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E. LESTER JONES, Superintendent

A STUDY OF MAP PROJECTIONS IN GENERAL

BY

OSCAR S. ADAMS

Geodetic Computer

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PREFACE.

In this publication an attempt has been made to treat in simple form some of the fundamental ideas that underlie the subject of map projections in general. There has been no intention to develop any phase of the subject at any length, but merely to give briefly some suggestions under the different headings that, it is hoped, may be found helpful to those who wish to get an understanding of the subject.

In the preparation of the publication considerable reference was made to the work on Map Projections, by Arthur R. Hinks, published by the Cambridge University Press. That work is well suited to the needs of cartographers who wish to have an account in the English language of most of the projections in common use without entering too deeply into the mathematical side of the subject.

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A STUDY OF MAP PROJECTIONS IN GENERAL.

By Oscar S. Adams,

Geodetic Computer, U. S. Coast and Geodetic Survey.

GENERAL STATEMENT.

The difficulty that one has to contend with in making a map of any portion of the earth's surface arises from the fact that the surface of the earth is spherical in shape, and hence it is nondevelopable; that is, it can not be spread out in a plane without some stretching, some tearing, or some folding. If a spherical surface were used upon which to make the map, any extent of surface could be mapped with an unvarying scale. Globes are, of course, representations of the entire surface of the earth, and if carefully made should have a scale constant at all points. Unfortunately, globes of any size are very unwieldy, and such maps are very hard to use in practical work. Sections of spherical surface for mapping parts of the earth's surface would be open to the same objection. In fact, it is much better to use a plane surface for the map and admit the defects rather than to try to avoid them and consequently to fall into greater difficulties.

Points upon the earth's surface are located by their latitude and longitude, or, in other words, their positions are referred to the meridians and parallels. If we can determine a system of lines in the plane to represent the meridians and parallels, all points upon the surface could be plotted with reference to these lines. Any lines drawn arbitrarily could be used for this purpose; but in practice it is the custom to employ some orderly arrangement for these lines. The aim, then, is to determine an orderly arrangement of lines in the plane that will give a one-to-one correspondence between these lines and the meridians and parallels. This orderliness is generally expressed in the terms of some mathematical formula, and, in fact, almost every projection that is ever used can be stated in such terms.

The number of ways in which this orderly arrangement can be determined is infinite. It could not be expected that all of these methods would be equally good or that any one would be the best for all purposes. It is well, then, to make some study of the different things that should be considered in a projection. In entering upon this study we find that there are, in the main, four things to be considered:

1. The accuracy with which a projection represents the scale along the meridians and parallels.
2. The accuracy with which it represents areas.
3. The accuracy with which it represents the shape of the features of the area in question.
4. The ease with which the projection can be constructed.

REPRESENTATION OF SCALE.

The scale of a map in a given direction at any point is the ratio which a short distance measured on the map bears to the corresponding distance upon the surface of the earth. The definition must be limited to short distances, because the scale of a map will generally vary from point to point; hence we must limit ourselves to small elements of length in the way that is familiar to every beginner in calculus.

We must be careful, in comparing distances, to choose directions that really correspond to each other upon the earth and upon the map. The meridians and parallels intersect everywhere at right angles on the earth; but there are many map projections in which the corresponding lines do not intersect at right angles. On these projections, two directions at right angles upon the earth would not necessarily correspond to two directions at right angles upon the map. Confusion will be avoided if we confine ourselves as much as possible to the consideration of the scale along the meridians and parallels of the map which necessarily correspond to the meridians and parallels of the earth.

It would be desirable to have the scale of the map correct in every direction at every point. If this could be done, the plane map would be a perfect representation of the spherical surface of the earth; but, since this is impossible, the scale can not be correct all over the map.

We can, however, choose some one direction and hold the scale constant in that direction; as, for instance, along the meridian or along the parallel. If this were done, the scale in any other direction would be wrong at most points.

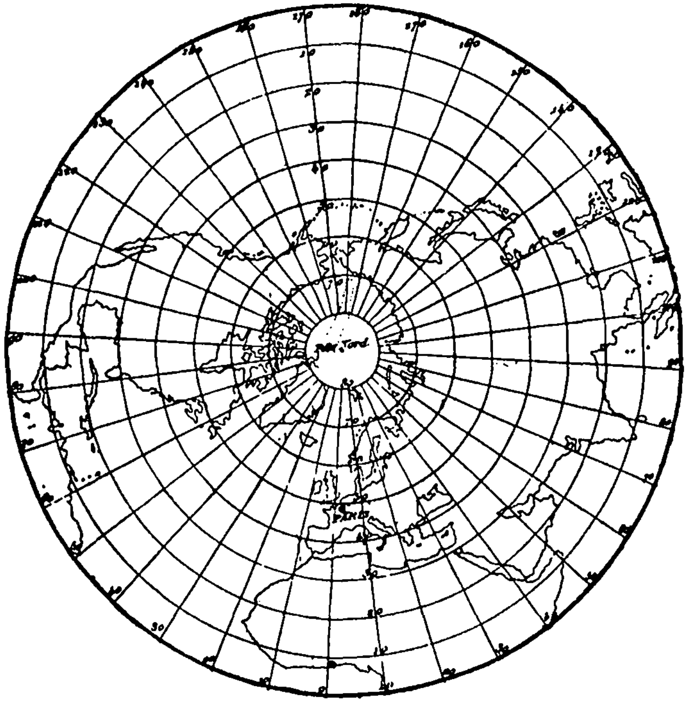

FIG. 1.—Zenithal equidistant projection of the Northern Hemisphere.

