$

$$\cos \theta = \frac{u^2 - \Gamma^2(\lambda)}{u^2 + \Gamma^2(\lambda)}.$$

Hence

$$\frac{2\rho^2}{a^2 \cos \varphi}\frac{du}{d\varphi}\frac{u^2 - \Gamma^2(\lambda)}{[u^2 + \Gamma^2(\lambda)]^2} - \frac{2\rho u}{a^2 \cos \varphi}\frac{1}{u^2 + \Gamma^2(\lambda)}\frac{d\rho}{d\varphi} = \frac{1}{\Gamma'(\lambda)}.$$

This is an equation that must be identically satisfied by the values of u (a function of φ) and $\Gamma(\lambda)$ (a function of λ). The right-hand member is independent of φ; hence the left-hand member must also be independent of φ. The condition will be identically satisfied if u equals a constant and if $\frac{2\rho}{a^2 \cos \varphi}\frac{d\rho}{d\varphi}$ is equal to a constant.

If u is a constant, s is also a constant, and the projection would pass into one of the limiting cases of the polyconic projections.

The integration of the equation

$$2\rho\, d\rho = a^2 c \cos \varphi\, d\varphi$$

gives

$$\rho^2 = \rho_0^2 + a^2 c \sin \varphi.$$

By assigning particular values to the constants ρ_0 and c, we may obtain Lambert's central equal area projection, Lambert's isospherical stenoteric projection (sometimes called Lambert's fifth), or, finally, Albers' projection. None of these are polyconic projections in the accepted sense, and hence no investigation of their properties will be given at this time.

No one of the strictly polyconic equivalent projections has ever become of practical importance, because they would generally be complicated both for computation and construction.

DISCUSSION OF THE MAGNIFICATION ON THE CONFORMAL DOUBLE CIRCULAR PROJECTION.

The values which we have found for k_m and k_p in any system of rectangular projections with circular meridians and parallels have now become equal to each other and we have for the ratio of the lengths at each point of a conformal projection

$$k = \frac{n \sin \theta}{\cos \varphi \tan \varphi' \sin \lambda'}.$$

It results from this equation that, upon any given parallel, k increases or diminishes at the same time as λ. When the value of $\sin \theta$ is substituted, we obtain

$$k = \frac{n \sec \varphi}{\sec \varphi' + \cos \lambda'} = \frac{n \sin p'}{\sin p\,(1 + \cos \lambda' \sin p')}.$$

A point of discontinuity is found when $\cos \lambda' \sin p' = -1$. Within the limits of the map this can happen only when $p' = \frac{\pi}{2}$ and $\lambda' = \pm \pi$. In the stereographic projection this point is the antipode of the center of the map. If n is less than unity it would fall outside of the map of the whole surface; but if n is greater than unity it would fall inside of the map of the earth's surface, since we should have $n\lambda = \pm \pi$.

For convenience we will write the above expression in the form

$$\frac{n}{k} = \sin p \left[\frac{1}{2}\left(\tan \frac{p'}{2} + \cot \frac{p'}{2}\right) + \cos \lambda'\right].$$

In this expression we need only to replace λ' by $n\lambda$ and $\tan \frac{p'}{2}$ by $\left(\cot \frac{p_0}{2} \tan \frac{p}{2}\right)^n$ to obtain k directly as a function of p and λ. In order to see immediately what happens to k at the poles, we shall make this substitution and express the result in the form

$$\frac{n}{k} = \left(\cot \frac{p_0}{2}\right)^n \left(\sin \frac{p}{2}\right)^{1+n} \left(\cos \frac{p}{2}\right)^{1-n}$$
$$+ \left(\tan \frac{p_0}{2}\right)^n \left(\sin \frac{p}{2}\right)^{1-n} \left(\cos \frac{p}{2}\right)^{1+n} + \sin p \cos n\lambda.$$

We shall need the derivatives of k with respect to p of the first two orders; we have

$$\frac{\sin p}{k}\frac{\partial k}{\partial p}=\frac{n \cos p'}{1+\sin p' \cos \lambda'}-\cos p$$

or

$$\frac{\partial}{\partial p}\left(\frac{1}{k}\right)=-\cot p'+\frac{1}{n}\ (\operatorname{cosec} p'+\cos \lambda')\cos p$$

$$n \sin p \sin p' \frac{\partial^2}{\partial p^2}\left(\frac{1}{k}\right)=n^2-n \cos p \cos p'$$
$$-\sin^2 p\ (1+\cos \lambda' \sin p'),$$

or

$$n \sin p \sin p' \left[\frac{1}{k^2}\frac{\partial^2 k}{\partial p^2}-\frac{2}{k^3}\left(\frac{\partial k}{\partial p}\right)^2\right]=\sin^2 p\ (1+\cos \lambda' \sin p')$$
$$+n \cos p \cos p'-n^2.$$

Let us first suppose $n<1$. Then at the two poles, that is, for $p=0$ and for $p=\pi$, we should have $k=\infty$; within the interval k would pass upon each meridian through a minimum. Denoting by a subscript m the value which applies for k a minimum, we should have, by equating to zero the first derivative of k with respect to p,

$$\frac{\cos p'_m}{1+\cos \lambda' \sin p'_m}=\frac{\cos p_m}{n}$$

$$k_m=\frac{\tan p'_m}{\tan p_m}$$

$$\sin p_m \sin p'_m\left[\frac{1}{k^2}\frac{\partial^2 k}{\partial p^2}\right]=\frac{\cos p'_m}{\cos p_m}-n.$$

The corresponding point is situated in the Northern Hemisphere.

The values which the above expression for $\frac{\sin p}{k}\frac{\partial k}{\partial p}$ assumes for $p=0$ and for $p=\pi$ are, respectively, $n-1$ and $1-n$, so that the first is negative and the second is positive. But for $p'=\frac{\pi}{2}$, $p(=p_0)>\frac{\pi}{2}$; hence the expression is positive for $p'=\frac{\pi}{2}$, and, in fact, it is positive for $p=\frac{\pi}{2}$. The point at which the minimum is found lies, therefore, in the Northern Hemisphere.

The values of p_m and p'_m for a given value of n on any given meridian would have to be determined by successive approximations until the equation containing p_m, p'_m, λ', and n would be satisfied by the value obtained. For particular meridians the equation becomes much simpler. Thus for the central meridian it becomes

$$\tan\frac{\varphi'_m}{2}=\frac{\sin\varphi_m}{n}.$$

When this value is substituted in the equation for the second derivative, we obtain

$$\sin p_m \sin p'_m \left[\frac{1}{k^2}\frac{\partial^2 k}{\partial p^2}\right]_m = n\left[\frac{1+\cos\phi_m - n^2}{n^2+\sin^2\varphi_m}\right].$$

It is upon this meridian that we obtain the smallest of all the minima.

Let us now suppose $n>1$. The conditions are now changed, since $k=0$ at the poles. The value of k upon each meridian passes through a maximum instead of a minimum; this maximum is found in the Southern Hemisphere and lies between the colatitude p_0 and the South Pole. This is shown by the fact that $\frac{\sin p}{k}\frac{\partial k}{\partial p}$ is equal to $n-1$ for $p=0$, a positive result; for $p=p_0$, $p'=\frac{\pi}{2}$, and the value is $-\cos p_0$, still positive, since $p_0>\frac{\pi}{2}$; for $p=\pi$ the value becomes $1-n$, a negative result. Hence the maximum lies between the straight line parallel and the South Pole.

When n is slightly greater than unity, it may happen that, starting at zero, the value of k would pass through a maximum in the Northern Hemisphere; then it would fall to a minimum in the same hemisphere, and finally pass through a maximum in the Southern Hemisphere to return to zero at the South Pole. This depends upon whether $\frac{\cos p'_m}{\cos p_m}$ becomes greater than n; this may well happen if n is but slightly greater than unity.

Lagrange proposed to profit by the fact that n and p_0 were arbitrary parameters to so determine them that k would vary as slowly as possible at a given point upon the meridian and upon the parallel in the vicinity of the principal place of the country the map of which he wished to construct. One part of the condition is fulfilled by making the meridian of the place become the central or straight line meridian, for in that case the derivative of k with

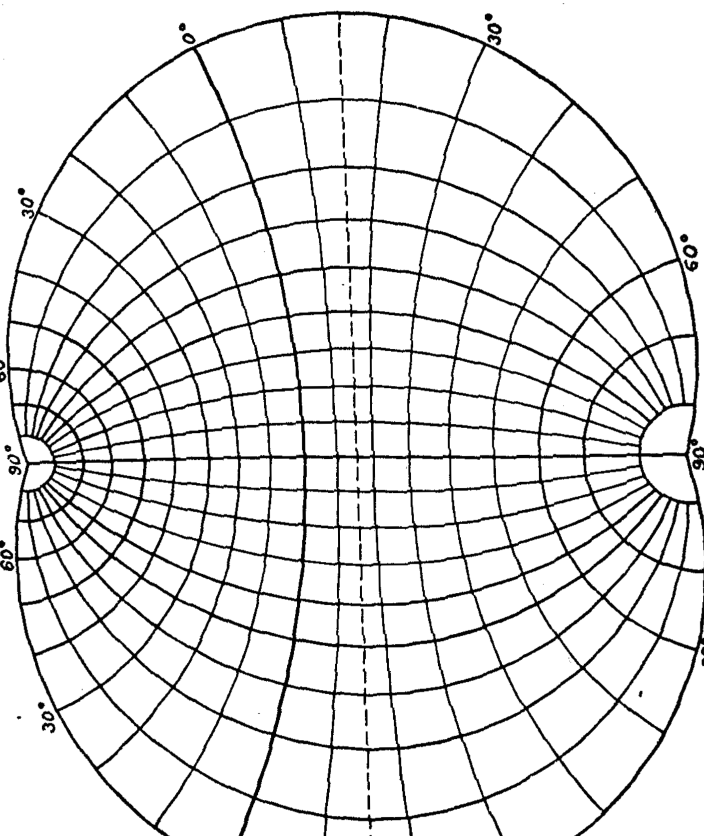

FIG. 33.—Lagrange's projection with Paris as center of least alteration.

respect to λ becomes zero for $\lambda = 0$. We can now equate to zero the first derivative of k with respect to p upon this meridian; it would merely be necessary to consider φ_m as the latitude of the given place. The second derivative will also become equal to zero if we take

$$n = \sqrt{1 + \cos^2 \varphi_m}.$$

Having thus found n, we would calculate φ'_m by means of the formula

$$\tan \frac{\varphi'_m}{2} = \frac{\sin \varphi_m}{n}.$$

Then we should have for the determination of p_o

$$\tan \frac{p_o}{2} = \tan \frac{p_m}{2}\left(\cot \frac{p'_m}{2}\right)^{\frac{1}{n}}.$$

For example, if the principal place was found on the Equator, we should have

$$\varphi_m = 0,\ n = \sqrt{2},\ \varphi'_m = 0, \text{ and } p_o = \frac{\pi}{2}.$$

The Equator would then be represented by a straight line and the system of projection would be defined by the equations

$$\lambda' = \lambda\sqrt{2}$$

$$\tan \frac{p'}{2} = \left(\tan \frac{p}{2}\right)^{\sqrt{2}}.$$

A special case considered by Lagrange is given by the values of definition

$$s = \cot \frac{\varphi}{2}$$

$$S = \cot \frac{\lambda}{2}.$$

Hence

$$\operatorname{cosec} \varphi' = \cot \frac{\varphi}{2}$$

$$\cot \lambda' = \cot \frac{\lambda}{2}$$

or

$$\lambda' = \frac{\lambda}{2}$$

$$n = \frac{1}{2}$$

$$\tan \frac{p'}{2} = \left(\tan \frac{p}{2}\right)^{\frac{1}{2}}.$$

Hence $p_0 = \frac{\pi}{2}$ and the Equator is represented by a straight line. The whole surface of the earth may be represented on a unit circle with the projection as defined, and the projection is so given in figure 34.

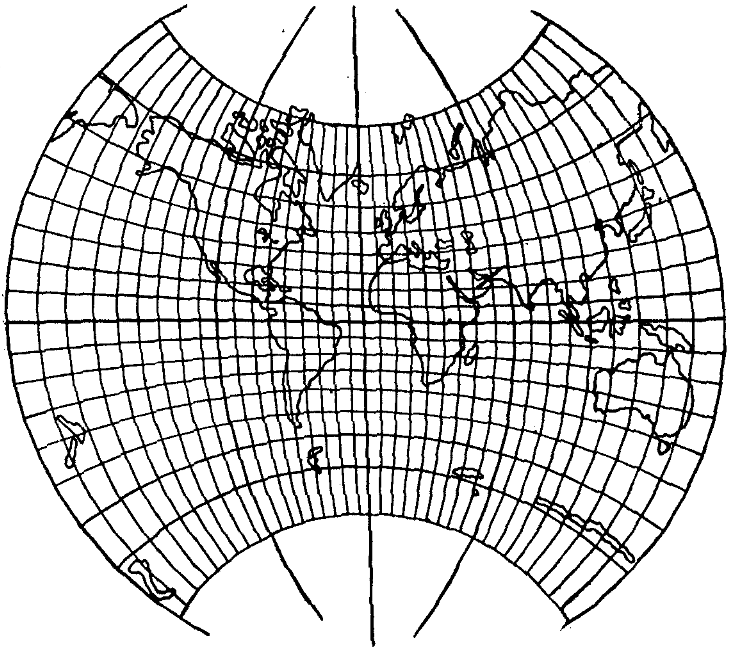

FIG. 34.—Lagrange's projection, earth's surface in a circle.

EQUIVALENT OR EQUAL-AREA POLYCONIC PROJECTIONS.

An equivalent or equal-area projection is one in which the proportion of areas is preserved constant; that is to say, that any portion of the map bears the same ratio to the region it represents that any other portion does to the region which it represents, or the ratio of area of any part is equal to the ratio of area of the whole representation. This is expressed analytically by the equation

$$k_m k_p \cos \psi = 1.$$

In the polyconic projection this becomes for the sphere

$$\frac{\rho}{a^2 \cos \varphi}\left(\frac{ds}{d\varphi}\cos\theta - \frac{d\rho}{d\varphi}\right)\frac{\partial\theta}{\partial\lambda} = 1.$$

Integrating partially with respect to λ and θ with φ remaining constant, we obtain

$$\frac{\rho}{a^2\cos\varphi}\left(\frac{ds}{d\varphi}\sin\theta-\frac{d\rho}{d\varphi}\theta\right)=\lambda,$$

no constant being added, since θ and λ vanish together. In this expression s and ρ are any function of φ that we may choose. θ would then be determined by the above equation. Inversely, if we give the relation which should obtain between θ, φ, and λ subject to the condition that λ should be a linear function of θ and $\sin\theta$, there would be an infinity of equal-area polyconic projections which would satisfy this relation. In fact, u and v being given functions of φ, the assigned relation would be

$$u\sin\theta-v\,\theta=\lambda,$$

in which

$$u=\frac{\rho}{a^2\cos\varphi}\frac{ds}{d\varphi}$$

$$v=\frac{\rho}{a^2\cos\varphi}\frac{d\rho}{d\varphi},$$

or

$$\rho^2=\rho_0{}^2+2a^2\int_0^\varphi v\cos\varphi\,d\varphi.$$

$$s=s_0+a^2\int_0^\varphi\frac{u}{\rho}\cos\varphi\,d\varphi$$

ρ_0 and s_0 denoting the two constants of integration.

