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AN INVESTIGATION OF THE LATITUDE OF UKIAH, CALIF., AND OF THE MOTION OF THE POLE

BY

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AN INVESTIGATION OF THE LATITUDE OF UKIAH, CALIF., AND OF THE MOTION OF THE POLE.

By WALTER D. LAMBERT, *Mathematician, U. S. Coast and Geodetic Survey.*

INTRODUCTION.

The following discussion of the observations of the International Latitude Service was occasioned by an article of Prof. Andrew C. Lawson,[1] in which he called attention to the fact that the latitude of Ukiah Observatory, California, maintained by the U. S. Coast and Geodetic Survey for the International Latitude Service, appears to be steadily increasing at the rate of about 0.″01 a year. The latitude of Lick Observatory appears to be increasing even more rapidly, although the progressive increase is complicated by an abrupt decrease of 0.″63 that seems to have occurred in the latter part of the year 1903. Obviously if the existence of progressive changes[2] of this order of magnitude can be established beyond question, the study of them becomes of great interest to geophysicists, astronomers, and geodesists.

The purpose of this paper is the discussion of the mass of data accumulated by the observatories of the International Latitude Service, in order to ascertain how far the large progressive change of latitude is peculiar to Ukiah alone among all the stations of the Latitude Service and how far the apparent change of latitude may be accounted for in other ways than by an actual shifting of the upper strata of the earth's crust in the vicinity of the station, this being Prof. Lawson's explanation. In order to reach any assured conclusion whatever, it is evidently necessary to compare Ukiah with other stations; accordingly all the latitudes of all stations of the International Service are here analyzed and discussed. The method of observation and the program of stars to be observed are the same at all stations of the International Service. The observations of the service therefore form a homogeneous whole peculiarly adapted to discussions of delicate questions of this sort; for one thing the errors of the star places have comparatively little effect on the results. The observations at Lick Observatory have not this advantage. No serious attempt, therefore, is here made to discuss the evidence from the Lick observations except to reproduce the passage on this subject from Prof. Lawson's article, together with some corrections and comments, chiefly those kindly supplied by the director of Lick Observatory.

[1] "The mobility of the Coast Ranges of California, an exploitation of the elastic rebound theory;" University of California Publications, Bull. Department of Geology, vol. 12, No. 7 Jan. 11, 1921.

[2] The word "progressive" is here used in much the same sense that the word "secular" is used in the theory of astronomical perturbations; that is, a progressive change is one proportional to the time, or nearly so. The word "secular," however, carries the connotation of almost indefinite continuance, as measured by ordinary human standards. The changes in latitude with which this report deals have not been proved to be of this character; the word "secular" has therefore been avoided and "progressive" substituted. (See p. 37.)

In the harmonic analysis of the latitudes the series used are necessarily short when measured by the number of periods used; the harmonic constants of any one component derived from such a short series are considerably affected by the presence of other components. The approximate formulas of the type used by this Survey in the ordinary harmonic analysis of short-period tides for clearing one component of the effects of another are not quite adequate in the present instance and a more accurate process is explained and used. It resembles the procedure given by Darwin for clearing one long-period tidal component from the effects of the others but is carried one step further by the solution of a set of linear equations in general terms. The derivation of the final formulas for numerical computation is more laborious than for the approximate formulas for dealing with short-period tides, but the numerical calculation with the more exact formulas, once they are obtained, is quite as simple as with the approximate ones.

In order not to interrupt the main discussion with matters primarily mathematical, the development of the formulas for clearing one harmonic component from the effects of others and some other mathematical developments are given separately in Chapter VI. This separation has necessitated a certain amount of repetition. On account of the general interest of the subject of latitude variation, the attempt has been made to render the details intelligible to a fairly wide circle of possible readers. For this reason some matters have been treated at greater length than would be necessary in a book intended solely for specialists.

Some results not directly connected with the problem in hand are given in Chapter IV. They were found to be deducible from results already obtained with such a comparatively small amount of extra labor that it seemed worth while to obtain and present them.

Miss Sarah Beall, of the division of geodesy, has given valuable help in plotting and reading curves and in making or checking much of the large mass of computation required in this work. Without her assistance the appearance of this investigation would have been much delayed. I. N. Beall and M. A. Crews, of the division of geodesy, particularly the latter, have also given extensive help with the computations.

While this publication was in the printer's hands the author undertook a revised presentation of the subject of polar motion with the purpose, among others, of meeting certain objections raised by Professor Schlesinger.[3] As a result it seemed justifiable to omit Tschardjui entirely from the determination of the polar motion. The rate of displacement of the pole from 1900 to 1917, inclusive, is then increased from 0.″0050, as found in chapter II, section 5 (p. 35), to 0.″0062 annually and the average direction of the motion of the North Pole comes out nearly along the meridian of 90° west. These figures should be considered as a revision of those contained in the present publication. For details see the author's paper "The interpretation of apparent changes of mean latitude at the international latitude stations," which will appear in the Astronomical Journal, Vol. 34, no. 804. The paper contains also a brief discussion of earth movements and of the elastic-rebound theory of earthquakes.

[3] F. Schlesinger, On Progressive Changes of Latitude, Astronomical Journal, vol. 34, p. 42. 1922.

Chapter I.—AN EXAMINATION OF PROF. LAWSON'S DATA.

The data from which Prof. Lawson deduced the progressive increase in the latitude of Ukiah are found in an article by Sir F. W. Dyson, Astronomer Royal,[1] and consist of curves showing the variation of the latitude after the annual portion[2] of the variation has been eliminated by calculation. The abscissa used in plotting the curve is the date; the ordinate, which we shall denote by $\Delta\phi$, is the difference between the observed latitude corrected for the annual variation and a certain initial latitude, which, for present purposes, may be quite arbitrary. The change in latitude still left in the curves is then due to the free oscillation of the earth, except in so far as the elimination of the annual portion may have been imperfect, or as small effects of other motions with periods unknown or uncertain may remain also. The curves are given for three stations only—Ukiah, Mizusawa, and Carloforte. The abscissas and ordinates from which the curves were plotted are not given in full. The data given for all stations consist of the epochs of maximum and minimum latitude and the differences between each maximum and the two minima adjacent; this is not sufficient to reconstruct the curves for Tschardjui, Cincinnati, and Gaithersburg, so all consideration of these stations is deferred until the next chapter.

The curves shown may be considered as somewhat irregular sine curves with variable amplitudes of oscillation. In the Ukiah curve an algebraic increase with the time of both the maximum and minimum ordinates is pretty evident. Prof. Lawson obtained the rate of progressive[3] increase by drawing a straight line as nearly as possible through the successive maxima and a similar line through the minima. The average slope of the two lines was taken as representing the progressive rate of change of the latitude of Ukiah. For this rate Prof. Lawson finds[4] 0.29 meter, or 0".0094 second of latitude per year. He then continues:

> In arriving at this figure it is to be noted that, since the mean position of the pole represented by the zero line is based upon observations of all stations including Ukiah, the zero line adopted diverges northerly from the true zero line, and the rate of creep inferred from the curve, 0.29 meter per year, is in reality somewhat less than the true rate. For, if Ukiah be moving northerly, the increase in latitude due to this cause should not enter into a determination of the mean position of the pole, that is the position of the zero line in relation to the curve.

The author doubts whether the statement quoted correctly interprets Sir F. W. Dyson's procedure. The only statement in regard to this is[5]—

> §6. The results for the six observatories given in A. N. 4841 were next examined separately. The mean annual terms for the interval 1900.0–1912.0 are:[6] * * *

[1] Monthly Notices of the Royal Astronomical Society, vol. 78 (1918), p. 452.
[2] The annual portion of the variation of latitude includes the annual part both of the motion of the pole itself and of all other changes in the latitude, which like the Kimura effect, have an annual periodic portion.
[3] For the sense in which the word "progressive" is used, see footnote No. 2, p. 1.
[4] See p. 436 of the article referred to on p. 1.
[5] Loc. cit. ante, p. 457.
[6] These values tabulated by Dyson, but here omitted, are in substantial accord with those found by an independent discussion and given on p. 51 of this publication. The exact values are not material to the question immediately under discussion.

§7. In the discussion of the free period as shown by the different observatories the annual terms given at the beginning of §6 were extracted and the figures then plotted. This is a rather complicated diagram to reproduce, and only the three observatories at Carloforte, Mizusawa, and Ukiah are given.

A reference to Astronomische Nachrichten No. 4841,[7] which is a long article by Przbyllok, shows that the stations are there tabulated separately and that the $\Delta\phi$'s for any station are the differences between the observed latitudes and some invariable initial latitude, the latter having nothing to do with the motion of the pole or with the latitudes of the other stations.

It might be objected, however, that the motion of the pole does enter to a certain extent into Przbyllok's $\Delta\phi$'s, for the so-called observed latitudes really depend both on instrumental readings and on the declinations used. The declinations of Przbyllok's definitive $\Delta\phi$'s are corrected declinations, the corrections deduced from a discussion of the differences between the observed latitudes (with the original uncorrected declinations) and the adjusted latitudes of the International Latitude Service, which depend on all stations. Thus to a certain extent Przbyllok's observed latitudes and hence his $\Delta\phi$'s for any station, which are presumably Dyson's also, depend on the latitudes of the other stations. An examination of the corrections actually used, as given at the foot of page 282 of Astronomische Nachrichten No. 4841, shows, however, that the mean correction to the declination is considerably less than 0.″001. Furthermore the corrections are the same for the early part of the series as for the later parts and so have no effect in increasing or diminishing any apparent progressive change of latitude such as occurs at Ukiah.

Dyson gives results extending beyond Przbyllok's tabulation, which does not go beyond 1912.0. Presumably Dyson's results are based on the provisional values in the Astronomische Nachrichten,[8] but whether any correction was applied to them to make them homogeneous with Przbyllok's results Dyson does not state. A lack of complete homogeneity would not be particularly important in Dyson's work, the purpose of which was to obtain an accurate value for the length of the 14-month period and which was not intended for the use to which Prof. Lawson has put it.

It appears then that neither Prof. Lawson's change of 0.29 meter or 0.″0094 a year nor the 0.″0081 presently to be deduced from the same data by another method should be augmented for any supposed change in the zero line of the $\Delta\phi$'s.

There may be some objection to the method of using only the maxima and minima of the latitude-variation curve in order to determine the rate of progressive change,[9] and it seems certain that the method of taking ordinates of the curve at small, equal intervals and finding the mean will give a more reliable result. It seemed desirable also to see how far Carloforte and Mizusawa might resemble Ukiah in showing a progressive change of latitude. Accordingly, the curves of all three stations were read at intervals of one month—i. e., one-twelfth of a year—and the means were taken over a period of 14

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[7] Vol. 202 (1916). The article is a careful discussion of the observations of the International Latitude Service.

[8] For references, see p. 8.

[9] For example, the form of a curve representing a periodic function may be such that the average of maximum and minimum ordinates differs from the mean of all ordinates; thus in careful tidal work a distinction must be made between mean sea level, which is found from the mean of closely spaced, equidistant ordinates of the curve drawn by an automatic tide gauge, and half-tide level, which is halfway between the mean ordinate of the high waters and the mean ordinate of the low waters.

months, or more specifically, 15 consecutive ordinates were used and half weight only was assigned to the first and to the fifteenth ordinate. The period used (14 months = 426.1 days) is so nearly the free period of the latitude variation that the mean over the 14 months should be practically unaffected by periodic changes of phase that run their course in the free period or its submultiples. For example, if we write the principal part of the free oscillation of the latitude in the form $A \cos(\kappa t - \alpha)$, where A and α are assumed to be constants, t the time, and κ a constant such that $\kappa t = 2\pi$ when $t = 432.5$ days, then the maximum error due to the incomplete elimination of this term from the mean of 15 readings taken as above is easily seen by formula (4) (p. 71) to be less than $0.015A$. Now A is $0''.20$ or less, so the systematic error is at most about $0''.003$ for any 14-month period.

The following table shows the means of the values of $\Delta\phi$ read from the curve for the period the middle of which is shown in the first two columns:

TABLE 1.—*Observed changes in latitude.*

Middle of period.		Mean value of $\Delta\phi$ for—		
Year.	Month.	Mizusawa.	Carloforte.	Ukiah.
		$''$	$''$	$''$
1900	7	−0.010	+0.004	+0.001
1901	9	− .011	+ .033	− .040
1902	11	− .042	+ .004	+ .001
1904	1	− .023	+ .022	+ .047
1905	3	− .040	+ .029	+ .054
1906	5	− .012	+ .021	+ .068
1907	7	− .025	+ .001	+ .086
1908	9	+ .004	+ .002	+ .047
1909	11	− .017	+ .016	+ .058
1911	1	− .047	+ .028	+ .070
1912	3	− .075	+ .013	+ .063
1913	5	− .088	+ .032	+ .055
1914	7	− .034	+ .032	+ .118
1915	9	− .032	+ .037	+ .152

The months shown are elapsed months, or more accurately elapsed twelfths of a Julian year; thus in the first line the "7" indicates that seven-twelfths of the year 1900 have elapsed, or the date is about August 1.

The values of $\Delta\phi$ in the table were plotted as ordinates against the time as abscissa and the results are shown in figures 1, 2, and 3. The attempt was then made to fit a straight line to the plotted points by the method of least squares; this corresponds to an assumed uniform rate of variation of the latitude.

Each mean value of $\Delta\phi$ gives an observation equation of the form

$$\Delta\phi = x + ty, \tag{1}$$

x being the adjusted value of $\Delta\phi$ at the time 1908+9 months and t the time reckoned from this epoch; y is the adjusted rate of change of the latitude. From the adjustment there was found with the 14-month period as the unit of time the following values, which are given with the probable errors attached:

For Mizusawa $x = -0''.0338 \pm 0''.0042$ $\quad y = -0''.0029 \pm 0''.0010$,
for Carloforte $x = +0.0202 \pm 0.0022$ $\quad y = +0.0013 \pm 0.0005$,
for Ukiah $x = +0.0590 \pm 0.0048$ $\quad y = +0.0095 \pm 0.0012$.