These facts are very easily recognized in what is called the zenithal equidistant projection. If we take the pole as the center of the map (see fig. 1) and call the polar distance z, we can use a radius for the parallels $\rho = Rz$.

The variation of scale along the meridian is denoted by k_m, and in this case it becomes equal to unity (that is to say, the scale is constant); hence $k_m = 1$.

An arc of the parallel on the map corresponding to a difference of longitude would be expressed by $R\lambda z$. The length of this arc upon the earth would be $R\lambda \sin z$. Hence the ratio of increase of length or the magnification in this direction denoted by k_p is expressed by

$$k_p = \frac{z}{\sin z}.$$

Similarly, if we wish to hold the scale along the parallel, we should take

$$\rho = R \sin z.$$

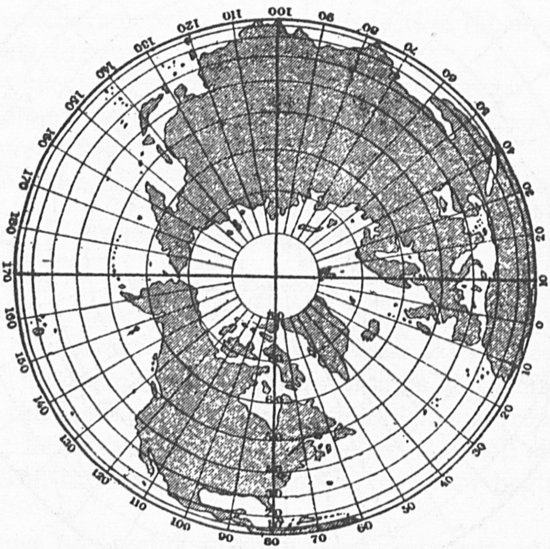

FIG. 2.—Orthographic projection of the Northern Hemisphere.

This is the orthographic projection on the plane of the Equator. (See fig. 2.) In this case the decrease of scale along the meridian would be expressed in the form

$$k_m = \frac{d\rho}{Rdz} = \frac{R \cos z\, dz}{R\, dz} = \cos z.$$

$$k_p = 1.$$

We should note that the term magnification is used ordinarily either in the case of an increase or of a decrease of scale. These simple cases show clearly what happens when we try to hold the scale in some given direction.

REPRESENTATION OF AREAS.

For some purposes, especially for political and statistical work, it is important that areas should be represented in their correct proportions. A projection that possesses this quality is called an equal-area, or an equivalent projection. In maps of this kind any portion whatever of the map bears the same ratio to the region it represents that the whole map bears to the whole region represented.

It can easily be seen how this quality can be attained in any projection. Let us suppose that AB and AC (see

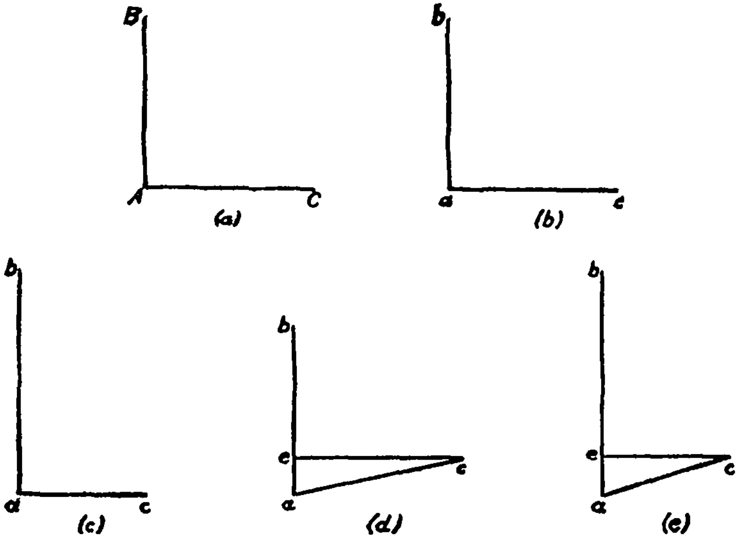

FIG. 3.—Three ways in which areas are preserved.

fig. 3) are two short distances at right angles to each other at any point on the earth. If the corresponding distances ab and ac [see fig. 3 (b)] upon the map were always in the same proportion and also always at right angles to one another, the projection would clearly be an equal-area projection. But these conditions can not be fulfilled, for, if they were fulfilled at every point, the map would be a perfect representation of the given region, which we have seen to be impossible. There are, however, three distinct ways in which the equality of areas may be preserved upon the map:

1. *ab* and *ac* may still be at right angles, but with the scale of one increased and of the other decreased, in inverse proportion [fig. 3 (*c*)].