There is no equivalent polyconic projection that is at the same time rectangular. In a rectangular polyconic projection we have

$$\frac{ds}{d\varphi}=\frac{\rho}{u}\frac{du}{d\varphi}$$

and

$$\tan\frac{\theta}{2}=\frac{\Gamma(\lambda)}{u}$$

$$\frac{\partial\theta}{\partial\lambda}=\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\sin\theta.$$

By substituting these values we obtain

$$\frac{\rho \sin \theta}{a^2 \cos \varphi}\left(\frac{\rho}{u}\frac{du}{d\varphi}\cos \theta - \frac{d\rho}{d\varphi}\right)\frac{\Gamma'(\lambda)}{\Gamma(\lambda)} = 1;$$

but

$$\sin \theta = \frac{2u\, \Gamma(\lambda)}{u^2 + \Gamma^2(\lambda)}$$

$$\cos \theta = \frac{u^2 - \Gamma^2(\lambda)}{u^2 + \Gamma^2(\lambda)}.$$

Hence

$$\frac{2\rho^2}{a^2 \cos \varphi}\frac{du}{d\varphi}\frac{u^2 - \Gamma^2(\lambda)}{[u^2 + \Gamma^2(\lambda)]^2} - \frac{2\rho u}{a^2 \cos \varphi}\frac{1}{u^2 + \Gamma^2(\lambda)}\frac{d\rho}{d\varphi} = \frac{1}{\Gamma'(\lambda)}.$$

This is an equation that must be identically satisfied by the values of u (a function of φ) and $\Gamma(\lambda)$ (a function of λ). The right-hand member is independent of φ; hence the left-hand member must also be independent of φ. The condition will be identically satisfied if u equals a constant and if $\frac{2\rho}{a^2 \cos \varphi}\frac{d\rho}{d\varphi}$ is equal to a constant.

If u is a constant, s is also a constant, and the projection would pass into one of the limiting cases of the polyconic projections.

The integration of the equation

$$2\rho\, d\rho = a^2 c \cos \varphi\, d\varphi$$

gives

$$\rho^2 = \rho_0^2 + a^2 c \sin \varphi.$$

By assigning particular values to the constants ρ_0 and c, we may obtain Lambert's central equal area projection, Lambert's isospherical stenoteric projection (sometimes called Lambert's fifth), or, finally, Albers' projection. None of these are polyconic projections in the accepted sense, and hence no investigation of their properties will be given at this time.

No one of the strictly polyconic equivalent projections has ever become of practical importance, because they would generally be complicated both for computation and construction.

Let us investigate the case in which the scale should be held constant along the parallels. We should then have

$$k_p = 1 \text{ and } k_m \cos\psi = 1,$$

or

$$\frac{1}{a}\left(\frac{ds}{d\varphi}\cos\theta - \frac{d\rho}{d\varphi}\right) = 1$$

$$ds\cos\theta - d\rho = a\,d\varphi$$

or

$$ds\cos\theta = d\rho + a\,d\varphi.$$

On any given parallel the right-hand member of this equation is a constant, since $d\rho$ is a function of φ; but θ is a function of φ and λ, for we have

$$k_p = \frac{\rho}{a\cos\varphi}\frac{\partial\theta}{\partial\lambda} = 1$$

or, by integration,

$$\theta = \frac{a\cos\varphi}{\rho}\lambda,$$

no constant being added, since θ and λ vanish together.

It follows that the left-hand member of the above equation must vanish identically; that is to say, $ds = 0$. The circles of parallels are, therefore, concentric and

$$d\rho = -a\,d\varphi,$$

or, by integration,

$$\rho = \rho_0 + a(\varphi_0 - \varphi).$$

This is Bonne's projection; but, of course, it is not a polyconic projection, since s is constant; that is, the parallel arcs are concentric. It appears, however, in the attempt to attain certain things by means of the equal-area polyconic projection and can be looked upon as a limiting case of the same.

If we assume

$$\rho = a\cot\varphi$$

$$s = a(\varphi + \cot\varphi),$$

then

$$\frac{d\rho}{d\varphi} = -a\operatorname{cosec}^2\varphi$$

$$\frac{ds}{d\varphi} = a(1 - \operatorname{cosec}^2\varphi) = -a\cot^2\varphi.$$

If these values are substituted in the equation of condition

$$\frac{\rho}{a^2 \cos \varphi}\left(\frac{ds}{d\varphi} \sin \theta - \frac{d\rho}{d\varphi}\theta\right) = \lambda,$$

we obtain for the determination of θ the equation

$$\theta - \cos^2 \varphi \sin \theta = \lambda \sin^3 \varphi.$$

In this case

$$k_m = \frac{1 - \cos^2 \varphi \cos \theta}{\sin^2 \varphi} \sec \psi$$

$$k_p = \frac{\sin^2 \varphi}{1 - \cos^2 \varphi \cos \theta},$$

so that we have as required

$$k_m k_p \cos \psi = 1,$$

and both k_m and k_p are equal to unity for $\theta = 0$.

If, on the other hand, we assume

$$\rho = a \cot \varphi$$

$$s = a \operatorname{cosec} \varphi$$

$$\frac{d\rho}{d\varphi} = -a \operatorname{cosec}^2 \varphi$$

$$\frac{ds}{d\varphi} = -a \cot \varphi \operatorname{cosec} \varphi$$

these values being substituted in the equation of condition give as the formula for θ

$$\theta - \cos \varphi \sin \theta = \lambda \sin^3 \varphi$$

and

$$k_m = \frac{1 - \cos \varphi \cos \theta}{\sin^2 \theta} \sec \psi$$

$$k_p = \frac{\sin^2 \varphi}{1 - \cos \varphi \cos \theta},$$

so that $k_m\, k_p \cos \psi = 1$ and $k_p = 1$ for $\theta = \varphi$, and $k_m = \sec \psi$ at the same point.

CONVENTIONAL POLYCONIC PROJECTIONS.

There is a class of projections that are not strictly equal-area, but which have the property that they preserve the area of the zones between the parallels and that of the lunes between the meridians. Any equal-area projection possesses this property, but it is not conversely true that any projection possessing this property is also an equal-area projection. Tissot calls projections of this class atractozonic. It can be rigidly proved that no rectangular polyconic projection can be an equal-area projection. We can, however, have an atractozonic projection in the polyconic class that also has circular meridians forming a rectangular net with the circular parallels.

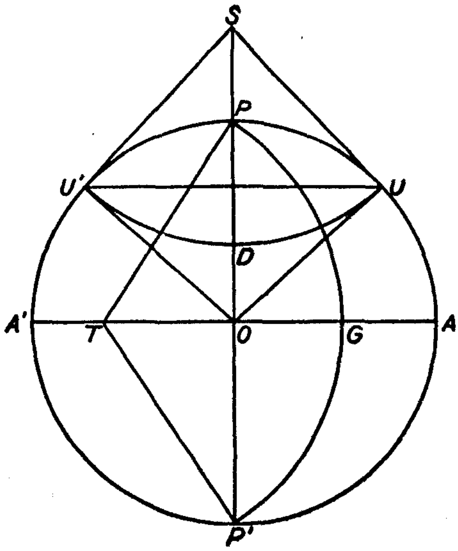

FIG. 35.—Geometrical relations of atractozonic projections.

In those that we shall study first we shall take the straight-line parallel of the map to represent the Equator, and the circumference described upon the line of poles of the map as diameter to represent the meridian the longitude of which is 90°, reckoned from the central meridian or the line of poles. We shall determine φ' as a function of φ in such a manner that, in the hemisphere limited by this meridian, the area of the half zone comprised between any two parallels will be preserved, and we shall determine λ' as a function of λ, so that the area of the lune formed by any two meridians may be preserved. The equal-area projections not only have the zones and lunes equal, but also in them the meridians of the earth and those of the map, respectively, divide each zone into proportional parts. This latter property is not found in the atractozonic projections.

In figure 35 we shall suppose the radius OA or OP equal to $\sqrt{2}$, so that the hemisphere and the circle which serves as its projection are equivalent, since the radius of the globe is taken as unity. The half zone with a base limited by the parallel of latitude φ has the area $\pi(1-\sin\varphi)$. It is projected upon the portion of the plane $PUDU'$ which the chord UU' divides into two segments of circles; the one UPU' is the difference between the sector $OUPU'$, measured by $\frac{1}{2}\ \overline{OP}^2$ times the arc UPU' or by $\pi-2\varphi'$, and the triangle OUU', which is measured by $\frac{1}{2}\ OU\times OU'\times \sin\angle UOU'$ or by $\sin 2\varphi'$; the other segment is the difference between the sector $SUDU'$ and the triangle SUU'; the angle USU' is equal to $2\varphi'$, and the radius SU of the parallel is equal to $\sqrt{2}\cot\varphi'$, so that the area of the segment is equal to $(2\varphi'-\sin 2\varphi')\cot^2\varphi'$. By equating the area of the zone to the area of the projection of the same, we obtain the relation

$$\pi-\pi\sin\varphi=\pi-2\varphi'-\sin 2\varphi'+(2\varphi'-\sin 2\varphi')\cot^2\varphi'$$

or

$$\frac{\pi}{2}\sin\varphi=\frac{\sin 2\varphi'-2\varphi'\cos 2\varphi'}{1-\cos 2\varphi'}.$$

According to the second condition, the area of the segment $OPGP'$ ought to be equal to that of the lune formed by the central meridian with the meridian of longitude λ. The angle PTG is the angle λ', so that $TP=\sqrt{2}\operatorname{cosec}\lambda'$. The area of the segment $OPGP'$ is equal to the area of the sector $TPGP'$, minus the area of the triangle TPP'.

$$TPGP'=\frac{1}{2}TP^2\times\text{arc } PGP'$$

$$=\frac{1}{2}\times 2\operatorname{cosec}^2\lambda'\times 2\lambda'$$

$$=2\lambda'\operatorname{cosec}^2\lambda'$$

$$\Delta TPP'=\frac{1}{2}TP\times TP'\sin\angle PTP'$$

$$=\frac{1}{2}\times 2\operatorname{cosec}^2\lambda'\sin 2\lambda'$$

$$TPP'=\operatorname{cosec}^2\lambda'\sin 2\lambda'.$$

Hence for the area of the segment we obtain

$$OPGP'=2\lambda'\operatorname{cosec}^2\lambda'-\operatorname{cosec}^2\lambda'\sin 2\lambda'.$$

The area of the lune upon the unit sphere is equal to 2λ; hence by equating this area to the area of the projection of the same we obtain

$$2\lambda=\frac{2\lambda'-\sin 2\lambda'}{\sin^2\lambda'}$$

or

$$\lambda=\frac{2\lambda'-\sin 2\lambda'}{1-\cos 2\lambda'}.$$

These two expressions may be written

$$\sin\varphi=\frac{\sin 2\varphi'-2\varphi'\cos 2\varphi'}{\pi\sin^2\varphi'}$$

$$\lambda=\frac{\lambda'}{\sin^2\lambda'}-\cot\lambda'.$$

By computing by means of the first equation the values of φ, which correspond to a sufficient number of values of φ', we could construct a table which, reciprocally, would make known the values of φ' corresponding to given values of φ. The second equation would make it possible to solve the same problem with respect to λ and λ'.

With these relations we obtain

$$\frac{d\varphi'}{d\varphi}=\frac{\pi}{4}\,\frac{\cos\varphi\,(1-\cos 2\varphi')^2}{\sin 2\varphi'\,(2\varphi'-\sin 2\varphi')}$$

$$\frac{d\lambda'}{d\lambda}=\frac{\sin^2\lambda'}{2\,(1-\lambda'\cot\lambda')}$$

$$k_m=\frac{\pi}{\sqrt{2}}\,\frac{\cos\varphi\sin\varphi'\tan\varphi'\sin\theta}{\sin\lambda'\,(2\varphi'-\sin 2\varphi')}$$

$$k_p=\frac{1}{\sqrt{2}}\,\frac{\sin\lambda'\sin\theta}{\cos\varphi\tan\varphi'\,(1-\lambda'\cot\lambda')}$$

or

$$k_m=\frac{\pi}{2\sqrt{2}}\,\frac{\cos\varphi\tan\varphi'}{2\varphi'-\frac{\pi}{2}\sin\varphi}\,\frac{1}{1+\cos\lambda'\cos\varphi'}$$

$$k_p=\frac{1}{\sqrt{2}}\,\frac{\cos\varphi'}{\cos\varphi}\,\frac{\sin^2\lambda'}{1-\lambda'\cot\lambda'}\,\frac{1}{1+\cos\lambda'\cos\varphi'}$$

$$=\frac{1}{\sqrt{2}}\,\frac{\cos\varphi'}{\cos\varphi}\,\frac{1}{1-\lambda\cot\lambda'}\,\frac{1}{1+\cos\lambda'\cos\varphi'}.$$

By setting aside the condition that the principal meridian should be represented by the circumference described upon the line of poles of the map as diameter, we could obtain a series of atractozonic projections instead of a single one, and in this group some would certainly be found the alterations of which would be less than those of the projection that we have just studied. We could still further increase the indetermination, and we could introduce two parameters in the place of one by not fixing in advance the parallel, the projection of which should be a straight line. This remark applies also to the remaining projections in this class.

In a rectangular circular projection, in place of determining φ' as a function of φ, so that the projection of each zone should be equivalent to the zone it represents, we can bring about that the ratio of the surfaces should be continually equal to unity along a given meridian or that the lengths should be preserved upon this meridian. Similarly, we could determine λ' as a function of λ in such a way that, upon a given parallel, the same conditions should be fulfilled. By combining each expression of φ' so obtained with one of the expressions for λ' we could form several kinds of projections, each of which would possess the two properties in question.

Let us continue to represent the principal meridian by the circumference described upon the line of poles of the map as diameter, the Equator by the diameter perpendicular to this line, and let us call R the radius of the circumference.

The ratio of surfaces at each point, in one of these rectangular circular projections, is

$$K=R^2\frac{\cos\varphi'}{\cos\varphi}\frac{1}{(1+\cos\lambda'\cos\varphi')^2}\frac{d\varphi'}{d\varphi}\frac{d\lambda'}{d\lambda}.$$

We now propose to bring about that it should remain equal to unity along the central meridian. For $\lambda=0$ we have $\lambda'=0$, and the derivative of λ' with respect to λ assumes a known value n, depending on the nature of the function of λ which has been adopted to represent the value of λ'. The condition is then

$$nR^2\frac{\cos\varphi'\,d\varphi'}{(1+\cos\varphi')^2}=\cos\varphi\,d\varphi$$

or, by integration,

$$\sin\varphi=\frac{nR^2}{2}\left(1-\frac{1}{3}\tan^2\frac{\varphi'}{2}\right)\tan\frac{\varphi'}{2}.$$

No constant of integration is added, since φ and φ' vanish at one and the same time. If each pole is to be a single point this equation must be valid for $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. This gives $nR^2=3$. If we wish that the ratio of surfaces should be equal to unity along the Equator, it would be necessary to have

$$n'R^2\frac{d\lambda'}{(1+\cos\lambda')^2}=d\lambda,$$

n' being the value of the derivative of φ' with respect to φ for $\varphi=0$. We deduce from this equation, by integration, the relation

$$\lambda=\frac{n'R^2}{2}\left(1+\frac{1}{3}\tan^2\frac{\lambda'}{2}\right)\tan\frac{\lambda'}{2},$$

no constant being added, since λ and λ' vanish together. Since the meridian of 90° of longitude is to be represented by the circumference described upon the line of poles of the map as diameter, it is necessary that this equation should be satisfied when we make in it at the same time $\lambda=\frac{\pi}{2}$ and $\lambda'=\frac{\pi}{2}$; we have then

$$n'R^2=\frac{3\pi}{4}.$$

We can unite the two conditions; then the mode of projection will be defined by the two relations which we have just obtained, the first between φ' and φ, the second between λ' and λ; in addition, n' will be found joined to n by the relation $nn'R^2=4$, which we obtain either by making $\varphi=0$ and $\frac{d\varphi'}{d\varphi}=n'$ in the first differential equation or by making $\lambda=0$ and $\frac{d\lambda'}{d\lambda}=n$ in the second. From this we conclude that

$$R=\frac{3}{4}\sqrt{\pi}.$$

The two equations are

$$\sin \varphi = \frac{1}{2}\left(3 - \tan^2 \frac{\varphi'}{2}\right)\tan \frac{\varphi'}{2}$$

$$\lambda = \frac{\pi}{8}\left(3 + \tan^2 \frac{\lambda'}{2}\right)\tan \frac{\lambda'}{2}.$$

k_m and k_p have now become

$$k_m = \frac{\sqrt{\pi}}{4} \frac{\cos \varphi \, (1 + \cos \varphi')^2}{\cos \varphi' \, (1 + \cos \lambda' \cos \varphi')}$$

$$k_p = \frac{1}{\sqrt{\pi}} \frac{\cos \varphi' \, (1 + \cos \lambda')^2}{\cos \varphi \, (1 + \cos \lambda' \cos \varphi')}$$

$$K = k_m k_p = \left[\frac{1}{2} \frac{(1 + \cos \lambda')\,(1 + \cos \varphi')}{1 + \cos \lambda' \cos \varphi'}\right]^2.$$

The latter formula can be written

$$K = \left[1 - \frac{1}{2} \frac{(1 - \cos \lambda')\,(1 - \cos \varphi')}{1 + \cos \lambda' \cos \varphi'}\right]^2.$$

In this form we see that K is everywhere less than unity, except on the Equator and upon the central meridian, and that the alteration of surface increases with the longitude and with the latitude. On the principal meridian we obtain

$$K = \cos^4 \frac{\varphi'}{2}.$$

Let us further examine how φ' ought to vary with φ in order that the areas should be preserved along the principal meridian. If we denote by n'' the value which the derivative of λ' with respect to λ takes for $\lambda = \frac{\pi}{2}$, we should have

$$\cos \varphi \, d\varphi = n'' R^2 \cos \varphi' \, d\varphi'$$

or, by integration,

$$\sin \varphi = n'' R^2 \sin \varphi',$$

no constant being added, since φ and φ' vanish simultaneously.