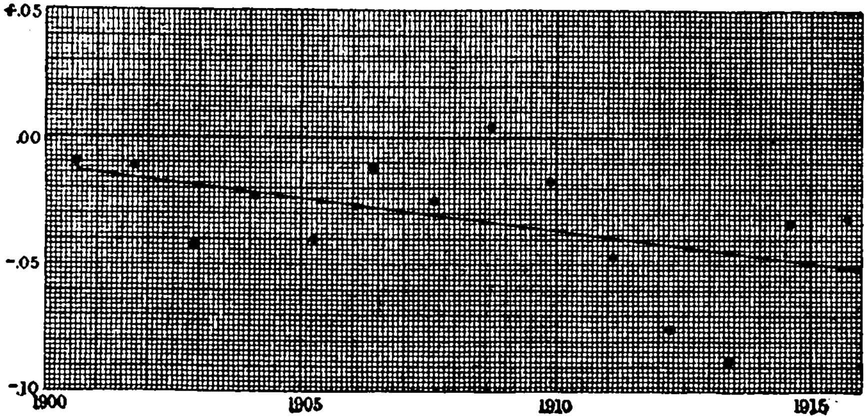

FIG. 1.—Progressive change of latitude at Mizusawa from Dyson's curves.

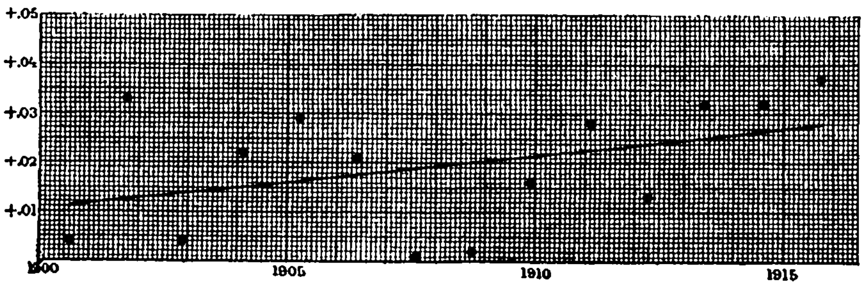

FIG. 2.—Progressive change of latitude at Carloforte from Dyson's curves.

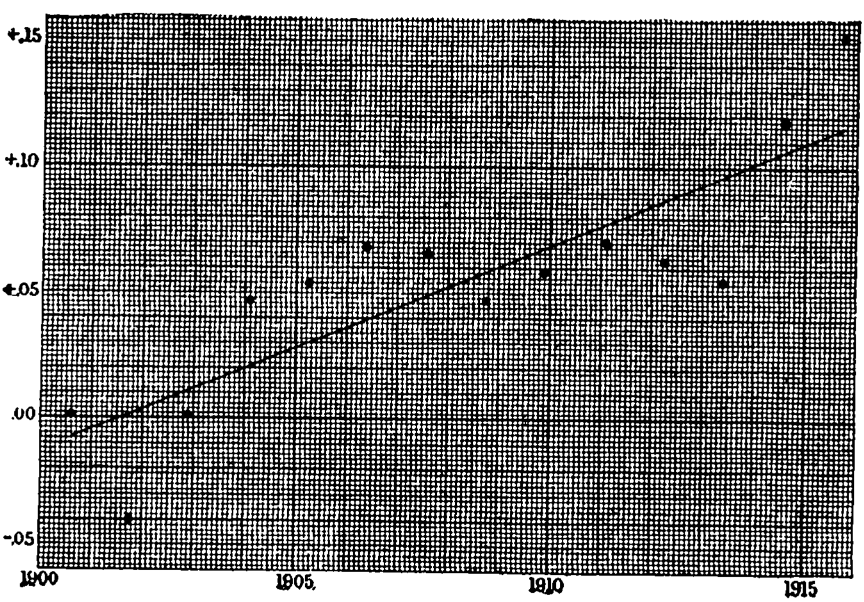

FIG. 3.—Progressive change of latitude at Ukiah from Dyson's curves.

To reduce the values of y and their probable errors to the year as unit they must be multiplied by $\frac{12}{14}$; we thus find:

$$
\begin{array}{ll}
\text{For Mizusawa} & y = -0''.0025 \pm 0''.0009, \\
\text{for Carloforte} & y = +0.0011 \pm 0.0004, \\
\text{for Ukiah} & y = +0.0081 \pm 0.0010.
\end{array}
$$

The annual rate of change found for Ukiah is thus slightly smaller than the rate (0''.0094) found by Prof. Lawson and this latter rate, as we have seen, should not be increased for any supposed change in the zero from which $\Delta\phi$ is reckoned. Even the smaller rate of change for Ukiah is, however, quite large enough to excite interest as being rather too large to be due to accidental error. Any attempt to explain this rate is, however, postponed until all the six stations of the International Latitude Service have been examined for similar rates. (See pp. 21–31.)

Chapter II.—THE STATIONS OF THE INTERNATIONAL LATITUDE SERVICE.

SECTION 1. INTRODUCTORY MATTER.

The stations of the International Latitude Service and their distribution in longitude are shown in the following table. West longitudes are positive.

TABLE 2.—*Stations of the International Latitude Service.*

Station.	General locality.	Longitude.	Observations ceased at end of year—
		° ′	
Mizusawa	Japan	−141 08	
Tschardjui [1]	Turkestan	− 63 29	1914.
Carloforte	Sardinia	− 8 19	
Gaithersburg	Maryland	+ 77 12	1914.
Cincinnati	Ohio	+ 84 25	1915.
Ukiah	California	+123 13	

[1] This is the spelling adopted in the publications of the International Latitude Service. The following forms have been found in works of reference published in Great Britain and the United States: Charchui Tscharjui, Charjooee, Chardshuil, and Charjui. For an English-speaking person the last-given form seems to be an adequate transliteration. The longitude is that of the old station. The longitude of the new station is −63° 35′.

The methods adopted for the discussion of the observations at these six stations are essentially the same as those of the preceding chapter but with some differences in detail. First of all the observed latitudes were plotted as ordinates against the dates expressed in years and decimals of a year as abscissas. The sources of the observed latitudes were the publications of the International Latitude Service and the Astronomische Nachrichten.[2] Pages 210 to 219 of the Resultate, Vol. III, and pp. 169–178 of Vol. V, contain the definitive results for the years 1900 to 1911, inclusive. The declination system of both volumes is that of Volume III, which is maintained for the sake of uniformity and convenience.[3] The same declination system is used in the provisional results given in the Astronomische Nachrichten.[4] Some of the star pairs have, however, been dropped as no longer available on account of their precessional motion, and have been replaced by other pairs. In working up the provisional results these new pairs were not used, and the results are therefore based on the original pairs that still remain available, amounting to about five-sixths of the complete program. The results for Tschardjui after July 23, 1909, when the new observatory was occupied, were reduced to the old observatory by applying to the latitudes observed after that date a constant correction of $-0.''225$, as given on page 3 of the Resultate (Vol. V).

[2] Resultate des Internationalen Breitendienstes: Vol. III, by Albrecht and Wanach, Berlin. 1909; vol. V, by Wanach, Berlin, 1916. Astronomische Nachrichten: vol. 198, No. 4749; vol. 201, No. 4802; vol. 203, No. 4858; vol. 205, No. 4908; vol. 208, No. 4969.
[3] Vol. V, p. 76.
[4] See Astronomische Nachrichten No. 4749, vol. 198.

The plotted latitudes for each station were then adjusted graphically by drawing a curve to conform as nearly as might be to the plotted points without showing sharp vertices or rapid alternations in slope. On these curves for the several stations all the subsequent work is based; they were therefore drawn with some care and the first draft of a curve was always revised by one other person and sometimes by two others. The observations are consistent enough so that there was very seldom room for any serious difference in judgment as to the proper form and location of the curve. In doubtful cases the doubt was decided in favor of smoothing out the observations too little rather than too much.

SECTION 2. HARMONIC ANALYSES.[5]

Harmonic analyses were made of the curves to determine expressions for the annual and the 14-month components at each station. The period of the annual component was assumed to be one Julian year, or 365.25 days. No distinction was made between leap years and common years. The period of the 14-month component was assumed to be 432.5 mean solar days; this is a mean between Dyson's result of 432.2 days[6] and Wanach's of 432.8 days.[7] This gives the 14-month, or free, period of the latitude variation as 1.18412 Julian years and the change of phase of the free oscillation in one month (one-twelfth of a Julian year) as $25^\circ.33526$ and in one year $304^\circ.0231$. One year equals 0.844509 of the free period and the change of phase of the annual component in one-twelfth of a free period (0.098677 year) is $35^\circ.5236$, and the change of phase in one free period is $426^\circ.2834$. These numbers are here set down for reference as needed. When the 14-month component or a 14-month period is spoken of hereafter in this report, this should be understood as an abbreviation for a 432.5-day component, etc., unless it is clear from the context that exactly $\frac{7}{6}$ of a Julian year is meant.

For the annual component the curve for a station was read at intervals of one-twelfth of a year, the first reading coinciding with the beginning of a year. For the 14-month component, representing the free oscillation, the period was also divided into 12 parts, the interval between readings being 0.0987 year, as above; the first reading of the series was always taken at the beginning of some year. Since we have approximately 5 free periods equal to 6 annual periods and 6 free periods equal to 7 annual periods, with about the same degree of approximation in both cases, the obvious lengths for a series to be analyzed harmonically are 6 or 7 years. Both lengths are here used. A 6-year series for the annual component includes 72 readings, the time between the first and the last reading being 5.92 years. For the 14-month component a 6-year series is taken to mean one that represents 5 free periods; it has 60 readings with an interval of 5.82 years between the first and the last reading. Similarly, a 7-year series has 84 readings for the annual component, with 6.92 years between the first and the last reading, and 72 readings for the 14-month component, with 7.01 years between the first and the last reading.

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[5] See sec. 2 of Chap. VI, p. 73.
[6] Monthly Notices Royal Astronomical Society, vol. 78 (1918), p. 462.
[7] Resultate, Vol. V, p. 209.

The mean reading corresponding to each twelfth of a period was found and these means were subjected to the usual process of the harmonic analysis for 12 ordinates. The values of s and c in the customary notation, say for the annual component (see p. 76), are affected by the presence of the 14-month component, and vice versa. To remove the effects, the formulas developed on pages 78–79 were used. To show the order of magnitude of the corrections so obtained the table of harmonic constants that follows gives the results both from the uncorrected and the corrected s's and c's. The epochs or ζ's for the 14-month component are all reduced to the beginning of the year 1900. The quantity $\zeta+\lambda$, where λ is the west longitude of the station, is added to show how nearly the motion of the pole is uniform and circular. If the motion were uniform and circular and in the same direction as the rotation of the earth and if all the results were errorless, the quantity $\zeta+\lambda$ would be constant for all longitudes. The length of the series analyzed is to be inferred from the dates in the second column; thus the dates 1900–05 indicate a 6-year series with the first reading for both components on 1900.00. The dates 1900–06 indicate a 7-year series beginning as before; the final reading for the 14-month component of such a series falls on 1907.01. The harmonic constants in the table below are not quite the definitive values resulting from this investigation (see p. 51) but they are accurate enough for the immediate purpose.

TABLE 3.—*Harmonic constants for the variation of latitude.*

[Assumed form, $R \cos (\kappa t-\zeta)$.]

ANNUAL COMPONENT.

Station and longitude.	Years of series inclusive.	Before clearing.		After clearing.		$\zeta+\lambda$
		Amplitude. R	Epoch. ζ	Amplitude. R	Epoch. ζ	
		″	°	″	°	°
Mizusawa, Japan. $\lambda=-141°\ 08'$	1900–05	0.113	13.3	0.106	15.7	234.6
	1900–06	.097	16.1	.107	15.9	234.8
	1906–11	.138	4.5	.132	10.4	229.3
	1906–12	.110	14.4	.123	9.2	228.1
	1910–15	.127	18.9	.140	17.4	236.3
	1911–17	.116	20.2	.111	13.1	232.0
	1912–17	.105	1.0	.100	7.1	226.0
Tschardjui, Turkestan. $\lambda=-63°\ 29'$	1900–05	.104	334.7	.101	340.0	276.5
	1900–06	.098	340.0	.105	335.9	272.4
	1906–11	.136	315.6	.133	322.3	258.8
	1906–12	.122	331.3	.131	323.7	260.2
	1908–14	.129	313.5	.134	321.4	257.9
	1909–14	.146	327.0	.148	319.8	256.3
Carloforte, Sardinia. $\lambda=-8°\ 19'$	1900–05	.101	271.3	.097	275.9	267.6
	1900–06	.094	283.0	.101	278.2	269.9
	1906–11	.119	270.2	.121	276.7	268.4
	1906–12	.104	284.6	.109	276.3	268.0
	1910–15	.107	273.1	.122	268.7	260.4
	1911–17	.108	279.1	.098	273.4	265.1
	1912–17	.085	263.9	.085	272.1	263.8
Gaithersburg, Maryland. $\lambda=+77°\ 12'$	1900–05	.027	175.8	.023	196.6	273.8
	1900–06	.024	227.0	.028	205.4	282.2
	1903–08	.024	266.9	.013	255.8	333.0
	1903–09	.023	152.7	.012	179.5	256.7
	1906–11	.058	168.7	.059	185.8	263.0
	1906–12	.049	203.2	.052	182.6	259.8
	1908–14	.057	146.7	.059	195.3	272.5
	1909–14	.070	176.3	.072	163.1	240.3

TABLE 3.—*Harmonic constants for the variation of latitude*—Continued.