2. Or *ab* and *ac* may no longer be at right angles; but, while the scale of one is maintained correct, that of the other is increased in such a proportion that the perpendicular distance between the lines of correct scale is also maintained correct in scale [fig. 3 (*d*)].

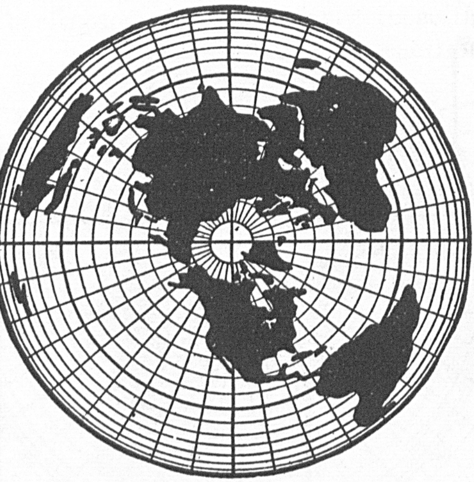

FIG. 4.—Lambert's zenithal equal-area projection of the sphere.

3. Or, finally, neither *ab* nor *ac* may be correct in scale, but they may make such an angle with each other that the scale along one and the perpendicular to it from the extremity of the other are in inverse ratio [fig. 3 (*e*)].

It is evident that any one of these conditions would preserve the proportionality of the areas.

LAMBERT'S ZENITHAL EQUAL-AREA PROJECTION.

The first of these conditions is clearly illustrated by Lambert's zenithal equal-area projection when the pole is taken as the center of the map. (See fig. 4.) Let us take the radius of the parallels equal to the expression

$$\rho = 2R \sin \frac{z}{2}.$$

The scale along the radius—that is, along the meridian—is given in the form

$$k_m = \frac{d\rho}{R\,dz} = \frac{R\cos\frac{z}{2}\,dz}{R\,dz} = \cos\frac{z}{2}.$$

The scale along the parallel is given by

$$k_p = \frac{\rho\lambda}{R\,\lambda\sin z} = \frac{2\,R\,\lambda\sin\frac{z}{2}}{2\,R\,\lambda\sin\frac{z}{2}\cos\frac{z}{2}} = \frac{1}{\cos\frac{z}{2}} = \sec\frac{z}{2}.$$

The two ratios are thus seen to be reciprocals of each other, as the conditions require.

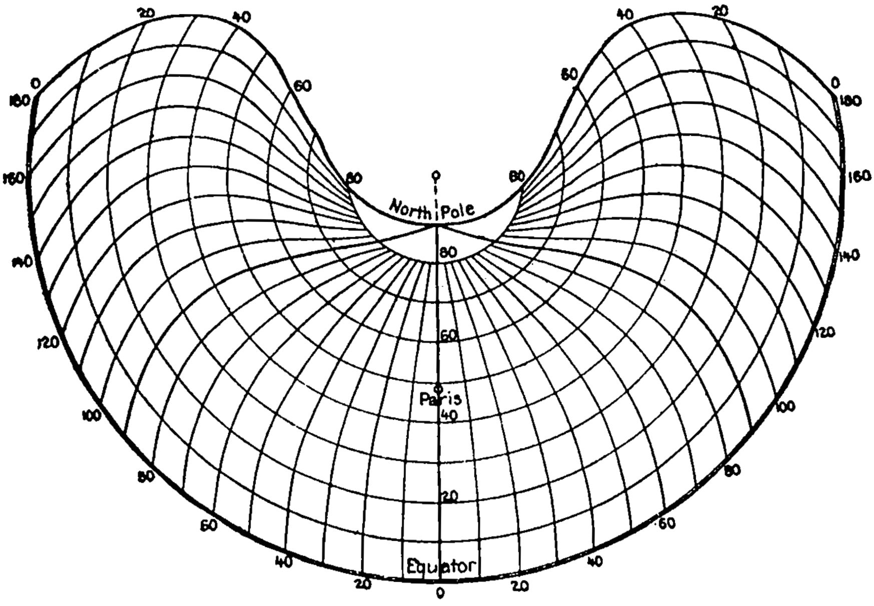

FIG. 5.—Bonne's projection for the Northern Hemisphere.

$$\left[\rho = \cot\phi_0 + \phi_0 - \phi,\ k_m = \sqrt{1+\lambda^2\left(\sin\phi - \frac{\cos\phi}{\rho}\right)^2},\ k_p = 1, \text{ and } \sin\chi = \frac{1}{\sqrt{1+\lambda^2\left(\sin\phi - \frac{\cos\phi}{\rho}\right)^2}}\right]$$

BONNE'S PROJECTION.