If φ and φ' are to become $\frac{\pi}{2}$ simultaneously, we shall have the condition

$$n'' R^2 = 1,$$

and in this case the pole will be represented by a single point. The equation then reduces to

$$\varphi' = \varphi.$$

If to this equation we add the following:

$$\lambda = \frac{\pi}{8}\left(3 + \tan^2 \frac{\lambda'}{2}\right) \tan \frac{\lambda'}{2},$$

we know that the surfaces will also be preserved along the Equator; this equation was derived from the differential equation

$$\frac{d\lambda'}{d\lambda} = \frac{4}{3\pi}(1 + \cos \lambda')^2$$

which gives $n'' = \frac{4}{3\pi}$ when in it we make $\lambda = \frac{\pi}{2}$, $\lambda' = \frac{\pi}{2}$, and $\frac{d\lambda'}{d\lambda} = n''$.

This value of n'' gives

$$R = \frac{1}{2}\sqrt{3\pi}.$$

The values for the magnification along the meridians and parallels now become

$$k_m = \frac{\sqrt{3\pi}}{2} \frac{1}{1 + \cos \varphi \cos \lambda'},$$

$$k_p = \frac{2}{\sqrt{3\pi}} \frac{(1 + \cos \lambda')^2}{1 + \cos \varphi \cos \lambda'},$$

and from these we derive

$$K = \left(\frac{1 + \cos \lambda'}{1 + \cos \varphi \cos \lambda'}\right)^2.$$

The ratio of surfaces is greater than unity everywhere except on the Equator and upon the principal meridian. The alteration increases with the latitude; on the other hand, it diminishes when the longitude increases. This is shown at once by writing the above expression in the form

$$K = \sec^2\varphi\left(1 - \frac{2\sin^2\frac{\varphi}{2}}{1+\cos\varphi\cos\lambda'}\right)^2.$$

Upon the central meridian, where the greatest alteration is produced, we have

$$K = \sec^4\frac{\varphi}{2}.$$

The conditions to insure that the areas should be preserved along the meridian of longitude λ_0 and along the parallel of latitude φ_0 give, respectively, the differential equations

$$A\sin^2\lambda'_0\frac{\cos\varphi'}{\cos\varphi}\frac{1}{(1+\cos\lambda'_0\cos\varphi')^2}\frac{d\varphi'}{d\varphi}=1$$

$$B\sin\varphi'_0\tan\varphi'_0\frac{1}{(1+\cos\varphi'_0\cos\lambda')^2}\frac{d\lambda'}{d\lambda}=1.$$

The integration of the first equation gives

$$\sin\varphi = A\left[\frac{\sin\varphi'}{1+\cos\lambda'_0\cos\varphi'} - 2\cot\lambda'_0\tan^{-1}\left(\tan\frac{\lambda_0}{2}\tan\frac{\varphi'}{2}\right)\right],$$

and from the second we get

$$\lambda = B\left[\frac{4}{\sin 2\varphi'_0}\tan^{-1}\left(\tan\frac{\varphi'_0}{2}\tan\frac{\lambda'}{2}\right) - \frac{\sin\lambda'}{1+\cos\varphi'_0\cos\lambda'}\right].$$

The quantities φ_0, φ'_0, λ_0, λ'_0 and the constants A and B are joined to each other by the four relations that are obtained by expressing that the first equation is satisfied for $\varphi=\varphi_0$ with $\varphi'=\varphi'_0$, as also for $\varphi=\frac{\pi}{2}$ with $\varphi'=\frac{\pi}{2}$ and the second for $\lambda=\frac{\pi}{2}$ with $\lambda'=\frac{\pi}{2}$, as also for $\lambda=\lambda_0$ with $\lambda'=\lambda'_0$.

The ratio of surfaces has now become

$$K=\left[\frac{(1+\cos\lambda'_0\cos\varphi')\,(1+\cos\varphi'_0\cos\lambda')}{(1+\cos\varphi'_0\cos\lambda'_0)\,(1+\cos\lambda'\cos\varphi')}\right]^2.$$

In the parentheses of the second member the factor which varies with φ' is

$$\frac{1+\cos\lambda'_0\cos\varphi'}{1+\cos\lambda'\cos\varphi'}=1+\frac{\cos\lambda'_0-\cos\lambda'}{\cos\lambda'+\sec\varphi'}.$$

We see, then, that upon each of the meridians for which we have $\lambda<\lambda_0$, the ratio K is less than unity and increases from the Equator to the pole; for $\lambda>\lambda_0$ we have $K>1$ and K increases from the pole to the Equator. We should see in a similar manner that, upon each parallel whose latitude is less than φ_0, K is smaller than unity and increases with the longitude, while, if φ is greater than φ_0, K will be greater than unity and will increase as the longitude decreases. Thus K attains a minimum K_1 at the center of the map, and another K_2 at the pole on the principal meridian; it attains a maximum K_3 at the pole on the central meridian; and, finally, a second maximum K_4 at the intersection of the Equator with the principal meridian; these values are

$$K_1=\left[\frac{(1+\cos\lambda'_0)(1+\cos\varphi'_0)}{2(1+\cos\lambda'_0\cos\varphi'_0)}\right]^2$$

$$K_2=\frac{1}{(1+\cos\lambda'_0\cos\varphi'_0)^2}$$

$$K_3=\left(\frac{1+\cos\varphi'_0}{1+\cos\lambda'_0\cos\varphi'_0}\right)^2$$

$$K_4=\left(\frac{1+\cos\lambda'_0}{1+\cos\lambda'_0\cos\varphi'_0}\right)^2.$$

Let us still consider the rectangular circular projection in which the hemisphere is represented by a complete circle, and let us now suppose that we wish to develop the central meridian with its true length. In order to do this we take the radius of the map equal to $\frac{\pi}{2}$. In figure 30 we have seen that the three points A', D, and U are in a straight line; hence the angle $OA'D$ is equal to the half of φ'. Moreover, we have here $OA'=\frac{\pi}{2}$ and $OD=\varphi$; the right triangle $OA'D$ will then give

$$\tan\frac{\varphi'}{2}=\frac{2\varphi}{\pi}.$$

If we also wish to develop the Equator with the true length, we should have in figure 31 $OG = \lambda$, and, since the angle OPG is equal to the half of λ', the triangle OPG will give in turn

$$\tan\frac{\lambda'}{2} = \frac{2\lambda}{\pi}.$$

From these two equations we obtain

$$\tan\frac{\theta}{2} = \frac{4\lambda\varphi}{\pi^2},$$

and also

$$\frac{d\varphi'}{d\varphi} = \frac{\sin\varphi'}{\varphi}$$

$$\frac{d\lambda'}{d\lambda} = \frac{\sin\lambda'}{\lambda},$$

so that we obtain

$$k_m = \frac{\pi}{2}\frac{\sin\theta}{\varphi\sin\lambda'} = \frac{\pi}{2}\frac{\sin\varphi'}{\varphi(1+\cos\lambda'\cos\varphi')}$$

$$k_p = \frac{\pi}{2}\frac{\sin\theta}{\lambda\cos\varphi\tan\varphi'} = \frac{\pi}{2}\frac{\sin\lambda'}{\lambda(1+\cos\lambda'\cos\varphi')}.$$

At the intersection of the Equator and the principal meridian, we have

$$k'_m = 2$$

$$k'_p = 1$$

$$K' = 2.$$

The Equator being developed with its true length, if we make the second condition no longer apply to the central meridian, but to the principal meridian, and if we wish that the arcs of this last have for projections arcs that are proportional to them, the relation between λ and λ' will remain the same, but that which exists between φ and φ' will be replaced by $\varphi' = \varphi$, which relations give

$$\tan\frac{\lambda'}{2} = \frac{2\lambda}{\pi}$$

$$\tan\frac{\theta}{2} = \frac{2\lambda}{\pi}\tan\frac{\varphi}{2}.$$

We have then

$$k_m = \frac{\pi}{2} \frac{\sin \theta}{\sin \varphi \sin \lambda'} = \frac{\pi}{2} \frac{1}{1 + \cos \lambda' \cos \varphi}$$

$$k_p = \frac{\pi}{2} \frac{\sin \theta}{\lambda \sin \varphi} = \frac{\pi}{2} \frac{\sin \lambda'}{\lambda(1 + \cos \lambda' \cos \varphi)}$$

$$K = \frac{\pi^2}{4} \frac{\sin \lambda'}{\lambda(1 + \cos \lambda' \cos \varphi)^2}.$$

This projection is sometimes called the stereographic projection with modified meridian.

NONRECTANGULAR CIRCULAR PROJECTIONS.

Let us always suppose that to each point of the globe there corresponds one point of the map, and only one, so that the circumferences which serve for the projections of the meridians all pass through two points P and P' in figure 36, which are the projections of the two poles. Let $APA'P'$ be the circumference described upon PP' as diameter, O its center, AA' the diameter perpendicular to PP', UDU' the projection of the parallel of latitude φ or of colatitude p, S the point in the prolongation of PP' which serves as the center for this projected parallel, V the middle point of the chord UU' common to the two circumferences $APA'P'$ and UDU'. Further, let PGP' be the projection of the meridian of longitude λ, reckoned from the central meridian projected into the line PP' and let T be the center of the circumference PGP'. Let us continue to define this last by the angle λ' at which it intersects PP', which is equal to the angle OTP, so that in the triangle OTP we have, as formerly, on taking OP as unity and on denoting by R and S, respectively, the radius TP and the distance OT,

$$R = \operatorname{cosec} \lambda', \; S = \cot \lambda', \; R^2 - S^2 = 1.$$

As to the projection UDU' of the parallel, we can define it by the two lengths r and s, as we have done up to this time, or by the two angles which the sides of the triangle OSU make with each other. Let us call the angle SOU, p'; its complement, φ'; the angle OSU, ϵ; and, finally, let γ denote the angle which one of the radii OU and SU makes with the prolongation of the other. Since we have $OU = 1$, the triangle OSU is determined by two of the

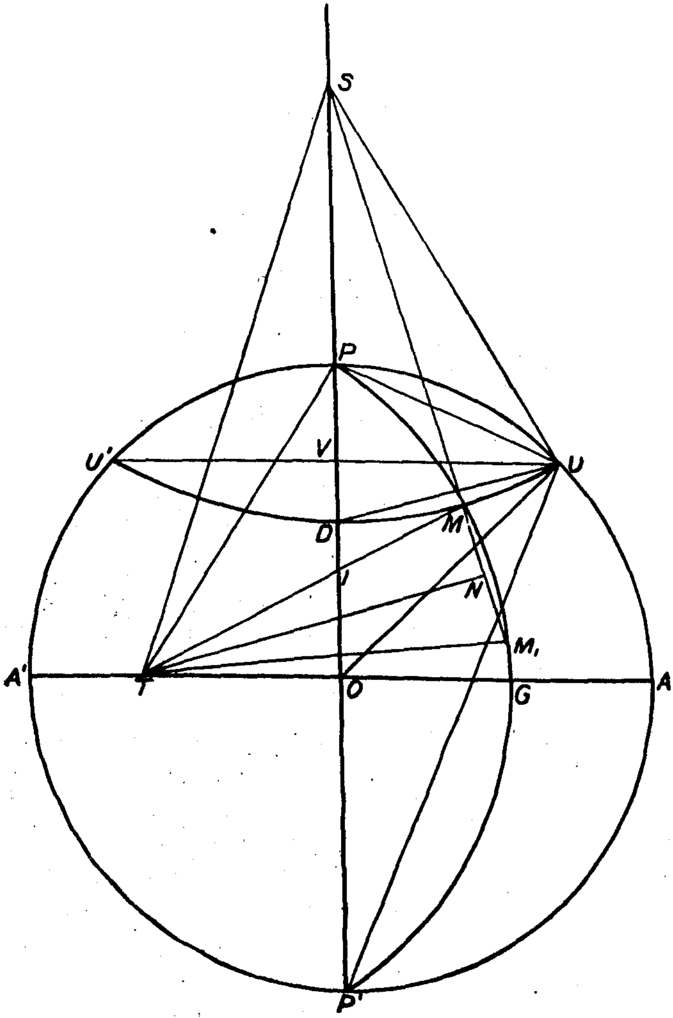

FIG. 36.—Geometrical relations of nonrectangular double-circular projections.

quantities r, s, p', ϵ, and γ and it is easy to express the three other quantities as well as the various lines of the figure in functions of the first two. We have especially

$$\gamma=\epsilon+p'$$

$$r=\frac{\sin p'}{\sin \epsilon}$$

$$s=\frac{\sin \gamma}{\sin \epsilon}$$

$$OD=s-r=\frac{\cos\left(\frac{\gamma+p'}{2}\right)}{\cos\frac{\epsilon}{2}}$$

$$s+r=\frac{\sin\left(\frac{\gamma+p'}{2}\right)}{\sin\frac{\epsilon}{2}}$$

The ratio of the two parts DP and DP' into which the line PP' is divided by the projection of the parallel is expressed very simply by means of p' and γ. In fact, this latter angle is equal to that of the two tangents at U to the two circumferences, which angle is divided into two parts by the chord UU', the one of which is the double of the angle DUU', and the other of the angle PUU'. The angle PUD is then equal to $\frac{\gamma}{2}$; but of the two complementary angles $PP'U$ and $P'PU$ the first is equal to $\frac{p'}{2}$. It comes about, then, in the triangles DPU and $DP'U$ that

$$DU\sin\frac{\gamma}{2}=DP\cos\frac{p'}{2}$$

$$DU\cos\frac{\gamma}{2}=DP'\sin\frac{p'}{2},$$

from which, by dividing member by member and on denoting the ratio by ξ,

$$\frac{DP}{DP'}=\xi=\tan\frac{p'}{2}\tan\frac{\gamma}{2}.$$

The alteration ψ of the angle of the meridians with the parallels is the excess of the angle SMT over $\frac{\pi}{2}$. In order to obtain it simply, let us note that, M_1 being the second point of intersection of SM with the circumference PMP', we have

$$SM \times SM_1 = SP \times SP',$$

if M is displaced by changing the meridian but, remaining on the same parallel, SM is constant; then the same is true of SM_1; consequently, also of MM_1. Then the projection MN of the radius TM of the variable meridian of the map upon the radius SM of the fixed parallel has a constant length. At the point M this length is expressed by $R \sin \psi$ or by $\frac{\sin \psi}{\sin \lambda'}$, and, at the point U, by $\cos \gamma$; it thus results that

$$\sin \psi = \cos \gamma \sin \lambda'.$$

In the triangle OST the angle at S, which we will call σ, may be immediately obtained, for we have

$$\tan \sigma = \frac{S}{s}.$$

Let us now designate by θ the angle OSM and by δ the angle OTM, which we shall need for calculating the ratios k_m and k_p. The triangle STM gives

$$\sin (\theta + \sigma) = \frac{R}{TS} \cos \psi$$

$$\cos (\delta + \sigma) = \frac{r}{TS} \cos \psi;$$

but we have in the triangle OST

$$TS = \frac{S}{\sin \sigma} = \frac{s}{\cos \sigma},$$

so that we have

$$\sin (\theta + \sigma) = \frac{R}{S} \sin \sigma \cos \psi$$

$$\cos (\delta + \sigma) = \frac{r}{s} \cos \sigma \cos \psi$$

or

$$\sin(\theta+\sigma)=\frac{\sin\sigma\cos\psi}{\cos\lambda'}$$

$$\cos(\delta+\sigma)=\frac{\sin p'\cos\sigma\cos\psi}{\sin\gamma}.$$

It is, however, sufficient to calculate one of the angles θ and δ; we have, in fact,

$$\delta-\theta=\psi,$$

for, I being the point of intersection of TU with PP', the two triangles OIT and ISM have the angles at I equal, and, by expressing that the sum of the other angles are the same in the one triangle as in the other, we obtain the relation which we have just written.