ANNNUAL COMPONENT—Continued.

Station and longitude.	Years of series inclusive.	Before clearing. Amplitude. R	Before clearing. Epoch. ζ	After clearing. Amplitude. R	After clearing. Epoch. ζ	$\zeta+\lambda$
		″	°	″	°	°
Cincinnati, Ohio. $\lambda=+84°\ 25'$	1900–05	.014	159.8	.009	219.4	303.8
	1900–06	.022	246.9	.022	217.2	301.6
	1903–08	.041	263.7	.029	261.2	345.6
	1906–11	.052	187.6	.059	202.5	286.9
	1906–12	.061	216.9	.056	200.8	285.2
	1908–13	.065	182.9	.053	178.3	262.7
	1910–15	.086	108.8	.099	165.6	250.0
Ukiah, California. $\lambda=+123°\ 13'$	1900–05	.048	84.6	.037	85.7	208.9
	1900–06	.024	88.4	.035	92.8	216.0
	1906–11	.069	68.4	.055	76.0	199.2
	1906–12	.034	67.1	.053	67.8	191.0
	1910–15	.067	96.0	.082	96.9	220.1
	1911–17	.057	117.6	.057	101.6	224.8
	1912–17	.075	90.6	.064	100.5	223.7

14-MONTH COMPONENT.

Station and longitude.	Years of series inclusive.	Before clearing. Amplitude. R	Before clearing. Epoch. ζ	After clearing. Amplitude. R	After clearing. Epoch. ζ	$\zeta+\lambda$
		″	°	″	°	°
Mizusawa, Japan. $\lambda=-141°\ 08'$	1900–05	0.142	179.3	0.134	177.5	36.4
	1900–06	.123	176.4	.134	176.5	35.4
	1906–11	.218	165.8	.213	163.2	22.1
	1906–12	.215	163.6	.226	165.3	24.2
	1910–15	.201	174.8	.213	175.6	34.5
	1911–17	.177	175.8	.175	179.4	38.3
	1912–17	.172	187.8	.168	185.2	44.1
Tschardjui, Turkestan. $\lambda=-63°\ 29'$	1900–05	.135	106.8	.132	103.0	39.5
	1900–06	.115	98.9	.123	102.3	38.8
	1906–11	.209	100.7	.208	97.3	33.8
	1906–12	.211	95.0	.218	97.8	34.3
	1908–14	.229	109.1	.234	106.2	42.7
	1909–14	.241	103.7	.242	107.0	43.5
Carloforte, Sardinia. $\lambda=-8°\ 19'$	1900–05	.130	42.9	.128	39.4	31.1
	1900–06	.125	33.4	.130	36.3	28.0
	1906–11	.204	33.8	.208	31.2	22.9
	1906–12	.204	32.6	.207	35.0	26.7
	1910–15	.216	50.5	.224	52.2	43.9
	1911–17	.197	53.8	.192	55.7	47.4
	1912–17	.184	62.3	.185	60.2	51.9
Gaithersburg, Maryland. $\lambda=+77°\ 12'$	1900–05	.144	322.1	.143	321.3	38.5
	1900–06	.142	317.8	.144	318.6	35.8
	1903–08	.155	310.5	.155	310.6	27.8
	1903–09	.174	311.7	.174	311.4	28.6
	1906–11	.225	305.0	.226	303.7	20.9
	1906–12	.221	305.2	.224	306.4	23.6
	1908–14	.210	317.0	.212	315.4	32.6
	1909–14	.213	317.6	.215	319.4	36.6
Cincinnati, Ohio. $\lambda=+84°\ 25'$	1900–05	.156	317.3	.157	317.1	41.5
	1900–06	.153	310.5	.154	311.3	35.7
	1903–08	.159	299.2	.156	299.3	23.7
	1906–11	.207	297.5	.210	296.4	20.8
	1906–12	.206	296.2	.207	297.8	22.2
	1908–13	.198	303.1	.194	303.6	28.0
	1910–15	.207	323.2	.216	323.9	48.3
Ukiah, California. $\lambda=+123°\ 13'$	1900–05	.135	277.5	.131	275.7	38.9
	1900–06	.131	273.4	.134	273.0	36.2
	1906–11	.225	256.3	.222	255.5	18.7
	1906–12	.216	258.6	.221	258.4	21.6
	1910–15	.199	279.7	.206	279.7	42.9
	1911–17	.207	283.2	.208	284.5	47.7
	1912–17	.220	287.5	.217	286.2	49.4

Much might be said in interpretation of the results in Table 3, but the discussion may conveniently be postponed (see pp. 51 to 64) except for a few remarks bearing directly on the problem of detecting a progressive variation of latitude.

It is at once evident that the so-called "harmonic constants" are only approximately constant. Their precise values have a significance only as means for the period of time from which they were obtained. However, if we limit ourselves to comparing merely the harmonic constants of the various stations derived from observations covering one and the same period of time for all stations, there are still some useful conclusions to be drawn.

The values of $\zeta + \lambda$ for the annual component show such marked variations that it is evident that something besides the motion of the pole enters into them. This something is generally called the Kimura term; it has been variously explained and is doubtless due to a combination of causes. It is generally assumed in most discussions of the variation of latitude that the Kimura is the same at all stations on the same parallel at any given time. This approximate equality comes out from the discussions later on (see p. 54) but there is no reason except mathematical convenience for assuming that the equality is absolute.[8] This shows the advantage of dealing with each station by itself, quite apart from any motion of the pole. The apparent annual variation of latitude at any given station depends not only on the motion of the pole but on the individual Kimura term of that station. From the observations at one station alone we can not separate the two, but for the present purpose, namely, that of eliminating the annual portion of the variation at that station, we do not need to know how much is due to the motion of the pole and how much to the individual peculiarities of the Kimura term.

For the 14-month component, on the other hand, the values of R and $\zeta + \lambda$ show a tolerable constancy from station to station, which is favorable to the hypothesis that the variation of latitude in the 14-month period is due chiefly to the motion of the pole in an orbit, approximately circular, about its mean position. Any individual peculiarities of the station as respects the variation of latitude in the 14-month period seem to be quite small. (See p. 61.)

SECTION 3. MEAN LATITUDES BY CALENDAR YEARS AND BY 14-MONTH PERIODS.

Having obtained the harmonic constants, we could do just what was done in Dyson's paper; we could adopt a set of harmonic constants for a given period and a given station, compute the corresponding annual portion of the variation of latitude, and by subtracting this annual portion from the ordinates of the curve mentioned on page 8, which may be called the curve of the complete variation, we should get the ordinates of a new curve of exactly the same nature as Dyson's curves (p. 3). The new curve would presumably contain only the variation of latitude in the 14-month period (this statement being subject to the same qualifications as on p. 3) and

[8] The weight of authority seems to favor the opinion that the Kimura term is chiefly due to the seasonal variation of the refraction, and to changes in the refraction in passing from the observatory to the outside air rather more than to refractions due to seasonal barometric gradients. If this be true, it may be considered as certain that, owing to differences of climate and local topography, the Kimura term will not be absolutely alike at all stations.

by taking the means of its ordinates over a 14-month period the periodic part would disappear and the means would bring to light the progressive variation of latitude, if any. Furthermore, the process may be reversed; that is, the 14-month portion of the variation may be computed from the assumed harmonic constants and subtracted from the curve of the complete variation to obtain the ordinates of a new curve, which will, presumably, contain only the annual portion of the variation. By taking means over a year the annual periodic part of the variation should disappear and the yearly means should put in evidence the progressive variation of latitude, if any.

The only difference between the process just outlined and what was actually done is that the drawing and reading of a second actual curve were dispensed with. The ordinates of the original curve of the complete variation had already been obtained at suitable intervals in the process of making the harmonic analysis. The portion of the variation, annual or 14-monthly as the case may be, that is to be removed may be computed from the adopted harmonic constants and subtracted from the ordinates of the curve of complete variation, thus obtaining the ordinates of the new curve and the means of these new ordinates may be taken over the proper period, all without actually drawing the new curve. Furthermore, the computation may be simplified, for instead of applying our corrections individually to the curve of complete variation and taking the mean of the corrected result, we may simply apply the mean of the corrections to the mean of the uncorrected ordinates, both means covering the proper period. The calculation of the mean correction is rendered very simple by a formula of the type of (1) on page 71.

The most obvious weakness of any calculation of this sort is in the choice of the harmonic constants for a particular year or period of 14 months. Since the so-called constants are valid only as mean values over a given period of time and it is not easy to determine mathematically their values at any given instant, there is necessarily an arbitrary element in the assignment of values for any given year or 14-month period. The consequences of this uncertainty are not, however, so serious as might be supposed. The calculations will be given for this part of the investigation in more detail than for most of the other parts, and it will not be difficult to satisfy oneself that any reasonable choice of the harmonic constants would lead to substantially the same result for the progressive change of latitude, which is the quantity sought.

The more important stages of computations for the case where the mean latitude is taken by calendar years are given on pages 15 and 16. The first column under each station gives the calendar year over which the mean is taken and the second the mean itself. In this mean 13 values were used, already read off from the curve for the purposes of the harmonic analysis. The first and last of the 13 for the year 1900, for example, are for the dates 1900.00 and 1901.00. These two end values are given only half weight. The mean latitude evidently applies exactly to the middle of the year 1900. The third and fourth columns contain the amplitudes and epochs (R and ζ) of the 14-month component as adopted for the purpose of clearing the annual mean. The epochs are reduced to the beginning of 1900 so as to be comparable with those of the preceding table. The way in which these quantities were determined will be explained in the next paragraph. Given the values of R and ζ, the quantity to be added to the uncorrected mean to clear it of the outstanding effects of tho 14-month component is found to be

$$0.174 R \cos (332^\circ\!.0 - \zeta - 55^\circ\!.98 n) \tag{1}$$

(See p. 73.) In this expression n is the number of years after 1900; e. g., for the year 1907, $n=7$. The values of the above expression are given in the fifth column; the sixth column, which is the sum of the second and fifth columns, is the cleared mean which applies to the middle of the appropriate calendar year; the trend of these should bring to light the progressive variation of latitude, if any.

As has been said, the 14-month component of the variation of latitude appears to be due almost entirely to the motion of the pole, but for the present purposes it is enough to assume that the variations with the time in R and ζ at any station correspond to variations in the motion of the pole. In an article by Wanach [9] there are given the instantaneous values of the amplitude and phase of the polar motion in the 14-month period. Wanach's quantities are computed on the assumption that the annual component repeats itself exactly, an assumption known to be erroneous and one which is therefore quite sufficient to explain the decidedly erratic variations in the quantities tabulated by Wanach. These he calls R, the amplitude of the polar motion and P its initial phase reckoned from a point midway between maximum and minimum. Although the values of R and P given for every tenth of a year are so far from constant, their means over a calendar year are more stable and these means [10] are given in the following table. In taking means the end values were given half weight and the year 1917, which is incomplete, is treated as if complete.

TABLE 4.—*Mean amplitude and phase of the 14-monthly polar motion.*

Calendar year.	Amplitude, R.	Phase, P.	Calendar year.	Amplitude, R.	Phase, P.
	″	°		″	°
1900	0.133	63	1909	0.268	66
1901	.166	55	1910	.234	62
1902	.120	53	1911	.262	64
1903	.144	52	1912	.279	47
1904	.139	52	1913	.214	38
1905	.089	53	1914	.202	46
1906	.138	80	1915	.189	33
1907	.189	94	1916	.157	38
1908	.258	78	1917	.161	25

The values of R in Table 4 are directly comparable with those in Table 3 and if the means are taken over corresponding periods the two sets of values are nearly equal. The quantity P should be nearly equal to $90^\circ - (\zeta + \lambda)$ for the 14-component and this is seen to be very nearly true. An example will explain how the values of R and ζ used on pages 15 to 17 for clearing the annual mean were obtained from Tables 3 and 4.

Let us take the year 1902 at Mizusawa and compare it with the harmonic analysis at the same station for the years 1900–05. The

[9] Die Chandlersche und die Newcombsche Periode der Polbewegung. Pub. No. 34 (new series), Zentralbureau der Internationalen Erdmessung. Berlin, 1919, p. 23.

[10] The ordinary mathematical mean was taken. This process is not quite rigorous in such a case and tends to exaggerate a little the values of R, but for present purposes it is sufficiently accurate.

mean R for the years 1900–05 from Table 4 is 0."132. The analysis (Table 3) gives 0."134. In Table 4 the R for 1902 is 0."120 or 0."012 less than the mean, so it is assumed that the R for Mizusawa in 1902 should also be 0."012 less than 0."134 or 0."122. In a similar way the mean P for the period is 55° and for 1902 is 53° or 2° *less*. Since ζ and P vary in opposite directions from their mean values the ζ for Mizusawa in 1902 should be 2° *greater* than the ζ in Table 3 corresponding to the whole period; i. e., its value is $177°.5+2°=179°.5$, or 180° by the ordinary convention for rounding off to whole degrees. The period 1900–06 and the corresponding harmonic analysis might be used in a similar way and values for 1902 deduced. The results are practically identical with those given. The values, $R=$ 0."122 and $\zeta=180°$, are found in the third and fourth columns of the table below. Some years, such as 1911, are included in the periods covered by several harmonic analyses; in such cases the mean of the several comparisons, which generally differ but little from one another, is used.

TABLE 5.—*Mean latitude by calendar years.*

MIZUSAWA, JAPAN.