The second condition is illustrated by Bonne's projection. This projection (see fig. 5) consists of a system of concentric circles to represent the parallels; these circles are spaced proportionally to their true distances apart along the central meridian, which is a straight line, one of the radii of the system of circles. Any chosen parallel is the developed base of the cone tangent along this parallel. Along the remaining parallels the longitudinal distances are laid off in proportion to their true distances. The scale along the parallels is thus held constant, and the perpendicular distances of adjacent parallels is maintained true to scale. Therefore, the projection is equal-area, although the meridians and parallels do not intersect at right angles, except along the central meridian.

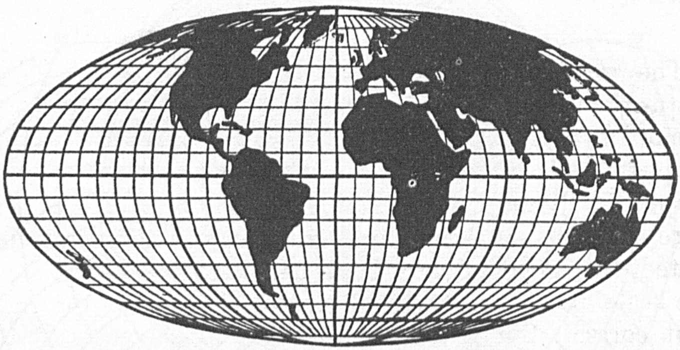

FIG. 6.—Mollweide's projection or homalographic projection of the sphere.

MOLLWEIDE'S PROJECTION OR HOMALOGRAPHIC PROJECTION.

In this projection the whole sphere is represented within an ellipse with major axis equal to twice the minor axis. (See fig. 6.) The major axis represents the Equator. It is divided into equal parts by the meridians, which are ellipses with the line of poles as one axis, and the other axis determined by the proper intersection of the meridian with the Equator. The parallels are straight lines, parallel to the Equator. The distance from the Equator of a parallel of latitude ϕ is equal to $r \sin \theta$ where r is the half of the polar line and θ is determined by the equation

$$\pi \sin \phi = 2\theta + \sin 2\theta.$$

If $r = \sqrt{2}\,R$ the formulas for the scale along the meridians and parallels become

$$k_m = \frac{\pi}{2\sqrt{2}} \cos \varphi \sec \theta \sqrt{1 + \frac{4}{\pi^2}\lambda^2 \tan^2 \theta}$$

$$k_p = \frac{2\sqrt{2}}{\pi} \sec \varphi \cos \theta$$

$$\sin \chi = \frac{1}{\sqrt{1 + \frac{4}{\pi^2}\lambda^2 \tan^2 \theta}}$$

χ being the angle between the meridian and parallel at their intersection.

REPRESENTATION OF SHAPE.

The representation of the shape of the geographical features of the earth, as nearly correctly as possible, is one of the most important functions of a map for ordinary purposes. It is evidently not possible to represent the shape of a large country correctly upon a map, for, if it were possible, the map would be perfect, which, as before stated, we know to be impossible. But, if at any point the scale along the meridian and the parallel is the same (not correct, but the same in both directions) and the parallels and meridians of the map are at right angles to one another, then the shape of any very small area on the map is the same as the shape of the corresponding small area of the earth. Such a projection is called conformal or orthomorphic, the latter term meaning "right shape." A projection of this kind is easily illustrated by the properties of the stereographic projection, with the pole as the center. (See fig. 7.) In this case we have

$$\rho = R \tan \frac{z}{2}.$$

The scale along the radius—that is, the meridian—is given by

$$k_m = \frac{d\rho}{R\ dz} = \frac{\frac{1}{2} R \sec^2 \frac{z}{2}\, dz}{R\ dz} = \frac{1}{2} \sec^2 \frac{z}{2}.$$

The scale along the parallel is

$$k_p = \frac{\rho \lambda}{R\lambda \sin z} = \frac{R \lambda \tan \frac{z}{2}}{2 R \lambda \sin \frac{z}{2} \cos \frac{z}{2}} = \frac{1}{2} \sec^2 \frac{z}{2}.$$

The scale is thus found to be the same in both directions at any point.

It is important to notice that the correctness of shape is limited to very small areas. Since the scale varies from

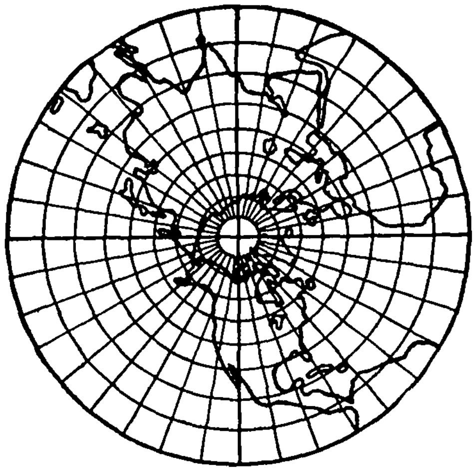

FIG. 7.—Stereographic projection of the Northern Hemisphere.

point to point, large areas are not correctly represented. The term orthomorphic is thus subject to misinterpretation, and for that reason the projections of this class are more generally called conformal projections. The conformal projections thus preserve the shape of small areas, although the scale varies from point to point upon the map. At first sight this preservation of shape appears to be an important property of this class of projections; but, if we remove the restriction to small areas, the general shape is often better preserved in projections which are not conformal than in those which are.