The rectangular coordinates of the point M with respect to the axes OA and OP are

$$x=r\sin\theta$$

$$y=R\sin\delta.$$

We now have

$$k_m=R\frac{\partial\delta}{\partial p}$$

$$k_p=\frac{r}{\sin p}\frac{\partial\theta}{\partial\lambda}.$$

By taking, with respect to p and with respect to λ, the derivatives of the logarithms of the two members of each of the relations which we have established between the different variables, we obtain $\frac{\partial\delta}{\partial p}$ and $\frac{\partial\theta}{\partial\lambda}$, which figure in the values of k_m and k_p; but it is more simple to obtain k_m by making use of the formula

$$k_m=\left(\frac{dr}{dp}-\frac{ds}{dp}\cos\theta\right)\sec\psi,$$

which has been demonstrated with regard to polyconic projections in general. Since the meridians are also circles with their centers upon the same straight line, we can form an expression for k_p by replacing in the expression for k_m, p by λ, r by R, s by S, and θ by δ, and by dividing by $\sin p$; this gives

$$k_p = \left(\frac{dR}{d\lambda} - \frac{dS}{d\lambda}\cos\delta\right)\frac{\sec\psi}{\sin p}.$$

The projection of TM upon OT being equal to TO plus the projection of SM, we have

$$R\cos\delta = S + r\sin\theta.$$

Substituting for $\cos\delta$, in the expression of k_p, the value which results from this last equation, and observing that $R\frac{dR}{d\lambda} - S\frac{dS}{d\lambda}$ is zero, since $R^2 - S^2$ is a constant, we have

$$k_p = -\frac{r\sin\theta}{R\sin p\cos\psi}\frac{dS}{d\lambda};$$

but

$$\frac{1}{R}\frac{dS}{d\lambda} = -\frac{1}{\sin\lambda'}\frac{d\lambda'}{d\lambda},$$

so that

$$k_p = \frac{r\sin\theta\sec\psi}{\sin\lambda'\sin p}\frac{d\lambda'}{d\lambda}.$$

The expression for k_m can be written

$$k_m = \left[\frac{d(s-r)}{d\varphi} - 2\frac{ds}{d\varphi}\sin^2\frac{\theta}{2}\right]\sec\psi.$$

Let us examine in particular what these ratios become upon the straight-line parallel of the map which we shall make, for example, correspond to the Equator. Let us call A the value which is assumed for $\varphi = 0$ by the derivative of OD or $s - r$ with respect to φ and $-B$ the limit toward which tends the ratio of $\frac{ds}{d\varphi}$ to $2r^2$ when φ tends toward zero. Since at the same time $r\theta$ tends toward OG or $\tan\frac{\lambda'}{2}$, we find that on the Equator

$$k_m = A + B\tan^2\frac{\lambda'}{2}$$

$$k_p = \frac{1}{2}\sec^2\frac{\lambda'}{2}\frac{d\lambda'}{d\lambda},$$

since $\psi = 0$ at that point.

The condition that the areas should be preserved along this line will then be

$$\frac{1}{2}\left(A+B\tan^2\frac{\lambda'}{2}\right)\sec^2\frac{\lambda'}{2}\frac{d\lambda'}{d\lambda}=1$$

or, by integration,

$$\left(A+\frac{B}{3}\tan^2\frac{\lambda'}{2}\right)\tan\frac{\lambda'}{2}=\lambda,$$

no constant being added, since λ and λ' vanish simultaneously.

There is an infinity of circular projections with oblique angles that are atractozonic. If we suppose the meridian of 90° of longitude represented by the circumference described upon the line of poles as diameter, these projections are furnished by the following equations:

$$2\varphi'+\sin 2\varphi'-(1+\cos 2\varphi')\frac{2\epsilon-\sin 2\epsilon}{1-\cos 2\epsilon}=\pi\sin\varphi$$

$$\frac{2\lambda'-\sin 2\lambda'}{1-\cos 2\lambda'}=\lambda.$$

The first leaves yet undetermined one of the two quantities φ' and ϵ as a function of φ; as to the second, it is incompatible with the condition of preservation of areas along the Equator, which proves that no circular projection with oblique angles can be equal-area in the complete sense.

PROJECTION OF NICOLOSI OR GLOBULAR PROJECTION.

In this projection the Equator and the central meridian are found developed in straight lines and with their true lengths; the principal meridian is represented by the circumference described upon the line of poles of the map as diameter; and, finally, the arcs of this meridian and the corresponding arcs of the circumference are proportional. We therefore have

$$\varphi' = \varphi$$

$$p' = p$$

$$\epsilon = \gamma - p$$

$$\xi = \frac{p}{\pi - p}$$

$$\tan\frac{\gamma}{2} = \frac{p}{\pi - p}\cot\frac{p}{2}$$

$$r = \frac{\pi \sin p}{2 \sin \epsilon}$$

$$s = \frac{\pi \sin \gamma}{2 \sin \epsilon}$$

$$\tan\frac{\lambda'}{2} = \frac{2\lambda}{\pi}^{*}$$

$$R = \frac{\pi}{2}\operatorname{cosec}\lambda'$$

$$S = \frac{\pi}{2}\cot\lambda'$$

$$\sin\psi = \cos\gamma\sin\lambda'$$

$$\tan\sigma = \frac{S}{s}$$

$$\sin(\theta + \sigma) = \frac{\sin\sigma\cos\psi}{\cos\lambda'}$$

$$\delta = \theta + \psi$$

$$k_m = \left[1 + 2\,\frac{\frac{\pi}{2}s\cos\varphi - r}{\frac{\pi}{2}\sin\varphi - \varphi}\sin^2\frac{\theta}{2}\right]\sec\psi$$

$$k_p = \frac{r\sin\theta}{\lambda\cos\varphi\cos\psi}.$$

*See p. 128.

The latter formula is very easily deduced, since by logarithmic differentiation we obtain

$$\frac{1}{\sin\lambda'}\,\frac{d\lambda'}{d\lambda}=\frac{1}{\lambda};$$

when this value is substituted in the general formula, we obtain the relation as given above. The formula for k_m is somewhat more complicated in its derivation. We have from the a priori conditions

$$s-r=\varphi$$

or

$$\frac{d}{d\varphi}(s-r)=1.$$

From the triangle *OSU* we obtain

$$r^2=s^2+\frac{\pi^2}{4}-\pi s\sin\varphi;$$

but

$$s-r=\varphi$$

$$(s-\varphi)^2=s^2+\frac{\pi^2}{4}-\pi s\sin\varphi$$

$$(\pi\sin\varphi-2\varphi)s=\frac{\pi^2}{4}-\varphi^2$$

or

$$s=\frac{\frac{\pi^2}{4}-\varphi^2}{\pi\sin\varphi-2\varphi}$$

$$\frac{ds}{d\varphi}=\frac{-2\varphi}{\pi\sin\varphi-2\varphi}-\frac{s\,(\pi\cos\varphi-2)}{\pi\sin\varphi-2\varphi}$$

$$=\frac{2r-\pi s\cos\varphi}{\pi\sin\varphi-2\varphi}.$$

When these values are substituted in the general formula on page 134, we obtain the value of k_m, as given above. A circle constructed upon the line of poles of the map as a diameter gives the projection of the principal meridian. A diameter perpendicular to this is the projection of the Equator. Both of these diameters are divided into equal parts and the projection of the principal meridian is divided into the same number of equal parts. The parallels are arcs through the divisions of the line of poles passing through the corresponding divisions of the principal meridian. The meridians are arcs passing through the poles and through the divisions of the Equator or the diameter perpendicular to the line of poles.

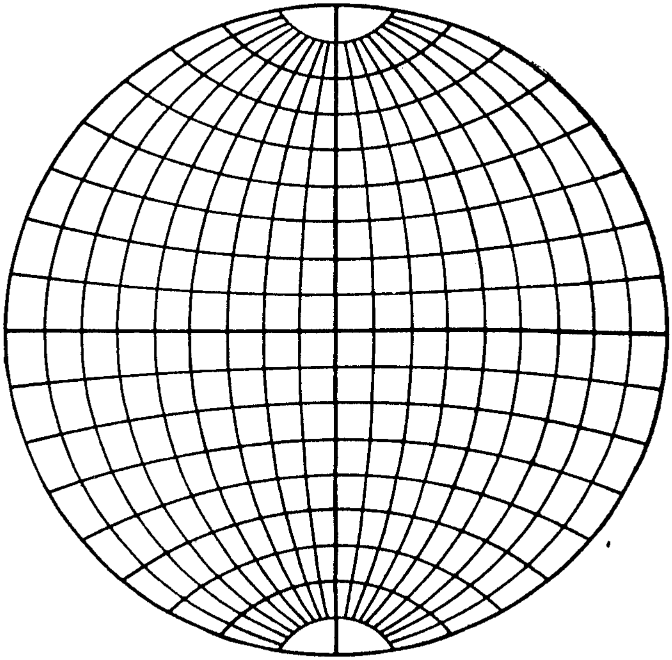

FIG. 37.—Nicolosi's projection or globular projection.

PROJECTION OF P. FOURNIER.

Another conventional projection is that proposed by P. Fournier in 1646, which is a polyconic projection with meridians that are ellipses. The Equator and the central meridian are developed with their true length on two straight lines perpendicular to each other; the central meridian serves as the major axis of all the ellipses for each of which the corresponding λ serves as the semiminor axis. The principal meridian is a circumference of a circle. The projections of the parallels intercept upon this circumference and upon the projection of the central meridian lengths proportional to the corresponding arcs of the globe.

In figure 38 let $APA'P'$ be a circumference the radius of which OP is equal to $\frac{\pi}{2}$; it will represent the principal meridian. Let PP' be the central meridian of the map

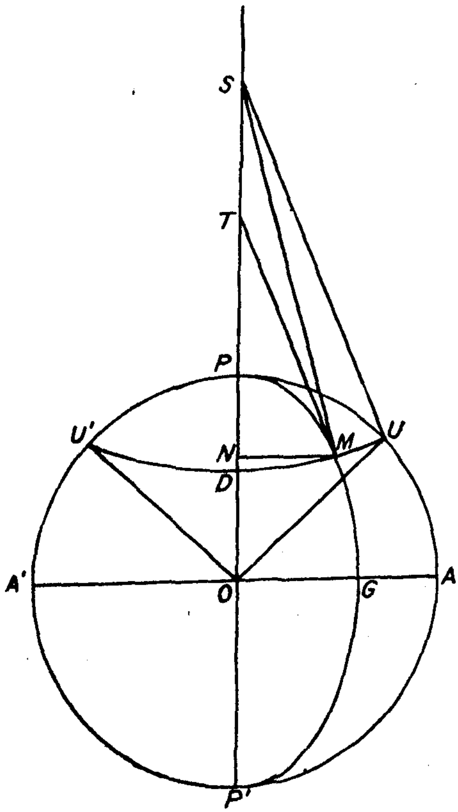

FIG. 38.—Geometrical relations of Fournier's projection.

and let AA' be the Equator. If we take OD equal to φ, and if we make the angles AOU and $A'OU'$ also equal to φ, the circumference passing through the three points U, D, U' will be the projection of the parallel of latitude φ. By taking OG equal to λ and constructing a half ellipse having for vertices P, G, and P' we shall obtain the projection of the meridian of longitude λ. Let M be the point where it intersects the parallel, and let S be the center for the latter; draw the abscissa MN of the point M and the tangent MT to the ellipse; also draw SU and SM.

The parallels are the same as those in the globular projection, so that we have, as before,

$$s-r=\varphi$$

$$r^2=s^2+\frac{\pi^2}{4}-\pi s \sin \varphi$$

or, by combining the two equations,

$$\varphi(r+s)-\pi s \sin \varphi+\frac{\pi^2}{4}=0$$

$$s=\frac{\frac{\pi^2}{4}-\varphi^2}{\pi \sin \varphi-2\varphi}.$$

By taking the derivatives of the two members of these equations with respect to φ we obtain

$$\frac{ds}{d\varphi}=\frac{2r-\pi s \cos \varphi}{\pi \sin \varphi-2\varphi}$$

$$\frac{dr}{d\varphi}=\frac{ds}{d\varphi}-1.$$

The angle OSM is still denoted by θ. The triangle SMN gives for the rectangular coordinates of M with O as an origin

$$x=r \sin \theta$$

$$y=s-r \cos \theta.$$

The elliptic meridian has the equation

$$\frac{x^2}{\lambda^2}+\left(\frac{2y}{\pi}\right)^2=1.$$

By substituting the above values of x and y in this equation, and then solving for $\cos \theta$, we find

$$\cos \theta=\frac{\pi\sqrt{4\lambda^4+2\pi\lambda^2(2s \sin \varphi-\pi)+\pi^2 r^2}-4\lambda^2 s.}{r(\pi^2-4\lambda^2)}.$$

By using this equation we can compute the angle θ as well as the values of x and y. If we denote by η the angle *OTM* formed by the tangent to the ellipse at *M* and the *Y* axis, we know that we have

$$\tan \eta = \frac{4\lambda^2 y}{\pi^2 x};$$

but the departure ψ of the angle of the meridian from an orthogonal intersection with the parallel is the angle *SMT*, which is equal to the difference between the angles *OTM* and *OSM;* we have then

$$\psi = \eta - \theta.$$

Everything is now known in the expression for k_m, namely

$$k_m = \left(\frac{ds}{d\varphi}\cos\theta - \frac{dr}{d\varphi}\right)\sec\psi.$$

By substituting the values this becomes

$$k_m = \left(1 + 2\frac{\pi s \cos\varphi - 2r}{\pi \sin\varphi - 2\varphi}\sin^2\frac{\theta}{2}\right)\sec\psi,$$

an expression that has the same form as in the case of the globular projection; but, of course, the angles θ and ψ have different values from what they had in that projection.

$$k_p = r\left(\frac{\partial\theta}{\partial\lambda}\right)\sec\varphi.$$

By differentiating the equation for cos θ with respect to λ we obtain the value of $\frac{\partial\theta}{\partial\lambda}$ which may be reduced to a convenient form by substituting for sin θ its value *in terms of* x and y; this form is much more readily obtained by differentiating the expressions for x and y with respect to λ, and then the differentiation of the equation of the ellipse partially with respect to λ will furnish the equation for determining $\frac{\partial\theta}{\partial\lambda}$. In this way we get

$$\frac{\partial x}{\partial\lambda} = r\cos\theta\frac{\partial\theta}{\partial\lambda} = (s-y)\frac{\partial\theta}{\partial\lambda}$$

$$\frac{\partial y}{\partial\lambda} = r\sin\theta\frac{\partial\theta}{\partial\lambda} = x\frac{\partial\theta}{\partial\lambda}$$

and

$$\frac{x}{\lambda^2}\frac{\partial x}{\partial\lambda} - \frac{x^2}{\lambda^3} + \frac{4y}{\pi^2}\frac{\partial y}{\partial\lambda} = 0.$$

By solving these linear equations for $\frac{\partial\theta}{\partial\lambda}$ we obtain

$$\frac{\partial\theta}{\partial\lambda}=\frac{\pi^2 x}{\lambda\,[\pi^2 s-(\pi^2-4\lambda^2)\,y]}.$$

Hence

$$k_p=\frac{\pi^2 rx \sec\varphi}{\lambda\,[\pi^2 s-(\pi^2-4\lambda^2)\,y]}.$$

Upon the central meridian we have

$$\theta=0,\ \psi=0,\ k_m=1,$$

and

$$k_p=\sec\varphi\sqrt{1-\left(\frac{2\varphi}{\pi}\right)^2},$$

upon the principal meridian

$$\cos\theta=\frac{1}{r}\left(s-\frac{\pi}{2}\sin\varphi\right),$$

a relation that is evident from the figure.