Year.	Seconds of mean latitude, uncleared.	Harmonic constants, 14-month component. R	Harmonic constants, 14-month component. ζ	Clearance.	Seconds of mean latitude, cleared.
	"	"	°	"	"
1900	3.634	0.134	170	−0.022	3.612
1901	.627	.168	178	−.001	3.623
1902	.559	.122	180	+.016	3.575
1903	.562	.146	182	+.024	3.586
1904	.590	.140	182	+.007	3.597
1905	3.591	0.090	180	−0.010	3.581
1906	.613	.133	155	−.022	3.591
1907	.609	.180	142	−.029	3.580
1908	.589	.248	158	+.003	3.592
1909	.546	.258	170	+.043	3.589
1910	3.567	0.222	170	+0.030	3.597
1911	.559	.244	172	−.004	3.555
1912	.628	.257	178	−.041	3.587
1913	.627	.189	184	−.025	3.602
1914	.612	.174	176	−.001	3.611
1915	3.554	0.161	189	+0.026	3.580
1916	.571	.124	184	+.019	3.590
1917	.648	.129	198	−.003	3.645

TSCHARDJUI, TURKESTAN.

Year.	Seconds of mean latitude, uncleared.	R	ζ	Clearance.	Seconds of mean latitude, cleared.
1900	10.698	0.128	96	−0.012	10.686
1901	.716	.161	104	−.028	10.688
1902	.643	.115	106	−.008	10.635
1903	.663	.140	107	+.013	10.676
1904	.636	.134	107	+.023	10.659
1905	10.602	0.084	106	+0.009	10.611
1906	.536	.124	86	.000	10.536
1907	.617	.173	76	−.022	10.595
1908	.714	.242	89	−.038	10.676
1909	.684	.256	100	+.002	10.686
1910	10.672	0.223	104	+0.034	10.706
1911	.724	.250	102	+.039	10.763
1912	.737	.270	117	−.006	10.731
1913	.825	.208	124	−.034	10.791
1914	.886	.196	116	−.030	10.856

TABLE 5.—*Mean latitude by calendar years*—Continued.

CARLOFORTE, SARDINIA.

Year.	Seconds of mean latitude, uncleared.	Harmonic constants, 14-month component. R	Harmonic constants, 14-month component. ζ	Clearance.	Seconds of mean latitude, cleared.
	″	″	°	″	″
1900	8.909	0.130	31	+0.012	8.921
1901	9.010	.162	39	− .015	8.995
1902	8.956	.116	41	− .020	8.936
1903	8.935	.142	42	− .013	8.922
1904	8.946	.136	42	+ .010	8.956
1905	8.948	0.086	41	+0.015	8.963
1906	.911	.122	23	+ .019	8.930
1907	.904	.167	11	+ .009	8.913
1908	.954	.236	27	− .033	8.921
1909	.973	.246	39	− .037	8.936
1910	8.960	0.218	41	−0.001	8.959
1911	.922	.245	38	+ .034	8.956
1912	.932	.262	53	+ .038	8.970
1913	9.004	.201	60	− .004	9.000
1914	9.027	.189	52	− .027	9.000
1915	9.020	0.170	66	−0.025	8.995
1916	.027	.141	59	− .003	9.024
1917	.040	.145	74	+ .023	9.063

GAITHERSBURG, MD.

Year.	Seconds of mean latitude, uncleared.	R	ζ	Clearance.	Seconds of mean latitude, cleared.
1900	13.115	0.144	314	+0.024	13.139
1901	.178	.176	322	+ .021	13.199
1902	.238	.130	324	− .005	13.233
1903	.271	.149	326	− .025	13.246
1904	.217	.143	326	− .020	13.197
1905	13.215	0.093	324	+0.001	13.216
1906	.222	.137	297	+ .012	13.234
1907	.208	.186	284	+ .031	13.239
1908	.232	.249	298	+ .025	13.257
1909	.278	.254	309	− .023	13.255
1910	13.297	0.217	312	−0.038	13.259
1911	.307	.245	310	− .026	13.281
1912	.288	.250	327	+ .026	13.314
1913	.272	.184	334	+ .032	13.304
1914	.327	.172	326	+ .016	13.343

CINCINNATI, OHIO.

Year.	Seconds of mean latitude, uncleared.	R	ζ	Clearance.	Seconds of mean latitude, cleared.
1900	19.318	0.156	308	+0.025	19.343
1901	.274	.189	316	+ .025	19.299
1902	.392	.143	318	− .003	19.389
1903	.436	.159	318	− .025	19.411
1904	.443	.153	318	− .023	19.420
1905	19.292	0.103	317	−0.002	19.290
1906	.326	.132	289	+ .009	19.335
1907	.277	.174	274	+ .027	19.304
1908	.311	.232	289	+ .029	19.370
1909	.368	.235	302	− .017	19.351
1910	19.408	0.206	306	−0.036	19.372
1911	.442	.233	304	− .027	19.416
1912	.410	.240	320	+ .021	19.431
1913	.392	.178	329	+ .031	19.423
1914	.473	.188	326	+ .017	19.490
1915	19.548	0.175	330	−0.018	19.530

TABLE 5.—*Mean latitude by calendar years*—Continued.

UKIAH, CALIF.

Year.	Seconds of mean latitude, uncleared.	Harmonic constants, 14-month component.		Clearance.	Seconds of mean latitude, cleared.
		R	ζ		
	″	″	°	″	″
1900	12.078	0.133	268	+0.010	12.088
1901	.024	.166	276	+ .029	12.053
1902	.048	.120	278	+ .011	12.059
1903	.115	.146	279	− .011	12.104
1904	.142	.140	279	− .024	12.118
1905	12.136	0.090	278	−0.011	12.125
1906	.118	.133	250	− .006	12.112
1907	.079	.182	235	+ .013	12.092
1908	.028	.250	251	+ .043	12.071
1909	.103	.260	263	+ .012	12.115
1910	12.163	0.221	267	−0.027	12.136
1911	.205	.250	264	− .043	12.162
1912	.185	.274	282	− .007	12.178
1913	.127	.212	288	+ .030	12.157
1914	.182	.200	280	+ .034	12.216
1915	12.251	0.187	293	+0.005	12.256
1916	.258	.165	286	− .018	12.240
1917	.264	.169	300	− .028	12.236

A process the reverse of that set forth in Table 5 consists in taking the mean over a period of 14 months (more accurately 432.5 days) and clearing this mean of the effects of the annual component. In Chapter I the mean was taken over a period of exactly 14 months (1$\frac{1}{6}$ years) and the difference between this, and the more accurate value of 432.5 days, was shown to have a nearly negligible effect. The same thing could be done here, but the curve readings used would be the same readings as those forming the basis of Table 5. We get, therefore, a determination more nearly independent of the preceding if we utilize the curve reading taken independently for the harmonic analysis of the 14-month (i. e., 432.5-day) component. The chief disadvantage is that, owing to the practice of beginning the series to be analyzed at the beginning of a year, the 432.5-day periods covered do not run consecutively and without overlap.

Table 6 gives the results of the taking and clearing of means. Its arrangement differs only slightly from that of Table 5. The first column will be explained later. The second column, which corresponds to the first column in Table 5, gives the dates in years and decimals of the beginning and ending of the period covered. The third column is the mean of 13 readings covering this period, the initial and final readings having only half weight. The mean then corresponds to the middle of the period. The fourth and fifth columns contain the harmonic constants of the annual component adopted for clearing the mean. For lack of a better principle of choice the constants in these columns are simply the mean of all available determinations in Table 3 that cover the period in question. The sixth column contains the clearance of the mean. It is given by the formula

$$\text{Clearance} = 0.142\, R \cos\,(213^\circ.1 - \zeta + 360^\circ f). \qquad (2)$$

The R and ζ of formula (2) are those in columns four and five; the quantity f is the fraction of a year elapsed at the beginning of the period to be cleared; thus, for the first line of Mizusawa $f=0.000$, for the second line $f=0.184$, for the third line $f=0.368$, etc. The seventh column representing the cleared mean is the sum of the third and sixth columns.

TABLE 6.—*Mean latitudes by periods of 432.5 days.*

MIZUSAWA, JAPAN.

Group No.	Period included.	Seconds of mean latitude, uncleared.	Harmonic constants, annual component. R.	Harmonic constants, annual component. ζ.	Clearance.	Seconds of mean latitude, cleared.
		"	"	°	"	"
1	1900.000-01.184	03.633	0.106	16	−0.014	03.619
2	1901.184-02.368	.603	.106	16	− .002	.601
3	1902.368-03.552	.569	.106	16	+ .013	.582
4	1903.552-04.736	.585	.106	16	+ .012	.597
5	1904.736-05.920	.585	.106	16	− .003	.582
6	1905.920-07.104	03.604	0.117	13	−0.016	03.588
6	1906.000-07.184	.597	.121	12	− .016	.581
7	1907.184-08.368	.569	.128	10	.000	.569
8	1908.368-09.552	.580	.128	10	+ .016	.596
9	1909.552-10.736	.586	.132	12	+ .014	.600
9	1910.000-11.184	03.612	0.126	13	−0.017	03.595
10	1910.736-11.920	.576	.126	13	− .005	.571
10	1911.000-12.184	.572	.121	11	− .016	.556
10	1911.184-12.368	.557	.121	11	− .001	.556
11	1911.920-13.104	.607	.121	11	− .017	.590
11	1912.000-13.184	03.612	0.118	12	−0.016	03.596
11	1912.184-13.368	.602	.118	12	− .001	.601
11	1912.368-13.552	.572	.118	12	+ .015	.587
12	1913.184-14.368	.615	.117	13	− .001	.614
12	1913.368-14.552	.590	.117	13	− .015	.575
12	1913.552-14.736	03.602	0.117	13	+0.013	03.615
13	1914.368-15.552	.585	.117	13	+ .015	.600
13	1914.552-15.736	.590	.117	13	+ .013	.603
13	1914.736-15.920	.602	.117	13	− .004	.598
14	1915.552-16.736	.607	.117	13	+ .013	.620
14	1915.736-16.920	.629	.117	13	− .004	.625
15	1916.736-17.920	03.650	0.106	10	−0.004	03.646
15	1916.920-18.104	.665	.106	10	− .015	.650

TSCHARDJUI, TURKESTAN.

Group No.	Period included.	Seconds of mean latitude, uncleared.	R.	ζ.	Clearance.	Seconds of mean latitude, cleared.
1	1900.000-01.184	10.694	0.103	338	−0.008	10.686
2	1901.184-02.368	.676	.103	338	+ .008	.684
3	1902.368-03.552	.632	.103	338	− .014	.618
4	1903.552-04.736	.662	.103	338	+ .004	.666
5	1904.736-05.920	.638	.103	338	− .011	.627
6	1905.920-07.104	10.558	0.118	330	−0.014	10.544
6	1906.000-07.184	.543	.123	327	− .007	.536
7	1907.184-08.368	.608	.133	322	+ .014	.622
8	1908.000-09.184	.683	.136	322	− .006	.677
8	1908.368-09.552	.662	.136	322	+ .018	.680
9	1909.000-10.184	10.692	0.136	322	−0.006	10.686
9	1909.184-10.368	.682	.136	322	+ .014	.696
9	1909.552-10.736	.702	.136	322	.000	.702
10	1910.184-11.368	.721	.136	322	+ .014	.735
10	1910.368-11.552	.720	.136	322	+ .018	.738
10	1910-736.11.920	10.759	0.136	322	−0.018	10.741
11	1911.368-12.552	.731	.136	322	+ .018	.749
11	1911.552-12.736	.760	.136	322	.000	.760
11	1911.920-13.104	.756	.136	322	− .014	.742
12	1912.552-13.736	.741	.137	321	.000	.741
12	1912.736-13.920	10.780	0.137	321	−0.018	10.762
13	1913.736-14.920	.875	.141	321	− .018	.857
13	1913.920-15.104	.869	.141	321	− .015	.854

TABLE 6.—*Mean latitudes by periods of 432.5 days*—Continued.

CARLOFORTE, SARDINIA.

Group No.	Period included.	Seconds of mean latitude, uncleared.	Harmonic constants, annual component. R.	Harmonic constants, annual component. ζ.	Clearance.	Seconds of mean latitude, cleared.
		″	″	°	″	″
1	1900.000-01.184	8.921	0.099	277	+0.006	08.927
2	1901.184-02.368	.963	.099	277	+ .014	.977
3	1902.368-03.552	.923	.099	277	+ .005	.928
4	1903.552-04.736	.962	.099	277	− .010	.952
5	1904.736-05.920	.975	.099	277	− .013	.962
6	1905.920-07.104	8.927	0.107	277	−0.001	08.926
6	1906.000-07.184	.925	.110	277	+ .007	.932
7	1907.184-08.368	.907	.115	276	+ .016	.923
8	1908.368-09.552	.903	.115	276	+ .006	.909
9	1909.552-10.736	.962	.117	274	− .012	.950
9	1910.000-11.184	8.952	0.112	274	+0.008	08.960
10	1910.736-11.920	.970	.112	274	− .014	.956
10	1911.000-12.184	.954	.107	274	+ .007	.961
10	1911.184-12.368	.953	.107	274	+ .015	.968
11	1911.920-13.104	.965	.107	274	.000	.965
11	1912.000-13.184	8.957	0.103	272	+0.008	08.965
11	1912.184-13.368	.959	.103	272	+ .014	.973
11	1912.368-13.552	.980	.103	272	+ .004	.984
12	1913.184-14.368	.987	.102	271	+ .014	09.001
12	1913.368-14.552	.996	.102	271	+ .004	.000
12	1913.552-14.736	9.005	0.102	271	−0.011	08.994
13	1914.368-15.552	8.984	.102	271	+ .004	.988
13	1914.552-15.736	8.997	.102	271	− .011	.986
13	1914.736-15.920	9.000	.102	271	− .013	.987
14	1915.552-16.736	.026	.102	271	− .011	09.015
14	1915.736-16.920	9.031	0.102	271	−0.013	09.018
15	1916.736-17.920	.073	.092	273	− .012	.061
15	1916.920-18.104	.053	.092	273	.000	.053

GAITHERSBURG, MD.