REPRESENTATION OF TRUE BEARINGS AND DISTANCES.

Besides the general shape of a country, we wish to know how well the bearings of points with respect to each other

FIG. 8.—Conical equal-area projection of the sphere.

$$\left[\rho=2\sqrt{2}\sin\frac{z}{2}\ k_m=\sqrt{2}\cos\frac{z}{2},\ k_p=\frac{1}{\sqrt{2}}\sec\frac{z}{2}\right]$$

are preserved on the map. For example, if we have a map of the United States, we may wish to know not only the error in the distance between New York and Cincinnati, but also what is the error in the azimuth of this line. The

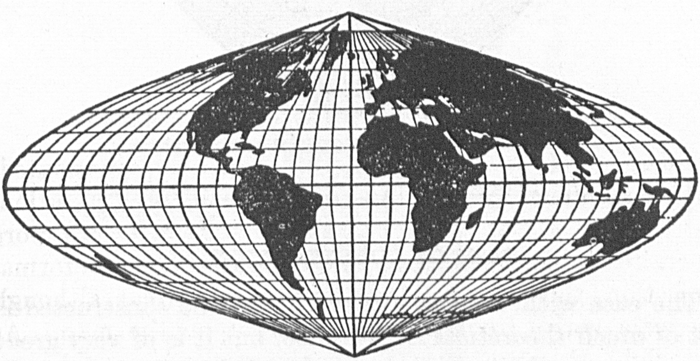

FIG. 9.—Sinusoidal equal-area projection of the sphere.

$$\left[x=\lambda\cos\phi,\ y=\phi,\ k_m=\sqrt{1+\lambda^2\sin^2\phi},\ k_p=1, \text{ and } \sin\chi=\frac{1}{\sqrt{1+\lambda^2\sin^2\phi}}\right]$$

consideration of small areas will not help us to answer either of these questions.

There is a class of projections which are sometimes called azimuthal from the fact that the azimuths or true bearings, from the center of the map to all points, are shown correctly. We have already called attention to the azimuthal equidistant, the azimuthal equal-area, and the azimuthal conformal projections.

We need to know not only how well the azimuths are preserved from the center but also from any given point. Instead of the term azimuthal the term zenithal is often used. This name probably arose from the fact that such a map of the celestial sphere has the zenith point as the central point of the map.

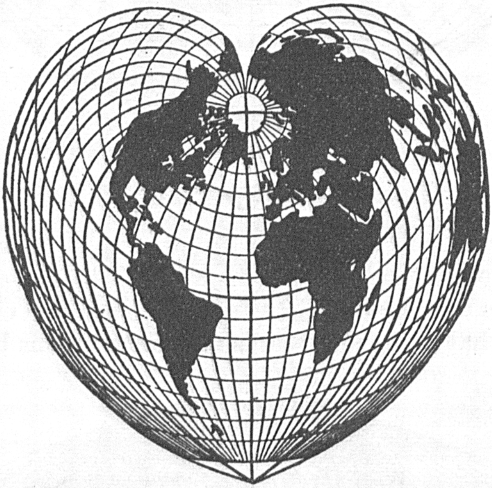

FIG. 10.—Werner's equal-area projection of the sphere.

$$\left[\rho = z,\ k_m = \sqrt{1+\lambda^2\left(\frac{\sin z}{z}-\cos z\right)^2},\ k_p = 1, \text{ and } \sin\chi = \frac{1}{1+\sqrt{\lambda^2\left(\frac{\sin z}{z}-\cos z\right)^2}}\right]$$

EASE OF CONSTRUCTION.

The ease with which a projection can be constructed is not of much theoretical importance, but it is of very great importance to the one who has in hand the actual construction of the projection. As a general rule, projections which are not built up of straight lines and circles are hard to draw. If the projection in question is such that it can not be constructed by graphical means, tables of coordinates have to be computed. We need to know, then, how readily the formulas lend themselves to computation.

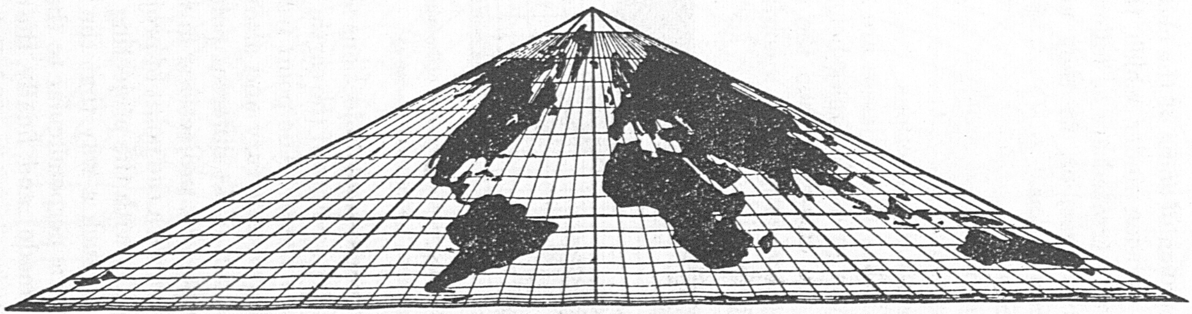

FIG. 11.—Collignon's equal-area projection of the sphere.