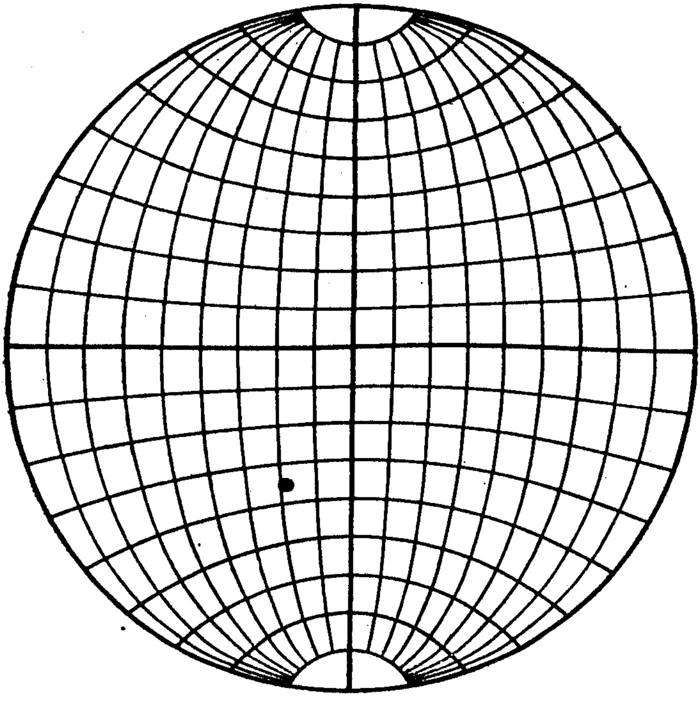

FIG. 39.—Projection of P. Fournier.

Also

$$\sin\psi = \frac{1}{\pi r}\left[\varphi\,(r+s) - \frac{\pi^2}{4}\right]$$

$$k_m = \frac{\sec\varphi}{s}\left[\left(\varphi - \frac{\pi}{2}\sin\varphi\right)\frac{ds}{d\varphi} + r\right]$$

$$k_p = \frac{r}{s}.$$

ORDINARY, OR AMERICAN, POLYCONIC PROJECTION.

This is the projection that is generally referred to in this country as the polyconic projection; but we have attempted to show that the polyconic projection class is an exceedingly broad one and that it contains examples of almost every kind of projections. The name American polyconic projection has been given to it by European writers chiefly because it has been extensively used by the United States Coast and Geodetic Survey; in fact, the projection seems to have been devised by Supt. F. R. Hassler to meet the requirements in the charting of the coast of the United States.

For convenience of reference we shall give again the differential formulas developed on pages 10–13:

$$\tan\psi = \frac{\rho\dfrac{\partial\theta}{\partial\varphi} + \dfrac{ds}{d\varphi}\sin\theta}{\dfrac{ds}{d\varphi}\cos\theta - \dfrac{d\rho}{d\varphi}}$$

$$k_m = \frac{(1-\epsilon^2\sin^2\varphi)^{3/2}}{a\,(1-\epsilon^2)}\left(\frac{ds}{d\varphi}\cos\theta \quad \frac{d\rho}{d\varphi}\right)\sec\psi$$

$$k_p = \frac{\rho\,(1-\epsilon^2\sin^2\varphi)^{1/2}}{a\cos\varphi}\,\frac{\partial\theta}{\partial\lambda}$$

$$K = \frac{\rho\,(1-\epsilon^2\sin^2\varphi)^2}{a^2\,(1-\epsilon^2)\cos\varphi}\left(\frac{ds}{d\varphi}\cos\theta - \frac{d\rho}{d\varphi}\right)\frac{\partial\theta}{\partial\lambda}.$$

The characteristics of this projection are that each parallel is the developed base of the cone tangent along the parallel in question; that the parallels are spaced along the central meridian in proportion to their true distances apart along this meridian; and, finally, that the scale is maintained constant along the parallels.

With these conditions we have

$$\rho = \frac{a \cot \varphi}{(1-\epsilon^2 \sin^2\varphi)^{1/2}}$$

$$s = a\,(1-\epsilon^2)\int_0^{\varphi} \frac{d\varphi}{1-\epsilon^2 \sin^2\varphi)^{3/2}} + \frac{a \cot \varphi}{(1-\epsilon^2 \sin^2\varphi)^{1/2}}$$

$$k_D = \frac{\rho\,(1-\epsilon^2 \sin^2\varphi)^{1/2}}{a \cos \varphi}\,\frac{\partial\theta}{\partial\lambda} = 1$$

or.

$$\frac{\partial\theta}{\partial\lambda} = \sin \varphi.$$

By intergration

$$\theta = \lambda \sin \varphi,$$

no constant of integration being added, since θ and λ vanish simultaneously. Since the parallels are represented by circles and since the scale along the parallels is to be maintained constant, the last relation can be obtained by equating an arc of the projection to an arc of the parallel; hence

$$\rho\theta = \frac{a\lambda \cos \varphi}{(1-\epsilon^2 \sin^2 \varphi)^{1/2}}$$

$$\frac{a \cot \varphi}{(1-\epsilon^2 \sin^2 \varphi)^{1/2}}\,\theta = \frac{a\lambda \cos \varphi}{(1-\epsilon^2 \sin^2 \varphi)^{1/2}}$$

or

$$\theta = \lambda \sin \varphi.$$

These values fully determine the projection, and all of the elements can at once be computed.

$$\frac{d\rho}{d\varphi}=-\frac{a\ \mathrm{cosec}^2\ \varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}+\frac{a\epsilon^2\cos^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$=\frac{-a\ \mathrm{cosec}^2\ \varphi+a\ \epsilon^2\ (1+\cos^2\varphi)}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$\frac{ds}{d\varphi}=\frac{a(1-\epsilon^2)}{(1-\epsilon^2\sin^2\varphi)^{3/2}}+\frac{-a\ \mathrm{cosec}^2\ \varphi+a\epsilon^2\ (1+\cos^2\varphi)}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$=\frac{a\ (1-\mathrm{cosec}^2\ \varphi)+a\epsilon^2\cos^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$=\frac{-a\cot^2\varphi+a\epsilon^2\cos^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$=\frac{-a\cot^2\varphi\ (1-\epsilon^2\sin^2\varphi)}{(1-\epsilon^2\sin^2\varphi)^{3/2}}$$

$$=\frac{-a\cot^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}$$

$$\frac{\partial\theta}{\partial\lambda}=\sin\varphi$$

$$\frac{\partial\theta}{\partial\varphi}=\lambda\cos\varphi.$$

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By substituting these values in the differential formulas we obtain

$$\tan\psi = \frac{\dfrac{a\cot\psi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}\lambda\cos\varphi - \dfrac{a\cot\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}\sin\theta}{-\dfrac{a\cot^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}\cos\theta + \dfrac{a\operatorname{cosec}^2\varphi - a\epsilon^2(1+\cos^2\varphi)}{(1-\epsilon^2\sin^2\varphi)^{3/2}}}$$

$$\tan\psi = \frac{\lambda\cos^2\varphi\sin\varphi - \cos^2\varphi\sin\theta}{-\cos^2\varphi\cos\theta + \dfrac{1}{1-\epsilon^2\sin^2\varphi} - \dfrac{\epsilon^2(1+\cos^2\varphi)\sin^2\varphi}{1-\epsilon^2\sin^2\varphi}}$$

$$= \frac{\lambda\sin\varphi - \sin\theta}{\sec^2\varphi - \cos\theta - \dfrac{\epsilon^2\sin^2\varphi}{1-\epsilon^2\sin^2\varphi}}$$

$$= \frac{\theta - \sin\theta}{\sec^2\varphi - \cos\theta - \dfrac{\epsilon^2\sin^2\varphi}{1-\epsilon^2\sin^2\varphi}}$$

$$k_m = \frac{(1-\epsilon^2\sin^2\varphi)^{3/2}}{a(1-\epsilon^2)}\left[-\frac{a\cot^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}}\cos\theta + \frac{a\operatorname{cosec}^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{1/2}} - \frac{a\epsilon^2\cos^2\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}\right]\sec\psi$$

$$= \frac{\sec\psi}{1-\epsilon^2}[-(1-\epsilon^2\sin^2\varphi)\cot^2\varphi\cos\theta + \operatorname{cosec}^2\varphi(1-\epsilon^2\sin^2\varphi) - \epsilon^2\cos^2\varphi]$$

$$= \frac{\sec\psi}{1-\epsilon^2}\left[\operatorname{cosec}^2\varphi - \epsilon^2 - \epsilon^2\cos^2\varphi - \cot^2\varphi(1-\epsilon^2\sin^2\varphi)\left(1-2\sin^2\frac{\theta}{2}\right)\right]$$

$$= \frac{\sec\psi}{1-\epsilon^2}\left[\operatorname{cosec}^2\varphi - \epsilon^2 - \epsilon^2\cos^2\varphi - \cot^2\varphi + \epsilon^2\cos^2\varphi + 2(\cot^2\varphi - \epsilon^2\cos^2\varphi)\sin^2\frac{\theta}{2}\right]$$

$$= \frac{\sec\psi}{1-\epsilon^2}\left[1-\epsilon^2 + 2(\cot^2\varphi - \epsilon^2\cos^2\varphi)\sin^2\frac{\theta}{2}\right]$$

$$k_p = 1$$

$$K = 1 + \frac{2(\cot^2\varphi - \epsilon^2\cos^2\varphi)\sin^2\dfrac{\theta}{2}}{1-\epsilon^2}.$$

When λ is small—that is, when the map is not extended far from the central meridian—an approximation in a series in terms of λ is very convenient. If we neglect θ^5 and higher powers, we obtain

$$\tan\psi=\frac{\theta-\theta+\frac{\theta^3}{6}-\cdots}{\sec^2\varphi-\frac{\epsilon^2\sin^2\varphi}{1-\epsilon^2\sin^2\varphi}-1+\frac{\theta^2}{2}-\cdots\cdot}$$

$$\tan\psi=\frac{\frac{\lambda^3\sin^3\varphi}{6}-\cdots\cdot}{\tan^2\varphi-\frac{\epsilon^2\sin^2\varphi}{1-\epsilon^2\sin^2\varphi}+\frac{\lambda^2\sin^2\varphi}{2}-\cdots\cdot}$$

or approximately

$$\tan\psi=\frac{\frac{\lambda^3\sin^3\varphi}{6}(1-\epsilon^2\sin^2\varphi)}{\tan^2\varphi(1-\epsilon^2\sin^2\varphi)-\epsilon^2\sin^2\varphi}$$

$$=\frac{\lambda^3}{6}\sin\varphi\cos^2\varphi\left(\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}\right)$$

$$=\frac{\lambda^3}{12}\sin 2\varphi\cos\varphi\left(\frac{1-\epsilon^2\sin^2\varphi}{1-\epsilon^2}\right).$$

For smaller values of ψ this can be still further approximated by the form

$$\psi=\frac{\lambda^3}{12}\sin 2\varphi\cos\varphi;$$

for the sphere k_m becomes

$$k_m=\sec\psi\ (\operatorname{cosec}^2\varphi-\cot^2\varphi\cos\theta).$$

To obtain an approximation we let $\sec\psi=1$ and we get

$$k_m=\left(\operatorname{cosec}^2\varphi-\cot^2\varphi+\cot^2\varphi\frac{\theta^2}{2}-\cdots\cdot\right)$$

$$=1+\frac{\lambda^2}{2}\cos^2\varphi.$$

In these approximations λ must of course be expressed in arc.

An approximation for k_m was determined by A. Lindenkohl, of the United States Coast and Geodetic Survey, that is remarkably close to the one given above. This was given in the form

$$E = +0.01\left(\frac{\lambda^\circ \cos\varphi}{8.1}\right)^2,$$

in which λ° is the distance from the central meridian in degrees of longitude. In this form E corresponds to the term $\frac{\lambda^2}{2}\cos^2\varphi$ in the first approximation.

The projection is generally plotted from computed coordinates of the intersections of the meridians and parallels. If we take as origin the interesection of the central meridian and the Equator, we shall have

$$x = \rho \sin\theta$$

$$y = s - \rho\cos\theta.$$

It is the more general practice to compute each parallel with its own origin; that is to say, by using as origin the intersection of the parallel in question with the central meridian.

In this case

$$x = \rho\sin\theta$$

$$y = \rho - \rho\cos\theta = 2\rho\sin^2\frac{\theta}{2} = x\tan\frac{\theta}{2}.$$

The θ angles have to be computed for each parallel that it is desired to map by computation. If these are to be at frequent intervals, it is customary to compute certain coordinates and then to interpolate the intervening values.

The meridional-arc values are tabulated in meters from minute to minute in the Polyconic Projection Tables, Special Publication No. 5, United States Coast and Geodetic Survey. If it is desired to refer the coordinates of the various parallels to a common origin, it is merely necessary to add the meridional-arc values reckoned from the chosen origin to the y values as determined above; this is true because the value of s is given as equal to the meridional arc from the Equator to the parallel of latitude φ, with the addition of the value of ρ in terms of φ. It is customary, however, in the construction of the projection to locate the various origins on the central meridian by their meridional-arc values and then to use the coördinates as originally computed. It is, in general, not necessary to compute the ρ_n values since the tabulated A factor values given in Special Publication No. 8, United States Coast and Geodetic Survey, are connected with them by the relation

$$A=\frac{1}{\rho_n \sin 1''}$$

or

$$\rho_n=\frac{1}{A \sin 1''}.$$

Hence

$$\log \rho_n = \text{colog } A + \text{colog sin } 1''.$$

The logarithms of the A factors in meters are tabulated for each minute of latitude in Special Publication No. 8, as referred to above. With these values as given the formula for ρ becomes

$$\rho=\rho_n \cot \varphi.$$

A great advantage of this projection consists in the fact that a universal table can be computed that can be used anywhere upon the earth's surface. Almost every other projection has special elements that must be determined for each projection. These elements are generally certain arbitrary constants that enter into the formulas for computation. The Mercator projection is another projection that can have a universal table.

If the whole earth's surface were mapped in one continuous projection it would be interesting to know what would be the length of the meridian that forms the outer boundary of the representation and also how many times the area has been increased. Such a projection of the sphere is shown in figure 40. By approximate measurement on a plate of such a projection it was found that the ratio of increase of length of the outer meridian was about 3.2 to 1.