Group No.	Period included.	Seconds of mean latitude, uncleared.	R.	ζ.	Clearance.	Seconds of mean latitude, cleared.
1	1900.000-01.184	13.143	0.026	201	+0.004	13.147
2	1901.184-02.368	.201	.026	201	+ .001	.202
3	1902.368-03.552	.254	.019	209	− .002	.252
4	1903.000-04.184	.244	.019	209	+ .003	.247
4	1903.552-04.736	.214	.019	209	− .003	.211
5	1904.184-05.368	13.192	0.019	209	+0.001	13.193
6	1905.368-06.552	.225	.031	201	− .004	.221
6	1905.920-07.104	.229	.031	201	+ .004	.233
6	1906.000-07.184	.232	.033	202	+ .005	.237
7	1906.552-07.736	.233	.033	202	− .004	.229
7	1907.184-08.368	13.260	0.039	200	+0.001	13.261
8	1907.736-08.920	.252	.039	200	.000	.252
8	1908.000-09.184	.245	.044	194	+ .006	.251
8	1908.368-09.552	.265	.044	194	− .006	.259
9	1908.920-10.104	.252	.044	194	+ .006	.258
9	1909.000-10.184	13.253	0.051	181	+0.006	13.259
9	1909.184-10.368	.257	.051	181	− .001	.256
9	1909.552-10.736	.263	.051	181	− .005	.258
10	1910.184-11.368	.259	.060	182	− .001	.258
11	1911.368-12.552	.307	.060	182	− .008	.299
11	1911.552-12.736	13.307	0.060	182	−0.005	13.302
12	1911.920-13.104	.302	.060	182	+ .009	.311
12	1912.552-13.736	.329	.061	180	− .005	.324
12	1912.736-13.920	.309	.061	180	+ .004	.313
13	1913.736-14.920	.325	.066	179	+ .005	.330
13	1913.920-15.104	.331	.066	179	+ .009	.340

TABLE 6.—*Mean latitudes by periods of 432.5 days*—Continued.

CINCINNATI, OHIO.

Group No.	Period included.	Seconds of mean latitude, uncleared.	Harmonic constants, annual component. R.	ζ.	Clearance.	Seconds of mean latitude, cleared.
		''	''	°	''	''
1	1900.000-01.184	19.340	0.016	216	+0.003	19.343
2	1901.184-02.368	.297	.016	216	+ .001	.298
3	1902.368-03.552	.325	.020	233	− .001	.324
4	1903.000-04.184	.419	.020	233	+ .003	.422
4	1903.552-04.736	.423	.020	233	.000	.423
5	1904.184-05.368	19.371	0.020	233	+0.002	19.373
5	1904.736-05.920	.316	.020	233	− .001	.315
6	1905.368-06.552	.299	.038	220	− .003	.296
6	1905.920-07.104	.332	.038	220	+ .004	.336
6	1906.000-07.184	.329	.042	220	+ .006	.335
7	1906.552-07.736	19.316	0.042	220	−0.006	19.310
7	1907.184-08.368	.323	.049	211	+ .003	.326
8	1907.736-08.920	.348	.049	211	.000	.348
8	1908.000-09.184	.360	.049	211	+ .007	.367
8	1908.368-09.552	.301	.049	211	− .005	.296
9	1909.184-10.368	19.350	0.067	187	−0.001	19.349
9	1909.552-10.736	.352	.067	187	− .007	.345
10	1910.000-11.184	.371	.067	187	+ .009	.380
10	1910.368-11.552	.408	.067	187	− .009	.399
10	1910.736-11.920	.406	.067	187	− .003	.403
11	1911.184-12.368	19.371	0.067	187	−0.001	19.370
11	1911.552-12.736	.430	.067	187	− .007	.423
12	1911.920-13.104	.416	.067	187	+ .010	.426
12	1912.368-13.552	.457	.069	182	− .009	.448
12	1912.736-13.920	.424	.069	182	− .004	.420
13	1913.552-14.736	19.488	0.076	172	−0.005	19.483
14	1914.736-15.920	.513	.099	172	+ .008	.521

UKIAH, CALIF.

Group No.	Period included.	Seconds of mean latitude, uncleared.	R.	ζ.	Clearance.	Seconds of mean latitude, cleared.
1	1900.000-01.184	12.089	0.036	89	−0.003	12.086
2	1901.184-02.368	.050	.036	89	− .005	.045
3	1902.368-03.552	.085	.036	89	− .001	.084
4	1903.552-04.736	.115	.036	89	+ .004	.119
5	1904.736-05.920	.120	.036	89	+ .004	.124
6	1905.920-07.104	12.116	0.045	81	−0.002	12.114
6	1906.000-07.184	.115	.048	79	− .005	.110
7	1907.184-08.368	.092	.054	72	− .007	.085
8	1908.368-09.552	.104	.054	72	.000	.104
9	1909.552-10.736	.127	.063	80	+ .008	.135
9	1910.000-11.184	12.149	0.062	86	−0.005	12.144
10	1910.736-11.920	.149	.062	86	+ .005	.154
10	1911.000-12.184	.175	.062	86	− .005	.170
10	1911.184-12.368	.176	.062	86	− .009	.167
11	1911.920-13.104	180	.062	89	− .001	.179
11	1912.000-13.184	12.185	0.064	92	−0.005	12.180
11	1912.184-13.368	.186	.064	92	− .009	.177
11	1912.368-13.552	.173	.064	92	− .003	.170
12	1913.184-14.368	.175	.068	100	− .010	.165
12	1913.368-14.552	.182	.068	100	− .004	.178
12	1913.552-14.736	12.195	0.068	100	+0.006	12.201
13	1914.368-15.552	.257	.068	100	− .004	.253
13	1914.552-15.736	.256	.068	100	+ .006	.262
13	1914.736-15.920	.279	.068	100	+ .009	.288
14	1915.552-16.736	.239	.068	100	+ .006	.245
14	1915.736-16.920	12.229	0.068	100	+0.009	12.238
15	1916.736-17.920	.212	.060	100	+ .008	.220
15	1916.920-18.104	.220	.060	100	+ .001	.221

SECTION 4. RATES OF CHANGE OF THE LATITUDES.

The results in the last column of Table 5 are plotted in figures 4*a*, 5*a*, 6*a*, 7*a*, 8*a*, and 9*a*. An inspection of these suggests that there is a progressive increase in latitude at the American stations, Gaithersburg, Cincinnati, and Ukiah, and that in the latter part of the period

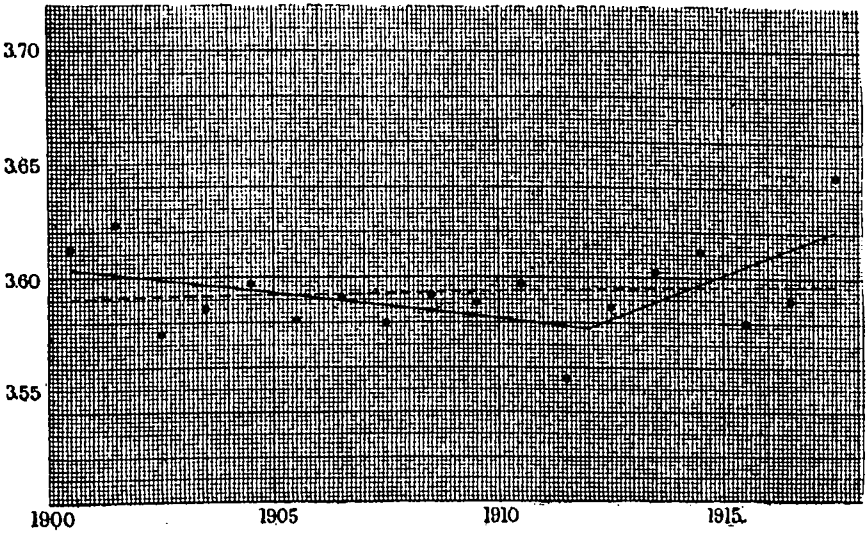

FIG. 4*a*.—Progressive change of latitude at Mizusawa—means by calendar years.

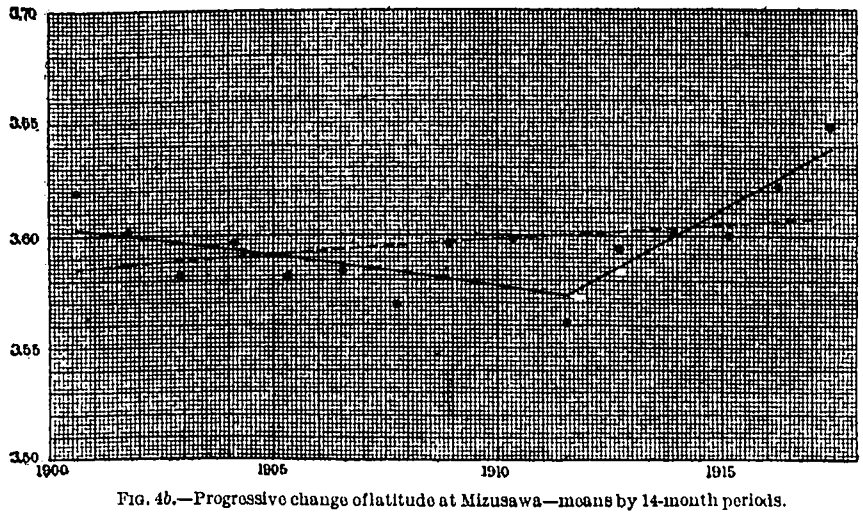

FIG. 4*b*.—Progressive change of latitude at Mizusawa—means by 14-month periods.

covered this increase becomes more rapid. Carloforte shows no clear indications of either an increase or a decrease during most of the time, but toward the end of the time there is a fairly well-defined tendency to increase. Mizusawa shows a slight decrease, if anything, at the beginning followed by an increase at the end. Tschardjui is very irregular with a sharp increase toward the end.

This changing rate of increase, which in the case of Mizusawa even reverses the sign of the progressive change of latitude, suggests that it may be desirable to modify the mathematical process used in Chapter I for determining mathematically these rates of change. After some consideration it was decided to assume the following form of observation equation:

$$\Delta\phi = x + t_1 y + t_2 z. \tag{3}$$

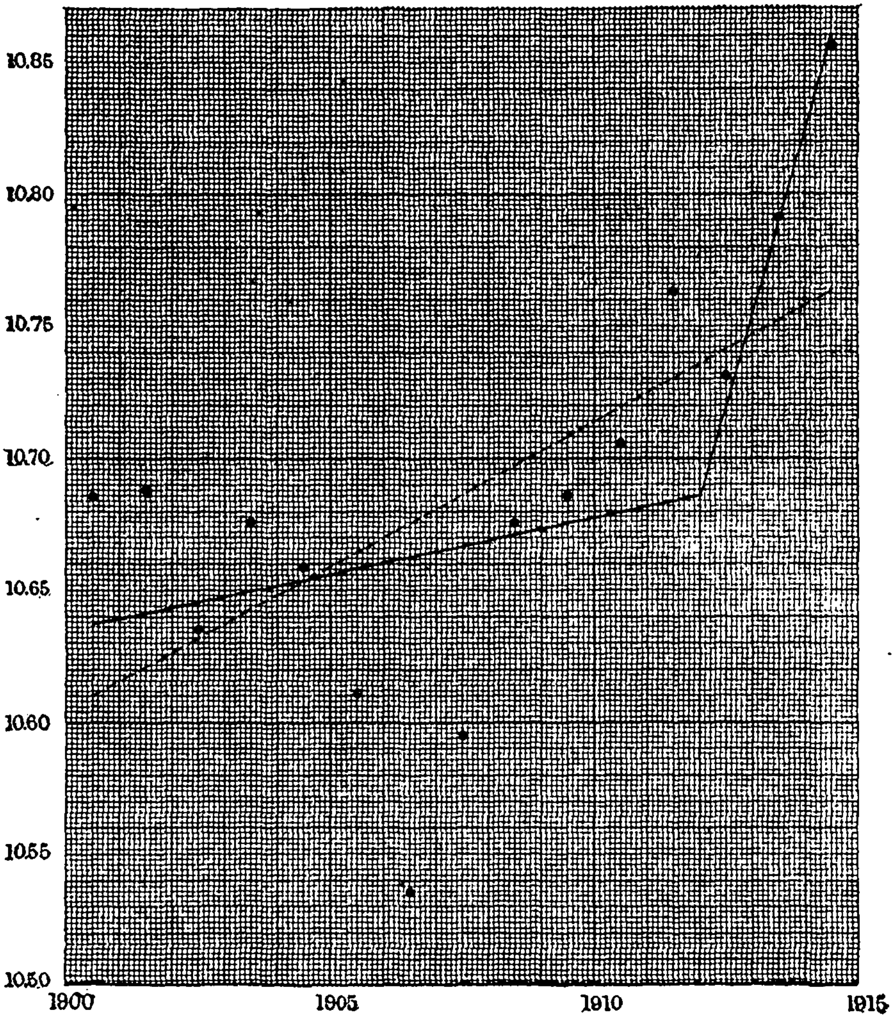

FIG. 5a.—Progressive change of latitude at Tschardjui—means by calendar years.