$$\left[x=\frac{2\sqrt{2}\,\lambda}{\sqrt{\pi}}\sin\frac{z}{2},\ y=\sqrt{\pi}\left(1-\sqrt{2}\sin\frac{z}{2}\right),\ k_m=\frac{\sqrt{\pi}}{\sqrt{2}}\sqrt{\frac{4}{\pi^2}\lambda^2+1}\cos\frac{z}{2},\ k_p=\frac{\sqrt{2}}{\sqrt{\pi}}\sec\frac{z}{2},\ \sin\chi=\frac{1}{\sqrt{\frac{4}{\pi^2}\lambda^2+1}}\right]$$

CLASSES OF PROJECTIONS.

We have already spoken of some of the classes of projections. The general division under which the subject of projections is commonly treated are as follows:

1. Perspective projections, or, as they are sometimes called, geometrical projections.
2. Conical projections.
3. Equal-area projections.
4. Conformal projections.
5. Azimuthal or zenithal projections.

It should be noted that these classes are not mutually exclusive, since a given projection may belong to two or sometimes three of the classes. Thus the stereographic projection is a perspective projection, a conformal projection, and a zenithal projection.

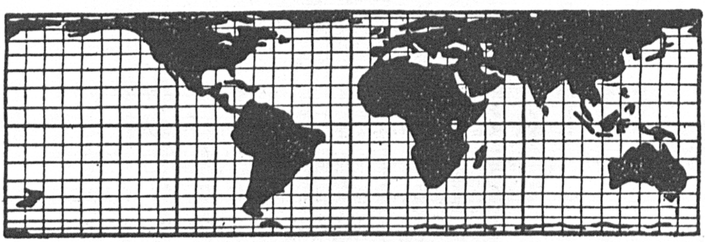

FIG. 12.—Cylindrical equal-area projection of the sphere.

[$x=\lambda$, $y=\sin\phi$, $k_m=\cos\phi$, and $k_p=\sec\phi$]

The perspective or geometrical projections are formed by the direct projection of the points of the earth, usually upon a plane tangent to the sphere. The point from which the projecting lines are drawn can vary and also the point of tangency of the plane can lie at different points upon the sphere. The most important projections in this class are the gnomonic projection, with the point of projection at the center; the stereographic, with the projecting point on the surface of the sphere in such a way that the line through the center of the sphere is perpendicular to the plane upon which the projection is made; and, finally, the orthographic projection, with the projecting point at infinity.

Instead of using a plane directly upon which to lay out the projection, the larger number of projections make use

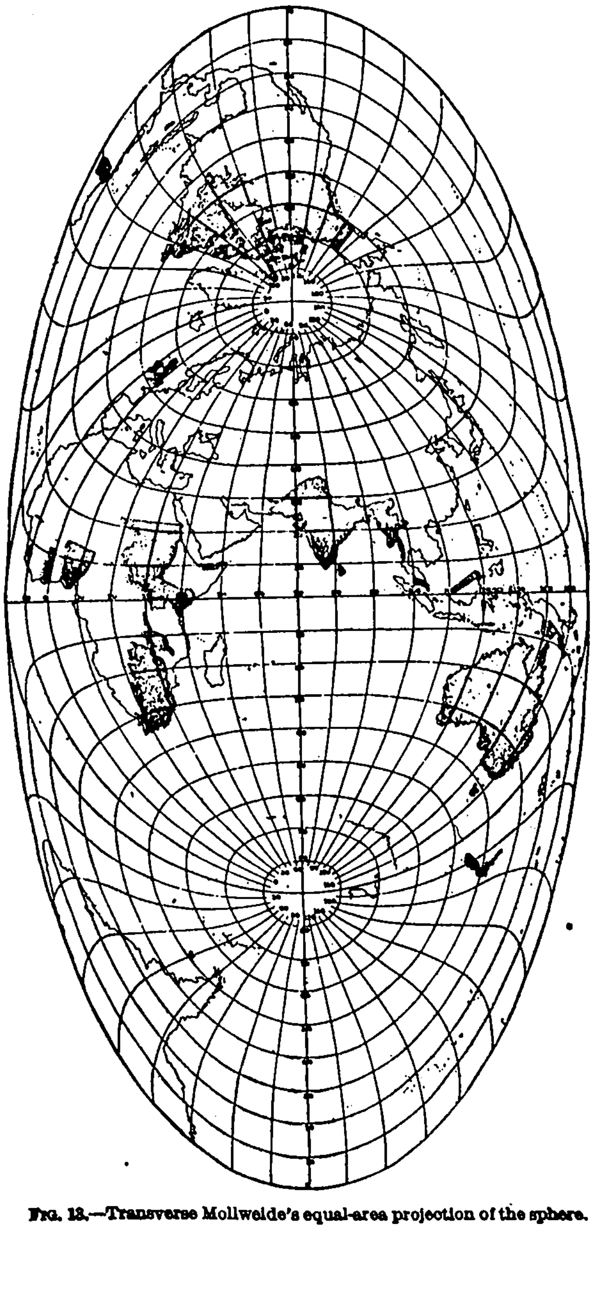

FIG. 13.—Transverse Mollweide's equal-area projection of the sphere.