The element of area of the representation being given in the form

$$dS=a^2 \; K \cos \varphi \; d\varphi \; d\lambda$$

for the sphere, we have

$$K=(\text{cosec}^2 \varphi - \cot^2 \varphi \cos \theta),$$

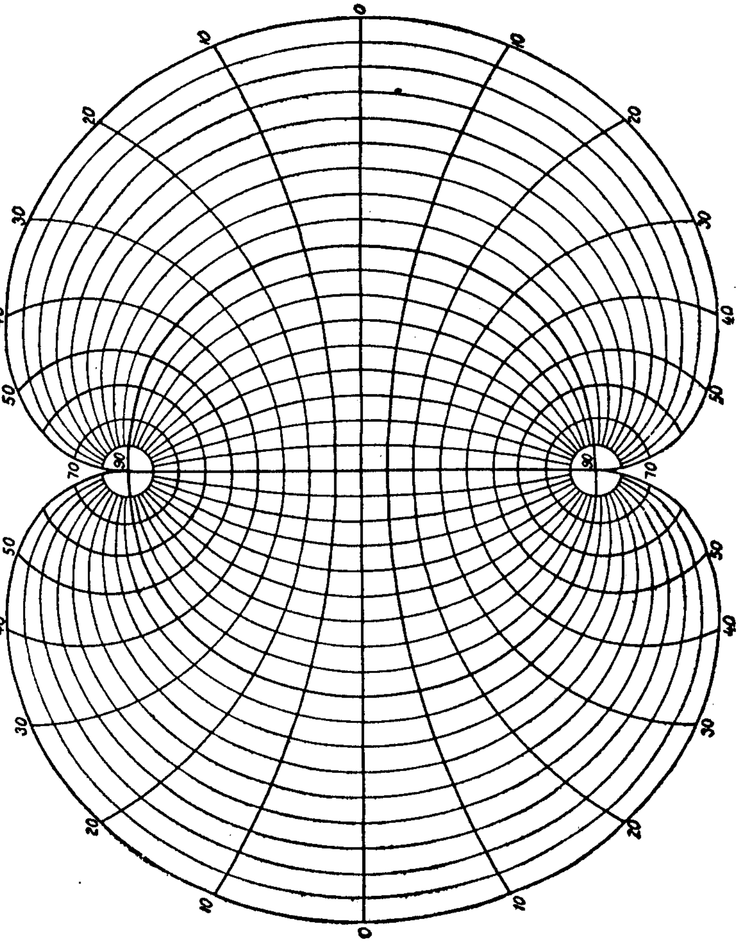

FIG. 40.—Ordinary or American polyconic projection of the entire sphere.

so that

$$dS = a^2 [\operatorname{cosec}^2 \varphi - \cot^2 \varphi \cos(\lambda \sin \varphi)] \cos \varphi \, d\varphi \, d\lambda.$$

One-fourth of the area is given by integrating between the limits $\lambda = 0$ to $\lambda = \pi$ and $\varphi = 0$ to $\varphi = \frac{\pi}{2}$. The total area S is therefore given by the formula

$$S = 4a^2 \int_0^{\frac{\pi}{2}} \cos \varphi \, d\varphi \int_0^{\pi} [\operatorname{cosec}^2 \varphi - \cot^2 \varphi \cos(\lambda \sin \varphi)] \, d\lambda$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} \left[\pi \operatorname{cosec}^2 \varphi - \frac{\cos^2 \varphi}{\sin^3 \varphi} \sin(\pi \sin \varphi) \right] \cos \varphi \, d\varphi$$

$$= 4a^2 [-\pi \operatorname{cosec} \varphi]_0^{\frac{\pi}{2}} - 4a^2 \int_0^{\frac{\pi}{2}} \frac{\cos^3 \varphi}{\sin^3 \varphi} \sin(\pi \sin \varphi) \, d\varphi.$$

In the latter integral let $x = \pi \sin \varphi$

then

$$\cos \varphi \, d\varphi = \frac{dx}{\pi},$$

and

$$-4a^2 \int_0^{\frac{\pi}{2}} \frac{\cos^2 \varphi}{\sin^3 \varphi} \sin(\pi \sin \varphi) \cos \varphi \, d\varphi$$

$$= -4a^2 \int_0^{\pi} \frac{1 - \frac{x^2}{\pi^2}}{\frac{x^3}{\pi^3}} \sin x \frac{dx}{\pi}$$

$$= -4a^2 \int_0^{\pi} \left[\frac{\pi^2}{x^3} - \frac{1}{x} \right] \sin x \, dx$$

$$= 4a^2\pi^2 \left[\frac{1}{2} \frac{\sin x}{x^2} + \frac{1}{2} \frac{\cos x}{x} \right]_0^{\pi} + (2\pi^2 + 4) \, a^2 \int_0^{\pi} \frac{\sin x}{x} \, dx.$$

Hence the value of S becomes

$$S = 4a^2[-\pi \operatorname{cosec} \varphi]_0^{\frac{\pi}{2}} + 2\pi^2 a^2 \left[\frac{\sin x}{x^2} + \frac{\cos x}{x} \right]_0^{\pi}$$

$$+ (2\pi^2 + 4) a^2 \int_0^{\pi} \frac{\sin x}{x} \, dx.$$

The integrated terms assume the form $\infty - \infty$ at the lower limit, and must be evaluated for that point. The last term of the expression is the transcendental function known as the integral sine; it is represented by the series

$$\int_0^x \frac{\sin x}{x}\,dx = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!} - \,\cdots\cdots\cdots$$

The value of this series for $x=\pi$ is approximately 1.852.

To aid in the evaluation of the integrated part, we shall restore the value of $x=\pi \sin \varphi$

$$\left[-4\pi \operatorname{cosec} \varphi + 2\,\frac{\sin(\pi \sin \varphi)}{\sin^2 \varphi} + 2\pi\,\frac{\cos(\pi \sin \varphi)}{\sin \varphi}\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{2 \sin(\pi \sin \varphi) + 2\pi \sin \varphi \cos(\pi \sin \varphi) - 4\pi \sin \varphi}{\sin^2 \varphi}\right]_0^{\frac{\pi}{2}}$$

$$\lim_{\varphi \doteq 0}\left[\frac{2 \sin(\pi \sin \varphi) + 2\pi \sin \varphi \cos(\pi \sin \varphi) - 4\pi \sin \varphi}{\sin^2 \varphi}\right]$$

$$=\lim_{\varphi \doteq 0}\left[\frac{2\pi \cos \varphi \cos(\pi \sin \varphi) + 2\pi \cos \varphi \cos(\pi \sin \varphi) - 2\pi^2 \sin \varphi \cos \varphi \sin(\pi \sin \varphi) - 4\pi \cos \varphi}{2 \sin \varphi \cos \varphi}\right]$$

$$=\lim_{\varphi \doteq 0}\left[\frac{2\pi \cos(\pi \sin \varphi) - \pi^2 \sin \varphi \sin(\pi \sin \varphi) - 2\pi}{\sin \varphi}\right]$$

$$=\lim_{\varphi \doteq 0}\left[\frac{-2\pi^2 \cos \varphi \sin(\pi \sin \varphi) - \pi^2 \cos \varphi \sin(\pi \sin \varphi) - \pi^3 \sin \varphi \cos \varphi \cos(\pi \sin \varphi)}{\cos \varphi}\right]$$

$=0.$

Therefore

$$\begin{aligned} S &= [-4\pi - 2\pi + (2\pi^2 + 4)\ 1.852]\ a^2 \\ &= [-6\pi + (2\pi^2 + 4)\ 1.852]a^2 \\ &= [-6\pi + 23.74 \times 1.852]a^2 \\ &= (-18.85 + 43.97)a^2 \\ &= 25.12\ a^2. \end{aligned}$$

Area of the sphere $=4\pi a^2 = 12.57\ a^2$.

$$\frac{\text{Area of map}}{\text{Area of sphere}} = \frac{25.12}{12.57} = 2 \text{ very nearly.}$$

The area is therefore increased approximately in the ratio of 2 : 1.

TISSOT'S INDICATRIX.

To represent one surface upon another we imagine that each surface is decomposed by two systems of lines into infinitesimal parallelograms, and to each line of the first surface we make correspond one of the lines of the second; then the intersection of two lines of the different systems upon the one surface and the intersection of the two corresponding lines upon the other determine two corresponding points; finally, the totality of the points of the second which correspond to the points of a given figure of the first forms the representation or the projection of this figure. We obtain the different methods of representation by varying the two series of lines which form the graticule upon one of the surfaces.

If two surfaces are not applicable to each other, it is impossible to choose a method of projection such that there is similarity between every figure traced upon the first and the corresponding figure upon the second. On the other hand, whatever the two surfaces may be, there exists an infinity of systems of projection preserving the angles, and, as a consequence, such that each figure infinitely small and its representation are similar to each other. There is also an infinity of others preserving the areas. However, these two classes of projections are exceptions. A method of projection being taken by chance, it will generally happen that the angles will be changed, except, possibly, at particular points, and that the corresponding areas will not have a constant ratio to each other. The lengths will thus be altered.

Let us consider two curves which correspond to each other on the two surfaces. In figure 41 let O and M be two points of the one, O' and M' the corresponding points of the other, and let OT be the tangent at O to the first curve. If the point M approaches the point O indefinitely, the point M' will approach indefinitely the point O', and the ratio of the length of the arc $O'M'$ to that of the arc OM will tend toward a certain limit; this limit is what we call the ratio of lengths at the point O upon the curve OM or in the direction OT. In a system of projection preserving the angles the ratio thus defined has the same value for all directions at a given point; but it varies with the position of this point, unless the two surfaces are applicable to each other. When the representation does not preserve the angles except at particular points, the ratio of lengths at all other points changes with the direction.

The deformation produced around each point is subjected to a law which depends neither upon the nature of the surfaces nor upon the method of projection.

Every representation of one surface upon another can be replaced by an infinity of orthogonal projections each made upon a suitable scale.

We note, first, that there always exists at every point of the first surface two tangents perpendicular to each other, such that the directions which correspond to them upon the second surface also intersect at right angles. In figure 42 let CE and OD be two tangents perpendicular to each other at the point O on the first surface; let $C'E'$ and $O'D'$ be the corresponding tangents to the second.

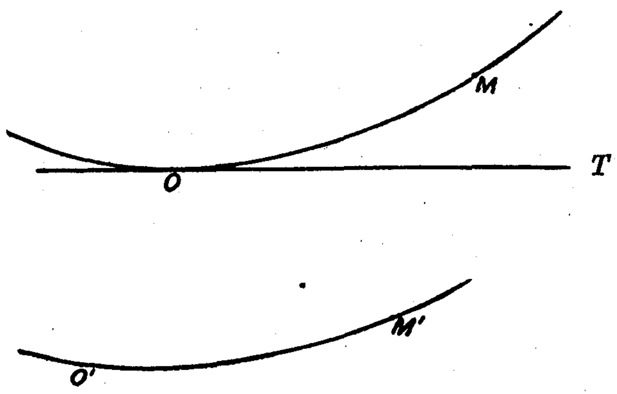

FIG. 41.—A curve and its projection.

Let us suppose that of two angles $C'O'D'$ and $D'O'E'$ the first is acute, and let us imagine that a right angle having its vertex at O turns from left to right around this point in the plane CDE, starting from the position COD and arriving at the position DOE. The corresponding angle in the plane tangent at O to the second surface will first coincide with $C'O'D'$ and will be acute; in its final position it will coincide with $D'O'E'$, and will be obtuse; within the interval it will have passed through a right angle. Therefore, there exists a system of two tangents satisfying the condition stated, except at certain singular points. From this property we conclude that in every system of representation there is upon the first of the two surfaces a system of two series of orthogonal curves whose projections upon the second surface are also orthogonal. The two surfaces are thus divided into infinitesimal rectangles which correspond the one to the other.

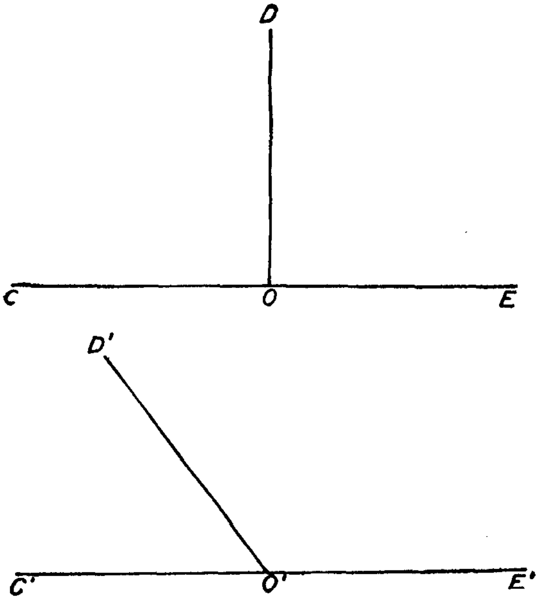

FIG. 42.—Two tangents at right angles and their projections.

This fact being established, let M be a point in figure 43 infinitely near to O upon the first surface and let $OPMQ$ be that one of the infinitesimal rectangles which we have just described that has OM as a diagonal. Let us move

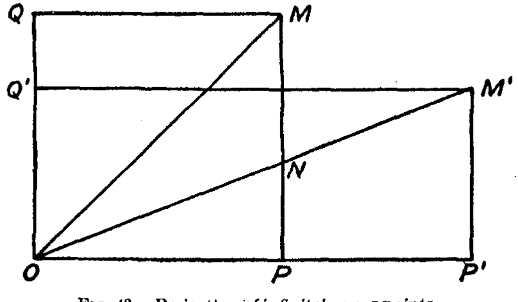

FIG. 43.—Projection of infinitely near points.

the second surface and place it so that the projections of the sides OP and OQ fall upon the sides themselves prolonged if necessary; then let $O'P'M'Q'$ be the rectangle corresponding to $OPMQ$; let us call N the point of intersection of the lines OM' and PM. We can consider this point as the orthogonal projection of the point that M would be if we should turn the plane of the rectangle $OPMQ$ through a suitable angle with OP as an axis. But this angle, which depends only upon the ratio of the two lines NP and MP, is the same whatever point M may be; for denoting, respectively, by c and d the ratios of the lengths in the directions OP and OQ—that is, on setting

$$\frac{OP'}{OP}=c \text{ and } \frac{OQ'}{OQ}=d,$$

we should have

$$\frac{NP}{M'P'}=\frac{OP}{OP'}=\frac{1}{c}, \text{ and } \frac{MP}{M'P'}=\frac{OQ}{OQ'}=\frac{1}{d},$$

and, consequently,

$$\frac{NP}{MP}=\frac{d}{c}.$$

Thus if M moves on an infinitesimal curve traced around O, we shall obtain the locus described by N by turning this curve through a certain angle around OP as an axis and by then projecting orthogonally upon the plane tangent at O. On the other hand, we have

$$\frac{OM'}{ON}=\frac{OP'}{OP}=c,$$

so that the locus of the points M' is homothetic to that of the points N; the center of similitude is O, and the ratio of similitude has the value c. The representation of the infinitesimal figure described by the point M is then in reality an orthogonal projection of this figure made on a suitable scale, or the figure formed by the points N and that formed by the points M' are formed by parallel sections of the same cone. Any geographic map can, therefore, be considered as produced by juxtaposition of orthogonal projections of all the surface elements of the country, provided that we vary from one element to the other both the scale of the reduction and the position of the element with respect to the plane of the map.

Of all the right angles which are formed by the tangents at the point O those of the lines OP and OQ and their prolongations are the only ones one side of which remains parallel to the tangent plane after the rotation which was described above; these are the only ones then which are projected into right angles. We can now state an addition to the proposition which has just been proved, and we can express the whole in the following form: At every point of the surface which we wish to represent there are two perpendicular tangents, and, if the angles are not preserved, there are only two, such that those which correspond to them upon the other surface also intersect at right angles. So that, upon each of the two surfaces, there exists a system of orthogonal trajectories, and, if the method of representation does not preserve the angles, there exists only one of them the projections of which upon the other surface are also orthogonal.

We shall denote, by first and second principal tangents, the two perpendicular tangents the angle between which is not altered by the projection. We shall continue to denote, respectively, by c and d the ratio of lengths in the directions of these tangents, and we shall suppose that c is greater than d.

If the infinitesimal curve drawn around the point O is a circumference of which O is the center, the representation of this curve will be an ellipse the axes of which will fall upon the principal tangents, and these will have the values $2c$ and $2d$, the radius of the circle being taken as unity. This ellipse constitutes at each point a sort of indicatrix of the system of projection.

In place of projecting orthogonally the circumference, the locus of the points M in figure 43, which gives the ellipse the locus of the points N, then increasing this in the ratio of c to unity, which gives the locus of the points M', we can perform the two operations in the inverse order. We should then in figure 44 obtain the point M' of the elliptic indicatrix which corresponds to a given point M of the circle by prolonging the radius OM until it meets at R the circumference described upon the major axis as diameter, and then by dropping a perpendicular from R upon OA, the semimajor axis, and, finally, by reducing this perpendicular RS, starting from its foot S in the ratio of d to c. The point M' thus determined will be the required point.