In this equation $\Delta\phi$ is the difference, observed latitude *minus* some fixed arbitrary latitude; the quantity x is the adjusted value of $\Delta\phi$ for the time 1912.00; y is the progressive rate of change of latitude before 1912.00; z is the rate after 1912.00; t_1 is the time reckoned from 1912.00 for dates before that time and is to be taken as zero after that time; t_2 is the time reckoned from 1912.00 for dates after that time and is to be taken as zero before then. These assumptions mean that the endeavor is made to fit the observed latitudes to two straight lines intersecting on the ordinate for 1912.00, the slopes of these lines being determined to fit the observations as closely as possible. The particular date 1912.00 was fixed on because that is the time when the change is made from the definitive results of the Resultate to the provisional results of the Nachrichten. The observed

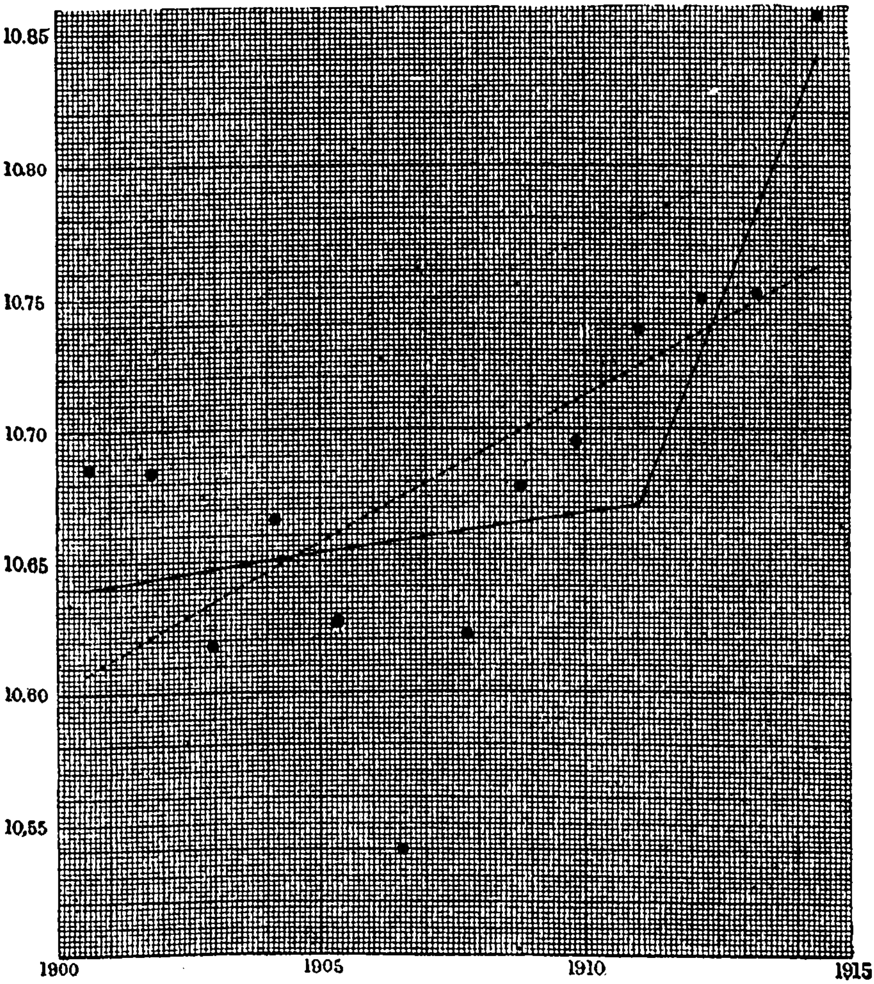

FIG. 5b.—Progressive change of latitude at Tschardjui—means by 14-month periods.

latitude depends on the declinations and these depend on the assumed proper motions in declination, which are not absolutely known.

The dropping out of some of the stars after 1912.00, already noted at the beginning of this chapter, involves a change in the mean error in the proper motions [11] and hence a change in the observed

[11] This expression is not to be taken in any sense as connected with the theory of least squares; what is meant is the mean of the actual errors in the proper motions.

latitudes. This change should affect all stations alike, barring errors of observations and the effect of the weather in causing some pairs to be missed at certain stations and other pairs at other stations. An inspection of the plotted observations suggests also that 1912.00 is not far from the time when the progressive rate of variation of the latitude seems to undergo a change, though if this time were to be

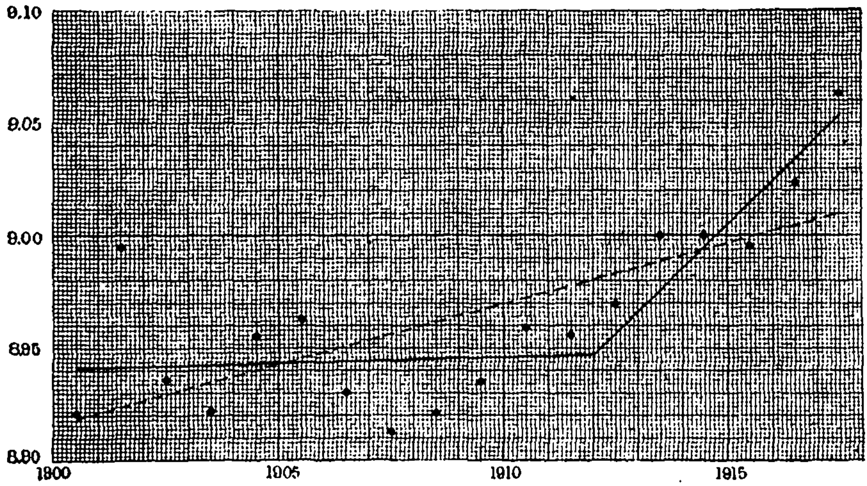

FIG. 6a.—Progressive change of latitude at Carloforte—means by calendar years.

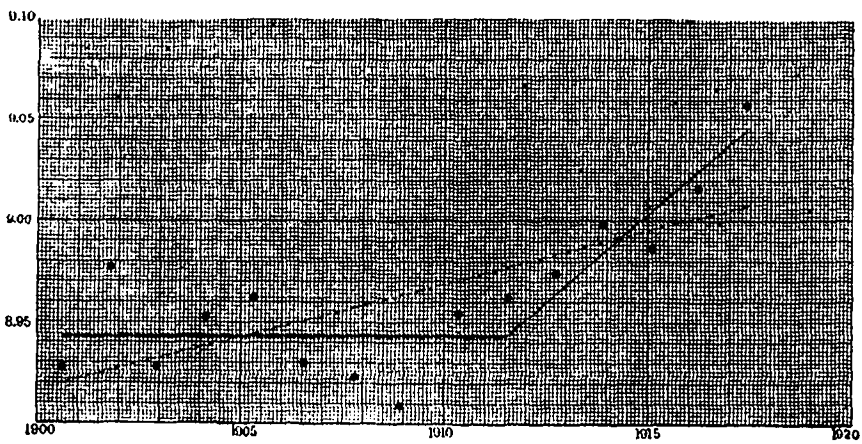

FIG. 6b.—Progressive change of latitude at Carloforte—means by 14-month periods.

determined by inspection of the plotted observations alone, it would be placed before 1912.00 rather than after it.

This may also be due to changes in the star program, for although all observed stars were used up to 1912.00, the later program was not identical with that used in 1900 and in introducing new pairs to replace those dropping out on account of precession, the reduction of the new pairs to the original declination system may not have been as accurate as was supposed.

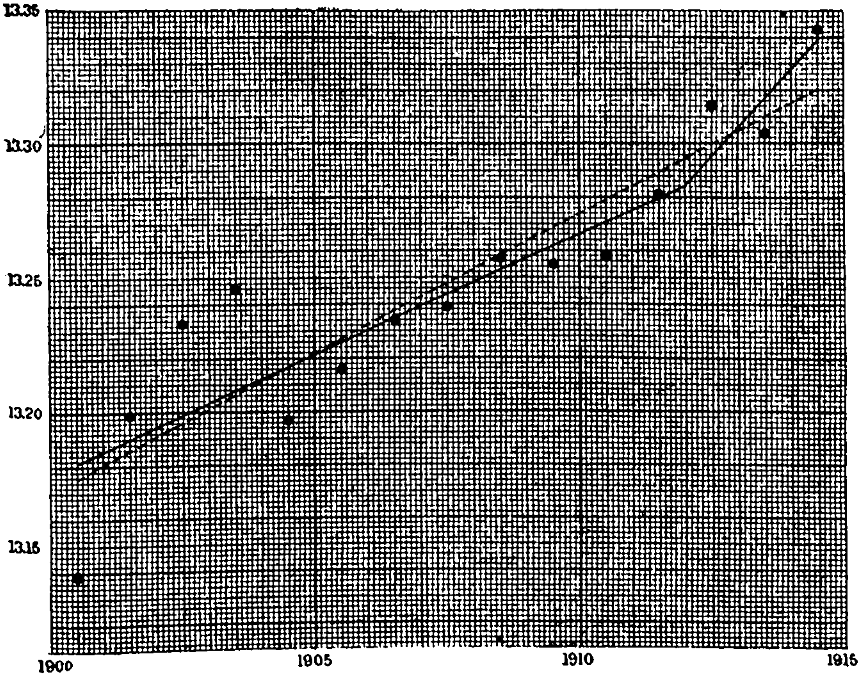

FIG. 7*a*.—Progressive change of latitude at Gaithersburg—means by calendar years.

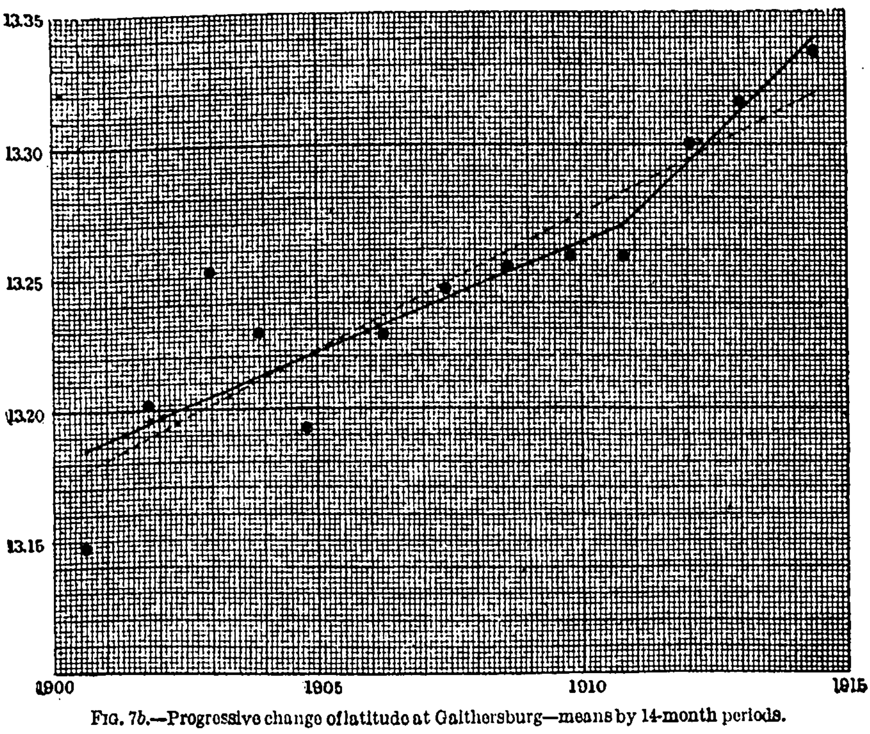

FIG. 7*b*.—Progressive change of latitude at Gaithersburg—means by 14-month periods.

On account of the small number of observations in which z is involved, it is not very well determined. If it is desired to disregard any possible change in the rate after 1912.00, it is easy to make $y=z$ in the observation equations and the normal equations, thus making all observations contribute to determining a single average rate for the whole period, as was done in Chapter I.

Normal equations for each station, based on observation equations of the form given by equation (3), were formed and solved for x, y, and z. The quantity z was also made equal to y and the resulting normal equations solved. The values of y and z, together with the probable errors, also the adjusted latitude of each station for 1912.00,

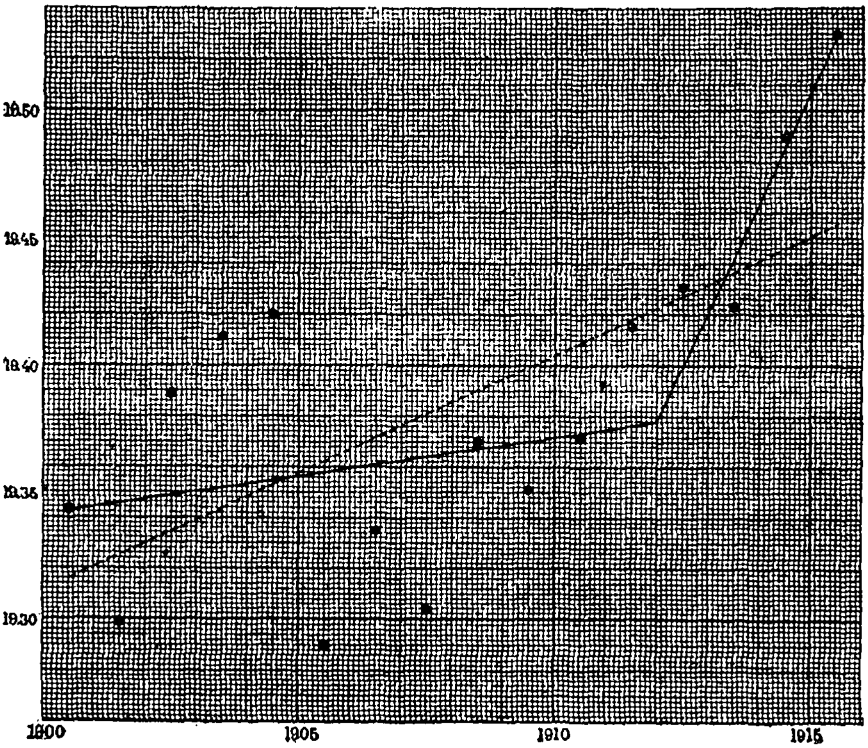

FIG. 8a.—Progressive change of latitude at Cincinnati—means by calendar years.

which depends directly on the value of x, are given in Table 8. The observations at all stations do not cover the same period, some stopping as early as the end of 1914. Since there may be some advantage in dealing with the same period for all stations, another set of solutions was made based on the observations from 1900 to 1914, inclusive. The results of these solutions are also given in Table 8. The discussion of all these results is deferred until something has been said about the analogous observations and normal equations based on means by 14-month periods.