of a developable surface tangent to the sphere and then spread this surface out upon a plane. The two surfaces suitable for this purpose are the cone and the cylinder. Since the cylinder is only a special case of the cone with the apex at infinity, the cylinder and the cone are both considered as belonging to the conic projections.

DEVELOPMENT OF THE MERCATOR PROJECTION.

The most important one of the cylinder projections is the Mercator. This is the conformal cylinder projection, in

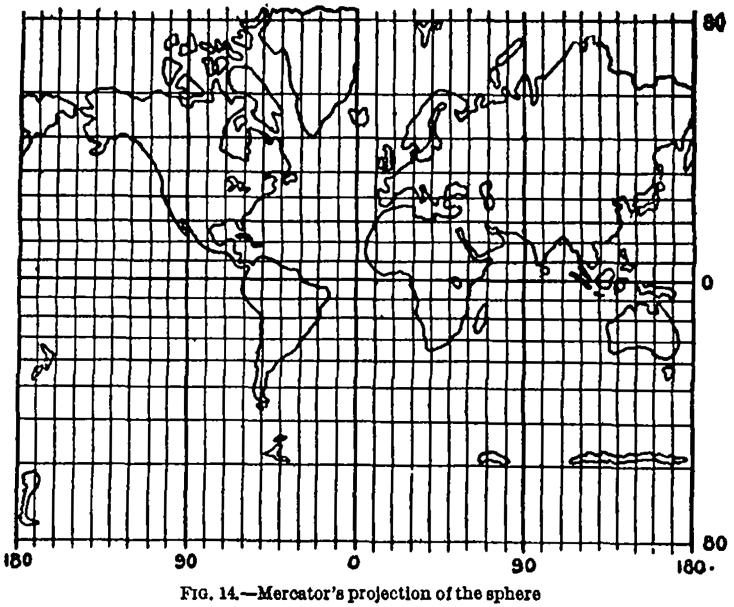

FIG. 14.—Mercator's projection of the sphere

which the meridians and parallels are represented by two systems of straight lines perpendicular the one to the other. We shall give a short and simple account of this projection, in which we shall consider the sphere to avoid complicated formulas. In order to avoid the frequent use of the statement of proportionality, we shall discuss the same as an equal representation; that is to say, one in which the ratio of scale along the equator is equal to unity. A projection is conformal (1) if all angles are preserved, or (2) if the scale is the same in all directions at any point. If the meridians and parallels are at right angles and the scale along both at any point is the same, the conditions for conformality will be fulfilled. A cylinder tangent at the equator would give for the intersections of the meridians

$$x = R\lambda.$$

This line is, accordingly, taken as the X axis, and the lines perpendicular to this axis will represent the meridians. Another system of lines parallel to this axis will represent the parallels. The only thing to determine, then, is the point of intersection of the parallels with the central meridian or Y axis.

An arc along a parallel is equal to $R\lambda \cos \varphi$. The partial derivative of this with respect to λ is equal to $R \cos \varphi$. But

$$\frac{dx}{d\lambda} = R.$$

Therefore, the element of length along the parallel is divided by $\cos \varphi$ to determine the length to be used upon the map. To keep the scale along the meridian the same, we must also divide this length upon the earth by $\cos \varphi$. But the element of length upon the earth along the meridian is given by

$$Rd\varphi.$$

Hence for the element of length upon the map we must have

$$dy = \frac{Rd\varphi}{\cos \varphi}.$$

This gives

$$y = R\int_0^\varphi \frac{d\varphi}{\cos \varphi} = R\int_0^\varphi \frac{\cos \varphi d\varphi}{\cos^2 \varphi} = R\int_0 \frac{\cos \varphi d\varphi}{(1 - \sin^2 \varphi)}.$$

$$y = \frac{R}{2}\int_0^\varphi \frac{\cos \varphi d\varphi}{1 + \sin \varphi} + \frac{R}{2}\int_0^\varphi \frac{\cos \varphi d\varphi}{1 - \sin \varphi}$$

$$y = \frac{R}{2} \log_e \frac{1 + \sin \varphi}{1 - \sin \varphi} = \frac{R}{2} \log_e \frac{1 + \cos\left(\frac{\pi}{2} - \varphi\right)}{1 - \cos\left(\frac{\pi}{2} - \varphi\right)}$$

$$y = \frac{R}{2} \log_e \frac{2 \cos^2\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)}{2 \sin^2\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)} = R \log_e \cot\left(\frac{\pi}{4} - \frac{\varphi}{2}\right);$$

or, finally,

$$y = R \log_e \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right).$$

We thus arrive at the formulas for the conformal cylinder projection

$$x = R\lambda.$$

$$y = R \log_e \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right).$$

We could have obtained these formulas in the following way: The element of length upon the sphere is given in the form,

$$ds^2 = R^2 \,(d\varphi^2 + d\lambda^2 \cos^2 \varphi),$$

or

$$ds^2 = R^2 \cos^2 \varphi \left(\frac{d\varphi^2}{\cos^2 \varphi} + d\lambda^2\right).$$