In figure 44 let us draw OM', and let us call, respectively, u and u' the angles AOM and AOM' which correspond upon the two surfaces. Inasmuch as the second is the

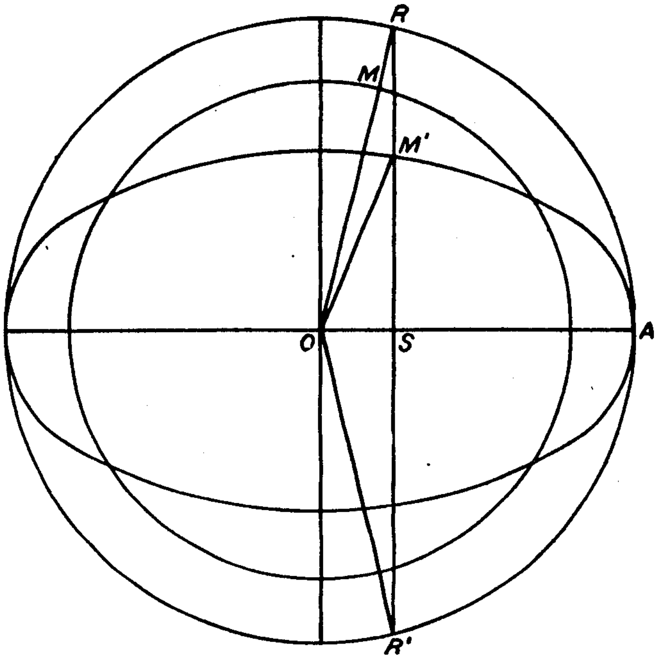

FIG. 44.—Tissot's indicatrix.

smaller of the two, we see that the representation diminishes all the acute angles one side of which coincides with the first principal tangent. Between u and u' we have, moreover, the relation

$$\tan u' = \frac{d}{c} \tan u,$$

since

$$\tan u = \frac{RS}{OS}$$

$$\tan u' = \frac{M'S}{OS},$$

and, consequently,

$$\tan u' = \frac{M'S}{RS} \tan u = \frac{d}{c} \tan u.$$

Let us prolong the line RS to R' and then join O and R'. The two triangles ORM' and $OR'M'$ give

$$\sin (u - u') = \frac{c-d}{c+d} \sin (u + u'),$$

which is obtained by equating two expressions for the ratio of the areas of the triangles. The same relation follows at once analytically from the tangent relation first given. The angle u increasing from zero to $\frac{\pi}{2}$, its alteration $u-u'$ increases from zero up to a certain value ω, then decreases to zero. The maximum is produced at the moment when the sum $u+u'$ becomes equal to $\frac{\pi}{2}$. Let U and U' be the corresponding values of u and u'. We find from the tangent formula that the following are their values:

$$\tan U = \frac{\sqrt{c}}{\sqrt{d}} \text{ and } \tan U' = \frac{\sqrt{d}}{\sqrt{c}}.$$

The quantity ω can be computed by any one of the formulas

$$\sin \omega = \frac{c-d}{c+d},$$

$$\cos \omega = \frac{2\sqrt{cd}}{c+d},$$

$$\tan \omega = \frac{c-d}{2\sqrt{cd}},$$

$$\tan \frac{\omega}{2} = \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}},$$

$$\tan\left(\frac{\pi}{4}+\frac{\omega}{2}\right) = \frac{\sqrt{c}}{\sqrt{d}} \text{ and } \tan\left(\frac{\pi}{4}-\frac{\omega}{2}\right) = \frac{\sqrt{a}}{\sqrt{c}},$$

From the last two equations since the sum of U and U' is equal to $\frac{\pi}{2}$ and their difference is equal to ω, we have

$$U = \frac{\pi}{4}+\frac{\omega}{2}, \text{ and } U' = \frac{\pi}{4}-\frac{\omega}{2}.$$

From the tangent relation we see that when we change u to $\frac{\pi}{2}-u'$ it is sufficient to change u' to $\frac{\pi}{2}-u$. The same substitutions being effected in $u+u'$, give for result $\pi-(u+u')$, so that the sine formula shows that the value of the alteration is not changed. Thus of two angles which are found to be changed by equal quantities each is the complement of the projection of the other.

If we wish to calculate directly the alteration which any given angle u is subject to, we should make use of one of the two formulas

$$\tan(u-u')=\frac{(c-d)\tan u}{c+d\tan^2 u}$$

$$\tan(u-u')=\frac{(c-d)\sin 2u}{c+d+(c-d)\cos 2u},$$

which follow immediately from the previous formulas by easy analytical reductions.

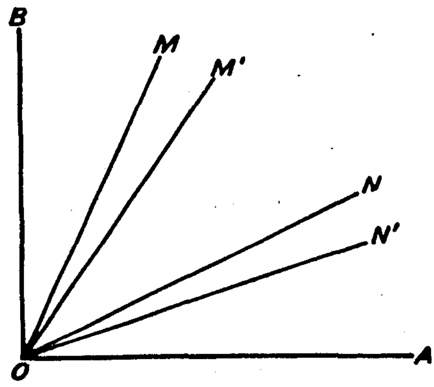

FIG. 45.—Angular change in projection, first case.

Let us now consider an angle MON in figures 45 and 46, which has for sides neither one nor the other of the principal tangents OA and OB. We can suppose the two directions OM and ON to the right of OB and the one of them OM above OA. According as the other ON will be above OA (fig. 45) or below OA (fig. 46), we should calculate the corresponding angle $M'ON'$ by taking the difference or the sum of the angles AOM' and AON', which would be given by the formula stated above. The alteration $MON-M'ON'$ would also in the first case be the difference, and in the second case would be the sum of the alterations of the angles AOM and AON. When the angle AON (fig. 45) is equal to the angle BOM', we know that its alteration is the same as that of the angle AOM, so that the angle MON will then be reproduced in its true magnitude by the angle $M'ON'$. Thus to every given direction we can join another, and only one other, such that their angle is preserved in the projection. However, the second direction will coincide with the first when it makes with OA the angle which we have denoted by U.

The angle the most altered is that which this direction forms with the point symmetric to it with respect to OA; it is represented upon the projection by its supplement. The maximum alteration thus produced is equal to 2ω.

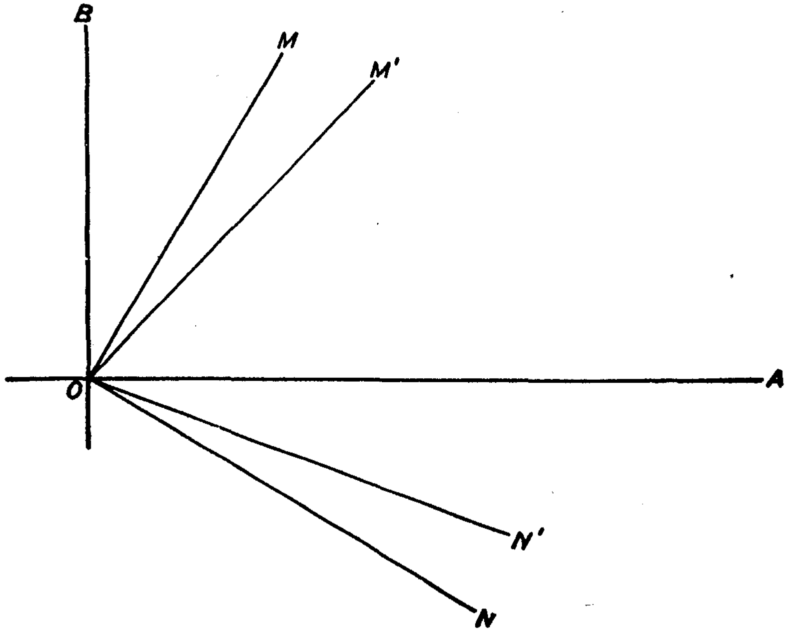

FIG. 46.—Angular change in projection, second case.

This can never be found applicable to two directions that are perpendicular to each other.

The length OM in figure 44 having been taken as unitv, the ratio of lengths in the direction OM is measured by OM'. Let us denote by r this ratio; we can calculate it by means of one of the formulas

$$r \cos u' = c \cos u$$

$$r \sin u' = d \sin u$$

or

$$r^2 = c^2 \cos^2 u + d^2 \sin^2 u.$$

We have also among r, u, and the alteration $u-u'$ of the angle u the relation

$$2r \sin (u-u') = (c-d) \sin 2u,$$

which expresses that, in the triangle ORM', the sines of two of the angles are to each other as the sides opposite.

The maximum and the minimum of r correspond to the principal tangents and are, respectively, c and d.

Let us call r and r_1 the ratios of lengths in two directions at right angles to each other and let ψ be the alteration that the right angle formed by these two directions is subjected to. From the well-known properties of conjugate diameters in the ellipse we have

$$r^2 + r^2_1 = c^2 + d^2$$

$$rr_1 \cos \psi = cd$$

or, in terms of the scales along the parallels and meridians, the semiaxes are given by the equations

$$c^2 + d^2 = k^2_{\mathrm{m}} + k^2_{\mathrm{p}}$$

$$cd = k_{\mathrm{m}} k_{\mathrm{p}} \cos \psi.$$

For all angles not changed by the projection the product of the ratios of lengths along their sides is the same. In fact, let OA (fig. 45) and OB be the two principal tangents; let MON be any angle whatever; and let $M'ON'$ be its projection. Let us denote by r' and r'' the ratios of lengths along OM and ON and by u and u' the angles AOM and AOM'.
Then

$$r' \cos u' = c \cos u$$

$$r'' \sin \angle AON' = d \sin \angle AON;$$

but we know that, when the alteration $MON - M'ON'$ is zero, the angle AON is the complement of u' and the angle AON' is the complement of u; so that the second equation gives

$$r'' \cos u = d \cos u'.$$

By multiplying these equations member by member we obtain

$$r' \, r'' = cd,$$

which proves the statement. It results from this property that the ratio of lengths in the two directions the angle of which undergoes the maximum alteration is equal to $\sqrt{cd}$; for the angle which is not altered and which has for side one of these two lines reduces to zero, and it has the same line for second side, so that $r' = r'' = \sqrt{cd}$.

In the ordinary, or American, polyconic projection we have

$$k_m = K \sec \psi$$

$$k_p = 1.$$

Hence

$$c^2 + d^2 = 1 + K^2 \sec^2 \psi$$

$$cd = K$$

or

$$c = \frac{1}{2}\left(\sqrt{1 + 2K + K^2 \sec^2 \psi} + \sqrt{1 - 2K + K^2 \sec^2 \psi}\right)$$

$$d = \frac{1}{2}\left(\sqrt{1 + 2K + K^2 \sec^2 \psi} - \sqrt{1 - 2K + K^2 \sec^2 \psi}\right).$$

By means of these formulas the semiaxes could be computed for any point on a continuous map of the sphere or of the ellipsoid if it is desired to take into account the eccentricity of the generating ellipse. As a good approximation for projections extending no farther from the central meridian than is usually the case, we may take

$$c = K \sec \psi = k_m$$

$$d = 1.$$

The effect of this approximation becomes barely perceptible in the third place of decimals for $\lambda = 45°$, so that the approximation is exceedingly good for projections of less extent in longitude.

With this approximation for the semiaxes it only remains to determine the angles through which the axes of coordinates should be turned to make them coincide with the directions of the axes of the ellipse. The angle through which the axes must be turned to make the x axis be tangent to the parallel at the point we shall denote by ξ; its value is given by the formula

$$\xi = \lambda \sin \varphi.$$

If γ is the angle between the conjugate axes, and if η is the angle between the major axis and the conjugate axis of x, we have from the theory of conjugate axes

$$\tan \eta \tan (\eta+\gamma) = -\frac{d^2}{c^2}.$$

By developing this expression we get

$$\tan \gamma = -\frac{d^2 + c^2 \tan^2 \eta}{(c^2 - d^2) \tan \eta};$$

but

$$\gamma = \frac{\pi}{2} + \psi.$$

Therefore

$$\cot \psi = \frac{d^2 + c^2 \tan^2 \eta}{(c^2 - d^2) \tan \eta}.$$

By solving this for $\tan \eta$ we get

$$\tan \eta = \frac{c^2 - d^2}{2c^2} \cot \psi - \sqrt{\frac{(c^2 - d^2)^2}{4c^4} \cot^2 \psi - \frac{d^2}{c^2}},$$

from which η can be determined. The angle between the minor axis and the conjugate minor axis is equal to $\eta + \psi$.

If ξ is counted positive for points east of the central meridian, the axes must be turned through the angle $\xi - \eta - \psi$. We shall then have

$$x' = x \cos (\xi - \eta - \psi) + y \sin (\xi - \eta - \psi)$$

$$y' = -x \sin (\xi - \eta - \psi) + y \cos (\xi - \eta - \psi).$$

For points west of the central meridian $\xi - \eta - \psi$ can be considered negative in the transformation formulas.

If geodetic azimuths are given, they should first be referred to the parallel as initial line; that is, they should be reckoned from the east around counterclockwise through north. If the $\eta + \psi$ angle is added to these azimuths we shall obtain the angle u. Since the elliptic indicatrix has the minor axis in the direction of the initial line, we have

$$\tan u' = \frac{c}{d} \tan u.$$

The ratio of scale is given by the equations

$$r \sin u' = c \sin u$$

or

$$r \cos u' = d \cos u.$$

If it is desired to determine the azimuth of the line from a point to a near point from their coordinates on the map, we have approximately

$$\tan u'' = \frac{y'}{x'},$$

x' *and* y' being the coordinates of one of the points with respect to the other as origin in the transformed system; that is, after the axes have been turned to make the axes of the ellipse coincide with the axes of coordinates. Then

$$\tan u = \frac{d}{c} \tan u''.$$

The azimuth reckoned from east to north is given by $\alpha = u + \xi - \eta - \psi$.

If the map does not extend more than 5 degrees beyond the central meridian, the angle η can be considered zero and the reductions become comparatively simple.

The theory of the elliptic indicatrix can be applied to any projection that has a change of scale at any point for different directions; that is, for any projection that is not conformal. It has been applied only to the ordinary polyconic projection in this publication, since for practical purposes that one is probably the most important of the nonconformal projections treated under the polyconic projections.

The appended tables of the elements of the ordinary polyconic projection are taken from Tissot's work. They are computed for the sphere but can safely be used for ordinary computation work. If more exact results are desired the computations should be made from the first by employment of the spheroidal formulas.

TABLES OF ELEMENTS OF THE ORDINARY OR AMERICAN POLYCONIC PROJECTION.

Values of ψ.

φ	λ						
	0°	15°	30°	45°	60°	75°	90°
°	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0	0 00	0 00	0 00	0 00	0 00	0 00	0 00
15	0 00	0 02	0 18	0 52	1 45	2 52	4 09
30	0 00	0 04	0 28	1 23	2 53	4 50	7 08
45	0 00	0 04	0 27	1 24	2 59	5 10	7 51
60	0 00	0 02	0 17	0 55	2 01	3 38	5 46
75	0 00	0 01	0 05	0 17	0 39	1 13	2 00
90	0 00	0 00	0 00	0 00	0 00	0 00	0 00

Values of k_m.

φ	λ						
	0°	15°	30°	45°	60°	75°	90°
°							
0	1.000	1.034	1.137	1.308	1.548	1.857	2.234
15	1.000	1.032	1.128	1.287	1.509	1.794	2.141
30	1.000	1.026	1.102	1.229	1.404	1.626	1.893
45	1.000	1.017	1.068	1.151	1.264	1.404	1.571
60	1.000	1.009	1.034	1.074	1.129	1.195	1.270
75	1.000	1.002	1.009	1.020	1.034	1.050	1.069
90	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Values of 2ω.

φ	λ						
	0°	15°	30°	45°	60°	75°	90°
°	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0	0 00	1 55	7 21	15 20	24 50	34 55	44 51
15	0 00	1 48	6 53	14 26	23 29	33 09	42 49
30	0 00	1 27	5 36	11 52	19 33	28 01	36 43
45	0 00	0 58	3 45	8 09	13 42	20 04	26 52
60	0 00	0 29	1 54	4 11	7 13	10 50	14 51
75	0 00	0 08	0 31	1 09	1 57	3 04	4 18
90	0 00	0 00	0 00	0 00	0 00	0 00	0 00

Values of c.