The cleared means by 14-month periods in the last column of Table 6 correspond to a number of overlapping periods not independent of one another, so that it would not be proper to use each period as the observed quantity in formation of observation equations of equal weight. The procedure adopted was to take the means by groups as indicated in the first column of the table; many of the groups contain only a single 14-month period. These groups were chosen so that the time interval between the middle of one group and the middle of an adjacent group should be as nearly 1.184 years (432.5 days) as practicable. There is still some overlapping, but it was not considered worth while to go deeply into the question of weights. The mean was taken for each group, both with respect to times and latitudes. The mean latitudes of the groups furnished the known quantities for the observation equations, to each of which

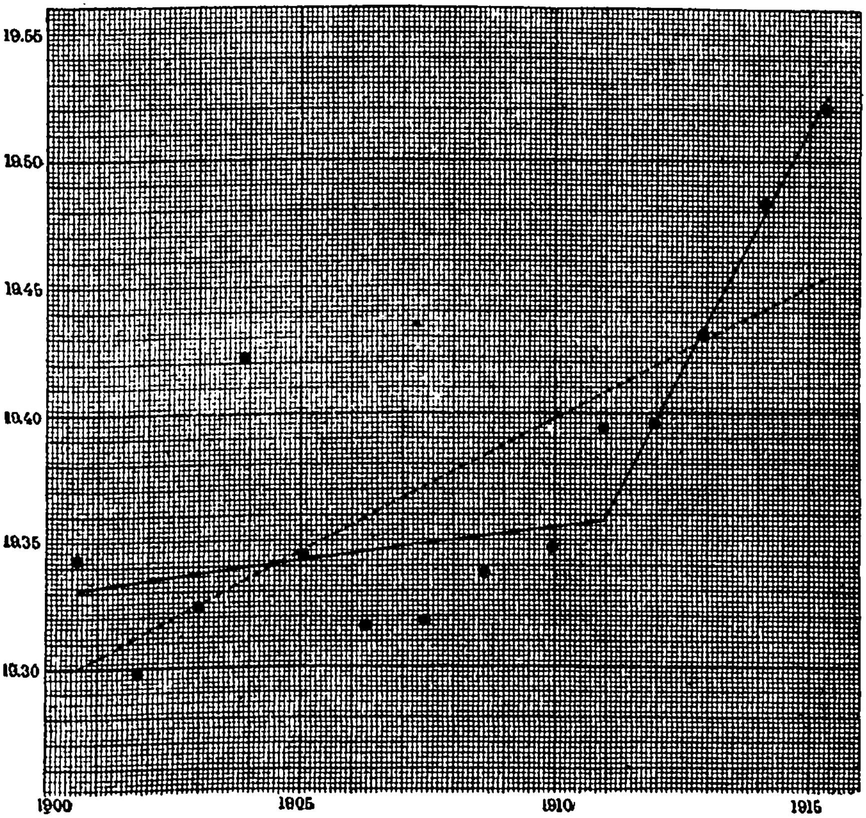

FIG. 8*b*.—Progressive change of latitude at Cincinnati—means by 14-month periods.

was attributed equal weight. In taking the means two very nearly coincident periods, such as are found in group 11 for Mizusawa, Carloforte, or Ukiah, with dates 1911.920–13.104 and 1912.000–13.184, are averaged together to form a single mean and this mean is treated as of equal weight with the other members of the group in forming the final mean for the entire group. The mean dates of each group and the corresponding mean latitudes are given in Table 7. It will be seen that Mizusawa, Carloforte, and Ukiah have identical groups. The mean latitudes in Table 7 are shown graphically in Figures 4*b* to 9*b*.

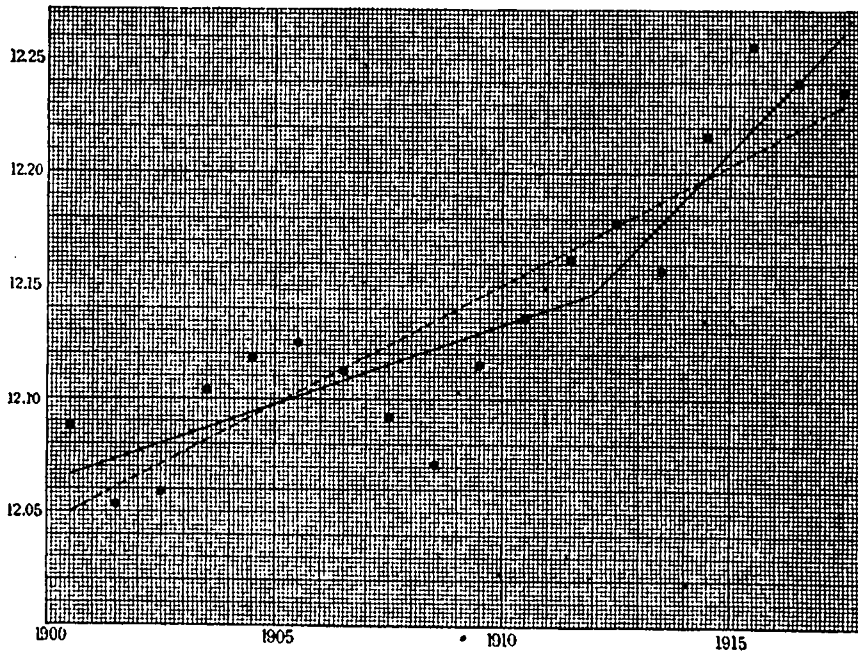

FIG. 9a.—Progressive change of latitude at Ukiah—means by calendar years.

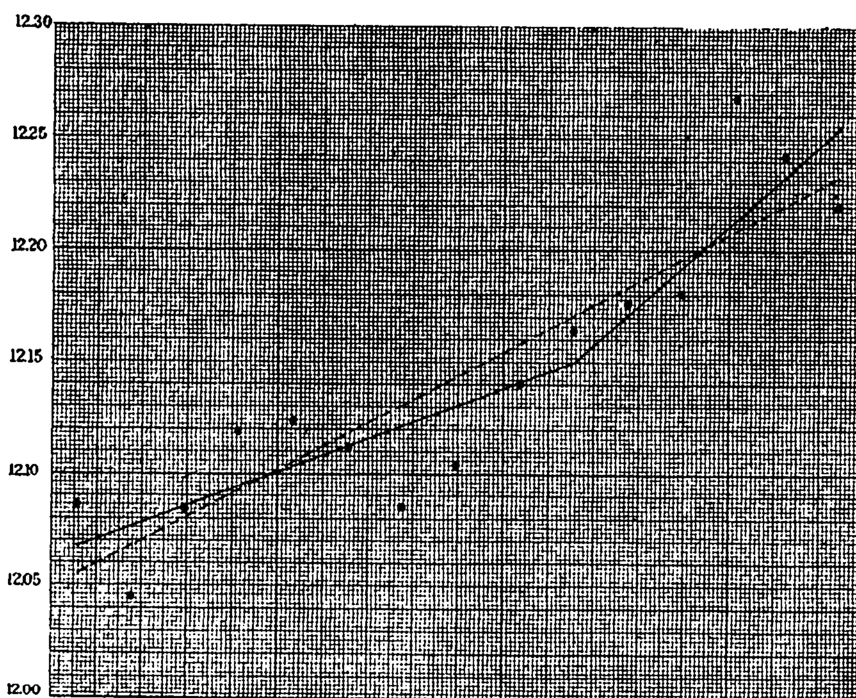

FIG. 9b.—Progressive change of latitude at Ukiah—means by 14-month periods.

TABLE 7.—*Data for observation equations to determine progressive variation of latitude—means by 14-month periods.*

No. of group.	Date of middle of group.	Seconds of mean latitude. Mizusawa.	Seconds of mean latitude. Carloforte.	Seconds of mean latitude. Ukiah.	Tschardjui. Date of middle of group.	Tschardjui. Seconds of mean latitude.	Gaithersburg. Date of middle of group.	Gaithersburg. Seconds of mean latitude.	Cincinnati. Date of middle of group.	Cincinnati. Seconds of mean latitude.
		″	″	″		″		″		″
1	1900.59	13.619	08.927	12.086	1900.59	10.686	1900.59	13.147	1900.59	19.343
2	1901.78	.601	.977	.045	1901.78	.684	1901.78	.202	1901.78	.298
3	1902.96	.582	.928	.084	1902.96	.618	1902.96	.252	1902.96	.324
4	1904.14	.597	.952	.119	1904.14	.666	1903.87	.229	1903.87	.422
5	1905.33	.582	.962	.124	1905.33	.627	1904.78	.193	1905.05	.344
6	1906.55	13.581	08.929	12.112	1906.55	10.540	1906.26	13.228	1906.26	19.316
7	1907.78	.569	.923	.085	1907.78	.622	1907.46	.245	1907.46	.318
8	1908.96	.590	.909	.104	1908.78	.678	1908.63	.254	1908.63	.337
9	1910.37	.598	.955	.140	1909.84	.695	1909.82	.258	1909.96	.347
10	1911.56	.561	.962	.164	1911.02	.738	1910.78	.258	1910.96	.394
11	1912.76	13.594	08.974	12.176	1912.21	10.750	1912.05	13.300	1911.96	19.396
12	1913.96	.601	.998	.181	1913.24	.752	1913.00	.316	1912.93	.431
13	1915.14	.600	.987	.268	1914.42	.856	1914.42	.335	1914.14	.483
14	1916.24	.622	09.016	.242	....	....	....	....	1915.33	.521
15	1917.42	.618	.057	.220	....	....	....	....	....	....

The observation equations based on Table 7 are, as before, in the general form:

$$\Delta\phi = x + t_1 y + t_2 z. \qquad (3)$$

The only difference in meaning of the symbols is due to a difference in the epoch at which the rate changes from y to z, from which t_1 and t_2 are therefore reckoned, and to which x refers. It would have been desirable, for the sake of uniformity, to have taken this epoch 1912.00 as before. By inadvertence this was not done and different epochs were used, all slightly earlier than 1912.000. The use of an earlier epoch may not be in itself a disadvantage, for it has been noticed that the change of rate may really occur before 1912.000, and as the difference between 1912.000 and the epochs used was not great, the original computations were adhered to. The epochs when the rate changes, as used in the observation equations and the normal equations, were 1911.56 for Mizusawa, Carloforte, and Ukiah; 1911.02 for Tschardjui; 1910.78 for Gaithersburg; and 1910.96 for Cincinnati. The times are reckoned from these epochs; t_1 is zero after the epoch stated for the station in question and t_2 zero before that time. The normal equations were formed as they were when the means were taken by calendar years and the corresponding solutions were made, namely: (1) y and z unequal, all observations used; (2) y and z unequal, only observation for years 1900 to 1914, inclusive, used; (3) y and z equal, all observations used; (4) y and z equal, only observations for years 1900 to 1914, inclusive, used. The results are given in Table 9 for comparison with the corresponding results in Table 8. The value of x does not give immediately the adjusted latitude in 1912.000, as that is not the epoch of the change of rate. For convenience in comparison, however, the latitude in 1912 is given, by applying a correction for the change between the epoch of the adjustment and 1912.000.

TABLE 8.—*Progressive change of latitude, by calendar years with results of the adjustment and probable errors.*

[The letter A after the name of a station indicates that all observations (see pp. 26 and 31) have been used; the letter B that the observations cover only the years 1900 to 1914, inclusive. When $y = z$ the value of y gives an average annual rate for the period covered and there is no epoch of change of rate. Probable errors (indicated by a ± sign) are in units of the fourth decimal place of seconds.]

y AND z UNEQUAL.

Station.	Epoch.	Latitude 1912.000	Annual rate of increase of latitude before epoch, y.	Annual rate of increase of latitude after epoch, z.	Probable error of a single observation, e.
		° ′ ″	″	″	
Mizusawa A	1912.000	39 08 03.577	−0.0023±8	+0.0080±20	±116
Mizusawa B	1912.000	03.575	− .0025±7	+ .0155±40	± 94
Tschardjui AB	1912.000	10.686	+ .0042±29	+ .0692±160	±370
Carloforte A	1912.000	08.948	+ .0006±11	+ .0191±26	±153
Carloforte B	1912.000	08.945	+ .0003±12	+ .0204±69	±161
Gaithersburg AB	1912.000	39 08 13.284	+0.0090±11	+0.0218±63	±147
Cincinnati A	1912.000	19.378	+ .0031±22	+ .0431±84	±288
Cincinnati B	1912.000	19.378	+ .0031±23	+ .0426±130	±300
Ukiah A	1912.000	12.147	+ .0070±12	+ .0209±29	±171
Ukiah B	1912.000	12.142	+ .0063±13	+ .0260±73	±168

y AND z EQUAL.

Station.	Lat. 1912.000.	Annual rate of increase of latitude, y (=z)	Probable error of a single observation, e.
	° ′ ″	″	
Mizusawa A	39 08 03.595	+0.0004±6	±138
Mizusawa B	03.589	− .0006±7	±114
Tschardjui AB	10.736	+ .0109±26	±437
Carloforte A	08.981	+ .0054±9	±206
Carloforte B	08.965	+ .0030±11	±186
Gaithersburg AB	39 08 13.294	+0.0103±9	±150
Cincinnati A	19.423	+ .0093±10	±350
Cincinnati B	19.408	+ .0072±20	±327
Ukiah A	12.172	+ .0106± 9	±196
Ukiah B	12.157	+ .0084±11	±179

TABLE 9.—*Progressive change of latitude, by 14-month periods, with results of the adjustment and probable errors.*

[For explanation see Table 8.]

y AND z UNEQUAL.