We then set

$$dx = R d\lambda$$

and

$$dy = \frac{R d\varphi}{\cos \varphi},$$

and thus arrive at the same formulas we obtained before.

CONICAL PROJECTIONS.

In all of the usual conical projections the meridians are straight lines converging to a point, the apex of the cone, and the parallels are concentric circles described with this point as a common center. The meridians are equally spaced, and make with one another angles which are a certain fraction n of the angles which the corresponding terrestrial meridians make with one another at the poles. The quantity n is called the constant of the cone. The conical projections thus meet the fourth requirement in that they are easy to construct either directly by drafting or by computation of coordinates.

The spacing of the parallels depends upon the particular property which we wish the projection to fulfill. One parallel (and sometimes two) is made of the true length; that is, if the map is to be on the scale of 1 part in 1 000 000, the length of the complete parallel on the map will be one one-millionth of the corresponding terrestrial parallel.

Lambert's conformal conic projection belongs to this class. The one usually employed has two standard parallels.

Besides the true conic projections, there is a class called the polyconic projections. In these projections the parallels are represented by a nonconcentric system of circles, but the centers of this system of circles lie upon a straight line called the central meridian. This meridian is the only one that is represented by a straight line. This class includes an unlimited number of projections, among them being the one called the ordinary or American polyconic. The name American polyconic has been given to it by

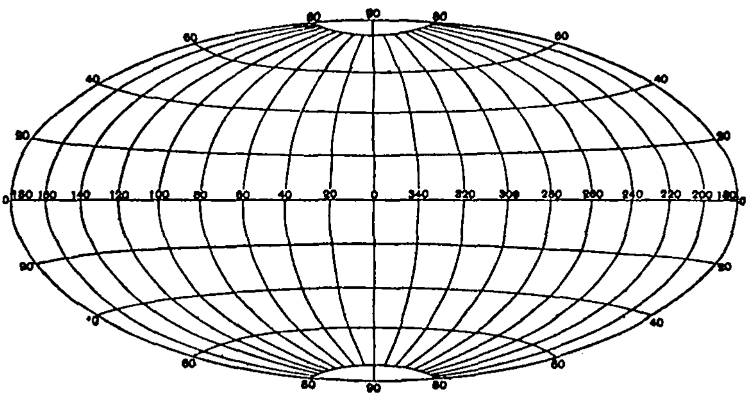

FIG. 15.—Aitoff's equal-area projection of the sphere.

European writers, chiefly because it has been extensively employed by the Coast and Geodetic Survey.

The azimuthal or zenithal projections are really special cases of conic projections in which the cone becomes flatter and flatter until it becomes a plane and the value of the constant n becomes equal to unity.

The true conical projections have a range of properties sufficiently wide to make them appear, at first sight, a very useful and valuable family—they may be made nearly true to scale over a fairly wide area, and consequently nearly equal-area and nearly conformal; or they may be made exactly equal-area and less conformal, or exactly conformal and less equal-area. The fact is, however, that the equal-area and conformal projections have been used very little, while until lately the ordinary conic with two standard parallels has been almost equally neglected. The simple conic has, however, been very much used for small atlas maps. Lambert's conformal projection with two standard parallels was used for the battle maps in France.

GENERAL CONCLUSION.

The subject of conformal representation has an interest and usefulness far wider than that due to its employment in map making. It has great uses in mathematical physics; and in differential geometry, it applies to the mapping of any surface upon any other, not being limited to the representation upon a plane. For these reasons this matter has been much studied by mathematicians, and they have been accused of adding to the difficulties of the subject of map projections by their labors. The real truth of the matter is that they have founded the whole subject upon general considerations that have great elegance, but that, no doubt, seem complicated to the one who wishes to know merely enough about the subject to be able to make use of the projections that come in this class. Fortunately most of the results that apply to map projections can be derived in a fairly simple way, as we have tried to show in the case of the Mercator projection.

The subject of map projections is, therefore, a very wide one, and some of the considerations have their roots extending far into the fertile soil of pure mathematics. A detailed study of even the few projections that have been practically used in map making would form a volume of quite respectable proportions. A careful working out of the results, for any one projection, would form a good exercise in the practical application of mathematical knowledge.

It is to be hoped that this discussion may have helped to make clear some of the points regarding projections. Any study of the subject is well worth while in the case of one who wishes some training in the applications of mathematics in which no great degree of difficulty is presented.

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