φ	λ						
	0°	15°	30°	45°	60°	75°	90°
°							
0	1.000	1.034	1.137	1.308	1.548	1.857	2.234
15	1.000	1.032	1.128	1.287	1.510	1.795	2.143
30	1.000	1.026	1.102	1.229	1.405	1.629	1.899
45	1.000	1.017	1.068	1.152	1.266	1.410	1.580
60	1.000	1.009	1.034	1.075	1.131	1.200	1.280
75	1.000	1.002	1.009	1.020	1.034	1.053	1.073
90	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Values of d.

φ	λ						
	0°	15°	30°	45°	60°	75°	90°
°							
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
15	1.000	1.000	1.000	1.000	0.999	0.998	0.997
30	1.000	1.000	1.000	0.999	0.997	0.994	0.989
45	1.000	1.000	1.000	0.999	0.996	0.992	0.984
60	1.000	1.000	1.000	0.999	0.997	0.993	0.987
75	1.000	1.000	1.000	1.000	1.000	0.998	0.995
90	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Values of K.

φ	λ						
	0°	15°	30°	45°	60°	75°	90°
°							
0	1.000	1.034	1.137	1.308	1.548	1.857	2.234
15	1.000	1.032	1.128	1.287	1.508	1.792	2.135
30	1.000	1.026	1.102	1.228	1.402	1.620	1.879
45	1.000	1.017	1.068	1.150	1.262	1.399	1.556
60	1.000	1.009	1.034	1.074	1.128	1.192	1.264
75	1.000	1.002	1.009	1.020	1.034	1.050	1.068
90	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TRANSVERSE POLYCONIC PROJECTION.

If the earth is considered as a sphere, there is no reason why the tangent cones that determine the projection should necessarily be tangent to the earth along parallels of latitude and should have their apexes in the axis of the earth. Any diameter prolonged might just as well serve as the line of apexes, and then the cones would be tangent

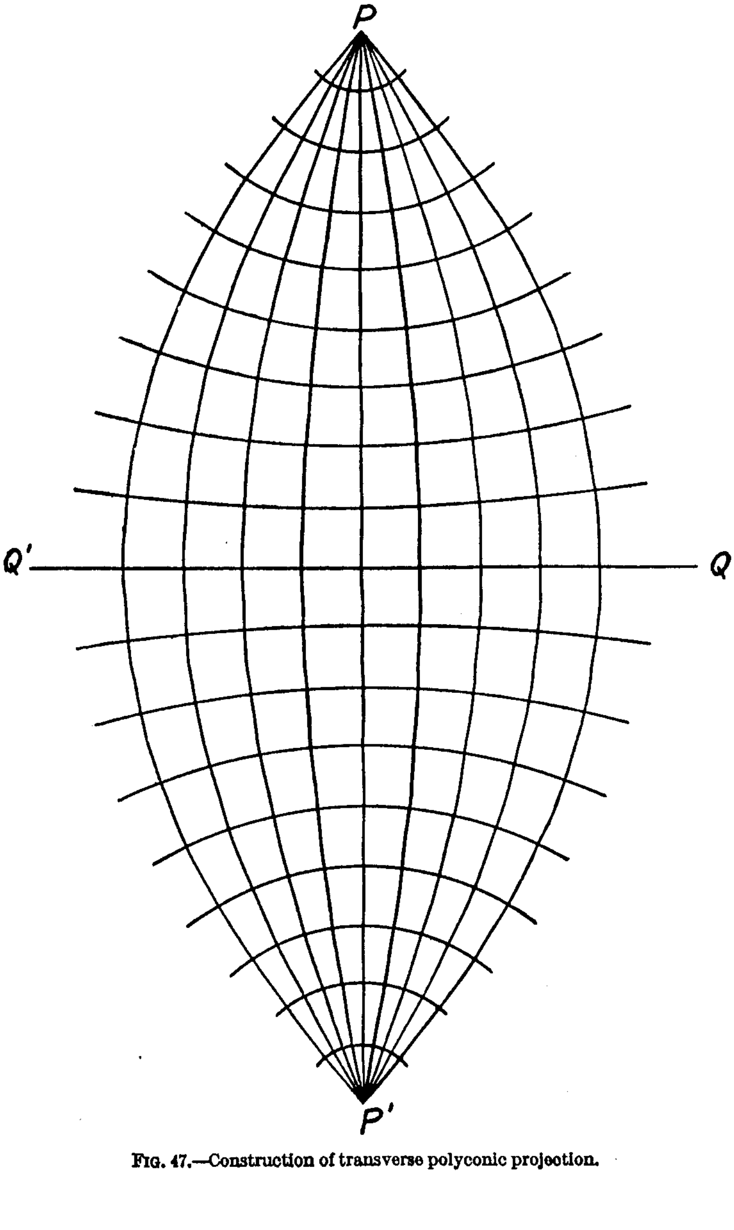

FIG. 47.—Construction of transverse polyconic projection.

along a system of small circles that would correspond to the parallels of latitude in the ordinary projection. Some great circle of the earth would correspond to the central meridian. By this scheme a map of great extent in longitude could be constructed without the usual trouble due to the longitudinal scale error. The error in scale in this case would appear along the great circles of the projection that correspond to the meridians in the ordinary projection.

The most feasible plan for the construction of such a projection would seem to be the following: Since such a map would, no doubt, be planned for a large section of the earth's surface, the ellipsoidal features would be negligible, and the ordinary tables could be employed, just as if they had been computed for the sphere. With these tables construct a projection in the usual way. After it is constructed turn the projection so that the poles fall

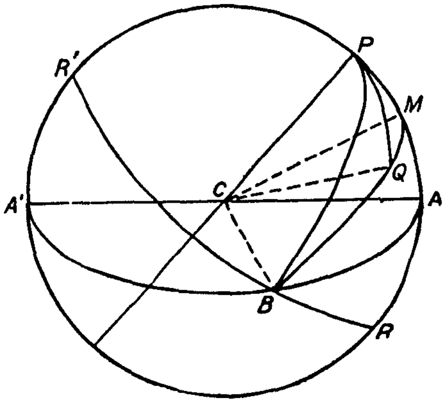

FIG. 48.—Transformation triangle for transverse polyconic projection.

upon the Equator and then by means of the formulas for the transformation of coordinates the intersections of the parallels and meridians can be computed in terms of the parameters that correspond to latitude and longitude on the ordinary projection. After the projection has been constructed and turned into the new position, the φ and λ values become what we shall denote by ψ and η. The values in degrees will be just the same as before, but they will have the new designation. Figure 47 represents such a scheme in outline. PP' is the central meridian, and QQ' represents the Equator in the projection as constructed. The projection is now turned and PP' becomes the chosen great circle, and QQ' becomes a meridian on the map; ψ

is measured to the right and left of QQ' and η is measured up and down from PP'.

In the figure 48 let P be the pole and let RBR' be the Equator and also let ABA' be the great circle that we wish to make correspond to the central meridian of the ordinary projection. BR and BA are quadrants, and AR measures the inclination of the given great circle to the plane of the Equator, and PMA becomes the Equator on the transverse projection. Let Q be the intersection that we wish to compute. We have $BQ=90°-\psi$; $QP=90°-\varphi$; $BP=90°$; $\angle BPQ=90°-\lambda$; $\angle ABR=\beta$; $\angle PBQ=90°-(\beta+\eta)$. By the trigonometry of the spherical triangle we obtain from these results the relations

$$\sin \psi = \sin \lambda \cos \varphi$$

$$\cos \psi \cos (\beta+\eta) = \cos \lambda \cos \varphi$$

$$\cos \psi \sin (\beta+\eta) = \sin \varphi,$$

or by combining the last two equations

$$\tan (\beta+\eta) = \sec \lambda \tan \varphi.$$

β is a constant the value of which is known from our choice of the great circle that is to form the center of the map; it is the value of the parallel of latitude to which the great circle is tangent.

By use of the equations

$$\sin \psi = \sin \lambda \cos \varphi$$

and

$$\tan (\beta+\eta) = \sec \lambda \tan \varphi$$

we can compute the ψ and η values for any intersections of the parallels and meridians that we may wish to determine. The points are then plotted on the projection as originally constructed; a smooth curve drawn through the points corresponding to a constant value of φ will represent the parallel of latitude φ, and, similarly, the smooth curve through the points corresponding to a constant value of λ will represent the meridian of longitude λ. After these curves are drawn, the original projection lines can be erased, and then only the meridians and parallels will appear on the projection. The folding plate represents such a projection of the North Pacific Ocean, showing the eastern coast of Asia in its relation to North America.

The projection was constructed by Mr. Chas. H. Deetz, cartographer of the United States Coast and Geodetic Survey, with the central great circle approximately the one joining San Francisco and Manila. Another projection of this kind was constructed by Mr. A. Lindenkohl, cartographer in the United States Coast and Geodetic Survey, consisting of a map of the United States based on the great circle intersecting the 95° meridian at 39° of latitude. In this projection $\beta = 39°$ and λ is reckoned from the 95° meridian.

The meridian that corresponds to the Equator in the projection as first constructed is an axis of symmetry for the map, so that the coordinates of the intersections need to be computed only for one-half of the map if the Equator of the original projection corresponds to one of the meridians that appear on the map, so that for each value of $+\lambda$ we may have another intersection for $-\lambda$, with the latitude the same in both cases. In the one constructed by Mr. Lindenkohl for the United States the meridians were constructed for every 5° of longitude, so that the meridian of 95° appeared upon the projection. If 94° had been chosen in place of 95°, we should have had a meridian to compute for a λ of 4° E. and one for a λ of 6° W., and so on for the others.

In the construction of the projection of which the folding plate is a copy the central great circle is the one that is tangent to the parallel of 45° of latitude at the point of its intersection with the 160° meridian west of Greenwich. Mr. Deetz (in the construction of his projection) computed the intersections of his original projection after it was turned into the new position in terms of latitude and longitude and then interpolated the even values of intersections on this projection. From the original three equations we obtain

$$\tan \lambda = \sec (\beta + \eta) \tan \psi$$

$$\sin \varphi = \sin (\beta + \eta) \cos \psi.$$

In the case under consideration $\beta = 45°$ and $\beta + \eta$ is the latitude of the intersection of any given great circle with the 160° meridian. $\beta + \eta$ is, therefore, constant for any given great circle. The amount of computation required is about the same for either method of procedure.

PROJECTION FOR THE INTERNATIONAL MAP ON THE SCALE OF 1 : 1 000 000.

The projection adopted for this map is a modified polyconic projection devised by M. Lallemand. The scale is slightly reduced along the central meridian, thus bringing the parallels closer together in such a way that the meridians 2° on each side of the center are made true to scale. Up to 60° of latitude the separate sheets are to include 6° of longitude and 4° of latitude. From latitude 60° to the pole the sheets are to include 12° of longitude; that is, two sheets are to be united into one. The top and bottom parallel of each sheet are constructed in the usual way; that is, they are circles constructed from centers lying on the central meridian, but not concentric. These two parallels are then truly divided. The meridians are straight lines joining the corresponding points of the top and bottom parallels. Any sheet will then join exactly along its margins with its four neighboring sheets. The correction to the length of the central meridian is very slight, amounting to only 0.01 inch at the most, and the change is almost too slight to be measured on the map.

In the resolutions of the International Map Committee, London, 1909, it is not stated how the meridians are to be divided; but, no doubt, an equal division of the central meridian was intended. Through these points circles could be constructed with centers on the central meridian and with radii equal to $\rho_n \cot \varphi$. In practice, however, an equal division of the straight-line meridians between the top and bottom parallels could scarcely be distinguished from the points of parallels actually constructed by means of radii or by coordinates of their intersections with the meridians. The provisions also fail to state whether, in the sheets covering 12° of longitude instead of 6°, the meridians of true length shall be 4° instead of 2° on each side of the central meridian; but such was, no doubt, the intention. In any case, the sheets would not exactly join together along the parallel of 60° of latitude.

The appended tables give the corrected lengths of the central meridian from 0° to 60° of latitude and the coordinates for the construction of the 4° parallels within the same limits. Each parallel has its own origin; i. e., where the parallel in question intersects the central meridian. The central meridian is the Y axis and a perpendicular to it at the origin is the X axis; the first table, of course, gives the distance between the origins. The y values are small in every instance. In terms of the parameters used throughout this publication these values are given by the expressions

$$x = \rho_n \cot \varphi \sin (\lambda \sin \varphi)$$

$$y = \rho_n \cot \varphi [1 - \cos (\lambda \sin \varphi)] = 2\rho_n \cot \varphi \sin^2\left(\frac{\lambda \sin \varphi}{2}\right).$$

In the tables as published in the International Map Tables, the x coordinates were computed by use of the erroneous formula

$$x = \rho_n \cot \varphi \tan (\lambda \sin \varphi).$$

The resulting error in the tables is not very great and is practically almost negligible. The tables as given below are all that are required for the construction of all maps up to 60° of latitude. This fact in itself shows very clearly the advantages of the use of this projection for the purpose in hand.

A discussion of the numerical properties of this map system is given by M. Ch. Lallemand in the Comptes Rendus, tome 153, page 559. He finds that the maximum error of scale of a meridian is 1 part in 1270, which corresponds to 0.35 mm. in the height, 0.44 m., of the sheet. The maximum error of scale of a parallel is 1 part in 3200, and the greatest alteration of azimuth is 6 minutes of arc. These errors are much smaller than those occasioned by the expansion and contraction of the sheet due to atmospheric conditions.

TABLES FOR THE PROJECTION OF THE SHEETS OF THE INTERNATIONAL MAP OF THE WORLD.

[Scale 1:1 000 000. Assumed figure of the earth: a=6378.24 km.; b=6356.56 km.]

TABLE 1.—*Corrected lengths on the central meridian, in millimeters*

Latitude.	Natural length.	Correction.	Corrected length.
° °			
From 0 to 4	442.27	−0.27	442.00
4 to 8	442.31	.27	442.04
8 to 12	442.40	.26	442.14
12 to 16	442.53	.25	442.28
16 to 20	442.69	.24	442.45
20 to 24	442.90	.23	442.67
24 to 28	443.13	.22	442.91
28 to 32	443.30	.20	443.19
32 to 36	443.68	.18	443.50
36 to 40	443.98	.17	443.81
40 to 44	444.29	.15	444.14
44 to 48	444.60	.13	444.47
48 to 52	444.92	.11	444.81
52 to 56	445.22	.09	445.13
56 to 60	445.52	−.08	445.44

TABLE 2.—*Coordinates of the intersections of the parallels and the meridians, in millimeters.*

Latitude.	Coordinates.	Longitude from central meridian.		
		1°	2°	3°
°				
0	x	111.32	222.64	333.96
	y	0.00	0.00	0.00
4	x	111.05	222.10	333.16
	y	0.07	0.27	0.61
8	x	110.25	220.49	330.74
	y	0.13	0.54	1.21
12	x	108.91	217.81	326.73
	y	0.20	0.79	1.78
16	x	107.04	214.08	321.13
	y	0.26	1.03	2.32
20	x	104.65	209.31	313.98
	y	0.31	1.25	2.81
24	x	101.76	203.52	305.31
	y	0.36	1.45	3.25
28	x	98.37	196.75	295.15
	y	0.40	1.61	3.63
32	x	94.50	189.01	283.56
	y	0.44	1.75	3.93
36	x	90.17	180.36	270.59
	y	0.46	1.85	4.16
40	x	85.40	170.82	256.29
	y	0.48	1.92	4.31
44	x	80.21	160.45	240.73
	y	0.49	1.95	4.38
48	x	74.63	149.29	224.00
	y	0.48	1.94	4.36
52	x	68.69	137.40	206.16
	y	0.47	1.89	4.25
56	x	62.40	124.83	187.31
	y	0.45	1.81	4.06
60	x	55.81	111.64	167.52
	y	0.42	1.69	3.80

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