Station.	Epoch.	Latitude. 1912.000.	Annual rate of increase of latitude before epoch, y.	Annual rate of increase of latitude after epoch, z.	Probable error of a single observation, e.
		° ′ ″	″	″	
Mizusawa A	1911.56	39 08 03.570	−0.0027± 7	+0.0111± 14	± 86
Mizusawa B	1911.56	03.580	− .0027± 8	+ .0122± 43	± 91
Tschardjui AB	1911.02	10.720	+ .0030±32	+ .0502±112	±356
Carloforte A	1911.56	08.951	− .0000±12	+ .0174± 23	±140
Carloforte B	1911.56	08.953	− .0001±16	+ .0241± 88	±187
Gaithersburg AB	1910.78	39 08 13.293	+0.0083±14	+0.0193± 45	±150
Cincinnati A	1910.96	19.397	+ .0026±20	+ .0378± 54	±229
Cincinnati B	1910.96	19.398	+ .0026±21	+ .0389± 79	±232
Ukiah A	1911.56	12.158	+ .0076±15	+ .0178± 31	±185
Ukiah B	1911.56	12.154	+ .0072±14	+ .0168± 72	±153

TABLE 9.—*Progressive change of latitude, by 14-month periods, with results of the adjustment and probable errors*—Continued.

y AND z EQUAL.

Station.	Lat. 1912.000.	Annual rate of increase of latitude, y $(=z)$	Probable error of a single observation, ε.
	° ′ ″	″	
Mizusawa A	39 08 03.601	+0.0013± 7	±142
Mizusawa B	03.585	− .0011± 7	±105
Tschardjui AB	10.736	+ .0112±28	±427
Carloforte A	08.979	+ .0052±10	±198
Carloforte B	08.962	+ .0025±12	±172
Gaithersburg AB	39 08 13.295	+0.0103±10	±157
Cincinnati A	19.419	+ .0105±18	±315
Cincinnati B	19.403	+ .0081±19	±289
Ukiah A	12.175	+ .0106±10	±197
Ukiah B	12.157	+ .0080±10	±151

The agreement between the corresponding items of Tables 8 and 9 seems to be entirely satisfactory. As was to be expected, Cincinnati and Tschardjui have probable errors considerably larger than those of the other stations. The fact that at all stations the increase in latitude is more rapid toward the end of the period than toward the beginning appears in two ways: First, the values of z are, without exception, greater than the values of y for the same station and, usually, much greater; second, when $z=y$, the "A" values of y are always smaller than the "B" values, or in other words, the longer the period covered the greater the average rate of increase, so that the rate must itself be increasing as time goes on. This state of affairs requires that, when the stations are taken together and y and z are equal, the same period of time should be involved for all stations; this means that either the "B" values should be taken when all stations are included or that if the "A" values are taken, only the three stations, Mizusawa, Carloforte, and Ukiah, should be used, as these three alone run from 1900 to 1917, inclusive, Cincinnati stopping with 1915. When y and z are unequal the "A" and the "B" values of y are practically identical. This is as it should be, for the addition of a few more observations after 1914 should have little effect on the fitting of a straight line to the observations between 1900 and 1911 or 1912; the only mathematical connection between the equations in which y occurs and those in which z occurs is the common value of x. The values of z are too poorly determined, as appears from their probable errors, to afford any very definite information.

SECTION 5. EXPLANATION OF THE CHANGE OF LATITUDE BY A MOTION OF THE POLE.

It appears at once from Tables 8 and 9 that the size of the increase which Prof. Lawson found in the latitude of Ukiah, 0″.0094 a year, or the rate of 0″.0081 found by a better method from Lawson's data, is in no way exceptional.

If we adopt Lawson's explanation, namely, a creep of the surface strata, and apply it to all stations, it is evident that the accurate determination of astronomical quantities will be attended with very great difficulties; it is hardly likely, moreover, that all stations except Mizusawa should show a tendency to creep toward the pole, and that even Mizusawa should promise to develop such a tendency. It is more natural to seek an astronomical explanation, particularly in the astronomical quantities that enter into the computed declinations on which the observed latitudes are based.

The reductions of a star from mean to apparent place include corrections for precession, nutation, aberration, and proper motion. The annual precession of a star in declination is of the form $n \cos \alpha$, n being practically constant and equal to about 20 seconds and α being the right ascension. Since the star program of the International Latitude Service is pretty uniformly distributed in right ascension and since the right ascension of the stars observed depends upon the season of the year, the effect of an error in n would appear as a term of annual period with an increasing amplitude and would be combined with other effects having the same period. The annual part of the variation of latitude has, however, been eliminated. The effect of an erroneous constant of aberration would be to give rise to an annual term with a smaller term of a six-month period, which would go out along with annual terms when means were taken by calendar years. An error in the constant of nutation gives rise to something like an annual term, though one varying slowly in amplitude and phase with the revolution of the moon's node. The effect of such an error should nearly disappear when the mean over a year is taken, and the mean of the small residual effect should nearly vanish in a period of 18 years.[12] The terms of short period in the nutation are practically without effect even in series much shorter than those here dealt with.

The effects of errors in the precession, aberration, and nutation are then practically periodic and are eliminated from the y's and z's of Tables 8 and 9. An error in the proper motion in declination would enter directly into the observed latitudes for its full amount. If the average annual proper motion used in computing the declinations were too great, the observed latitudes would show an annual increase equal to the error. Since the star program is the same for all stations, all stations would be affected alike by such an error in the proper motion, and indeed by any error in the star positions which may have been overlooked in the previous discussion, except in so far as the weather may cause the stars actually observed to be different at the different stations.

The values of y, however, are far from being alike, but show signs of being dependent on the longitude. When y and z are unequal, y starts with a negative value at Mizusawa and becomes nearly zero at Carloforte, the value for Tschardjui with its big probable error counting for little either one way or the other; on the American continent y is clearly positive. The same algebraic increase from Japan through Europe to America appears when we consider the values of y for the case when y and z are equal, though the value of y

[12] The literature of the variation of latitude abounds in articles dealing with the determination of the constant of aberration from observations for the variation of latitude, a subject closely connected with that of the Kimura term. The nutation constant has been deduced from the observations of the International Latitude Service by Przbyllok in a work entitled: "Die Nutationskonstante abgeleitet aus den Beobachtungen des Internationalen Breitendienstes" (publication No. 36, new series, of the Zentral-bureau der Internationalen Erdmessung). Berlin, 1920.

starts from a different value at Mizusawa. From a purely geometrical point of view the simplest way to make y a function of the longitude is to attribute it to a motion of the pole.

To formulate this idea mathematically let us consider the North Pole and take as origin of coordinates some mean position of the pole and as axes of x and y two lines in a plane tangent to the earth's surface at the origin and lying in the meridian planes of Greenwich and of 90° west longitude, respectively.[13] Let u and v be the annual rates of displacement of the pole resolved along the axes of x and y, respectively; let w be the annual increase of latitude common to all stations alike and due presumably, in great part, to errors in the assumed proper motions, but including any other effects that may produce such a change common to all stations; further let λ denote the west longitude of a station. Then the increase in latitude at a station is equal to w, the portion common to all stations, *plus* the component of polar motion toward the particular station. Stating this in an equation we get

$$u \cos \lambda + v \sin \lambda + w = y. \tag{4}$$

This is the type of observation equation which may be used to determine values of u, v, and w so as to represent, as nearly as possible, the observed values of y. The quantity z might be substituted for y; it is evident that z corresponds to a different program of stars so that the w obtained from the z's should differ from that obtained from the y's. The values of u and v, however, should be the same in both cases, on the supposition that the progressive motion of the pole continues unchanged. Unfortunately the probable errors of the z's are so large as to make a calculation of this sort of rather doubtful value. In one case, however, the z's have been used as a partial verification. (See p. 36.)

As is evident from the probable errors in Tables 8 and 9, the values of y are not all equally reliable; these probable errors are some guide as to the appropriate weights for the observation equations of form (4), but they are not necessarily an infallible guide. In forming the normal equations from the observation equations (4), two systems of weighting were adopted; one in which the weights were taken, in accordance with the theory of least squares, to be inversely proportional to the squares of the probable errors[14] of the y's in Tables 8 and 9, and the other in which the weights were arbitrarily taken as unity except for the stations Tschardjui and Cincinnati, to which were assigned weights of one-half. Since there are two systems of weighting, two periods according to which means are taken, the annual and the 14-month, and two values of y according as y and z are equal or unequal, there are eight possible solutions for u, v, and w. All these solutions were carried out and the results with probable errors attached are set down in Table 10.

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[13] These are the axes used in the publications of the International Latitude Service.

[14] That is, approximately. Convenient round numbers that satisfied this condition fairly well were used for the weights.

TABLE 10.—*Motion of North Pole.*

[Probable errors (indicated by a ± sign) are in units of the fourth decimal place of seconds.]

Solution No.	Annual component of southward motion of North Pole along— Meridian of Greenwich. u	Meridian of 90° west. v	Correction to average annual proper motion of stars in declination. w	Probable error of station of unit weight. e
	"	"	"	
1	+0.0003± 9	+0.0059± 9	+0.0019± 7	±12
2	+ .0014±15	+ .0050±13	+ .0043±10	±23
3	+ .0002±11	+ .0055±11	+ .0013± 8	±21
4	+ .0016±16	+ .0053±13	+ .0040±10	±20
5	+ .0007±16	+ .0043±14	+ .0026±11	±23
6	+ .0015±20	+ .0034±17	+ .0052±13	±29
7	+ .0002±15	+ .0047±13	+ .0022±10	±22
8	+ .0018±21	+ .0036±18	+ .0050±14	±30

CONDITIONS.

Solution No. 1.—Weights according to probable errors in Table 8, y and z unequal, means by calendar years.

Solution No. 2.—Weights according to probable errors in Table 8, y and z equal, means by calendar years.

Solution No. 3.—Weights according to probable errors in Table 9, y and z unequal, means by 14-month periods.

Solution No. 4.—Weights according to probable errors in Table 9, y and z equal, means by 14-month periods.

Solution No. 5.—Tschardjui and Cincinnati half weight, others unit weight, y and z unequal, means by calendar years.

Solution No. 6.—Tschardjui and Cincinnati half weight, other unit weight, y and z equal, means by calendar years.

Solution No. 7.—Tschardjui and Cincinnati half weight, others nit weight, y and z unequal, means by 14-month periods.

Solution No. 8.—Tschardjui and Cincinnati half weight, others unit weight, y and z equal, means by 14-month periods.

The value of e is given in Table 10 in order to afford some idea of how well the y's of the individual stations are represented. In all solutions either Gaithersburg or Ukiah, or both, had unit weight and the probable error of their y's may be compared with e. It is seen that the e's in Table 10 are, in general, larger than the probable errors of the y's in Tables 8 and 9. This is to be expected, since the probable errors of the y's come simply from the failure of the latitudes to conform exactly to a linear law of increase, each station being treated by itself and not being tied up with other stations by the hypothesis of a polar motion.

The stations that show large residuals on any scheme of weighting are Tschardjui and Cincinnati. The former is so irregular at all times that it is difficult to make any plausible suggestion that will bring it into line with the other stations, and our uncertainties are increased by the enforced change in the position of the station and the comparatively weak connection between the old and the new stations. Although Cincinnati shows an increase in latitude between 1900 and 1912, it is noticeably smaller than its neighbors, Gaithersburg and Ukiah. An examination of figures 7*a* and 7*b* shows that the latitudes of Cincinnati for the years 1902, 1903, and 1904 seem abnormally high, and these years being near the beginning of the period treated the high values would tend to bring down the rate of increase in latitude to its apparently low figure in Tables 8 and 9. A similar phenomenon, though not so marked, occurs at Gaithersburg and

Ukiah, as appears from figures 8*a* and 9*a*, but it is not easy to see any counterpart for it in the old-world stations such as would exist if it were due to the motion of the pole. It may then be ventured that the phenomenon is due to variations in refraction, stronger at Cincinnati than at Ukiah or Gaithersburg because the climate at the former place is more markedly continental than at the stations nearer the sea coast. The suggestion of refraction is only a guess and does not justify any arbitrary emendation of the observed results at Cincinnati. It seems, however, appropriate to call attention at this point to the importance of refraction when dealing with such minute quantities as we are here concerned with and to the existence of large and persistent anomalies in the latitude which it is difficult to attribute to anything except refraction, however the latter may have been brought about. Przbyllok, a careful student of the subject, says:[16]

> The existence and behavior of the zenithal refraction is of great significance in the determination of fundamental astronomical constants. The observations of the International Latitude Service have shown that the mean latitude of an evening may be in error by nearly a second of arc. From the figures in column 6 of Table 2 it may be seen that latitude determined by a month's observations may be affected by an error of nearly $\frac{1}{4}''$; even the mean of a year's work may give values erroneous by about $0''.1$.

Since we are especially interested in the latitude of Ukiah, it may be remarked that the values of u, v, and w, in Table 10, always represent the value of y satisfactorily, the difference between the "observed" in Tables 8 and 9 and the "adjusted" value being often less than the probable error in Tables 8 and 9 and never greatly exceeding it. The adjusted rate of increase at Ukiah deduced from u, v, and w is always less than the observed rate, except in one case. This is due to the influence of its neighbor, Cincinnati, which has a small rate of increase.

It is not easy to choose among all the solutions in Table 10, but in order to present a single definite statement as the outcome of this part of the investigation, we may arbitrarily take the mean values of u and v from all eight solutions. These means are:

$$u = +0''.0010$$
$$v = +0''.0047$$

The resultant of these two component motions is in magnitude

$$\sqrt{(0''.0010)^2 + (0''.0047)^2} = 0''.0048, \text{ or}$$

in round numbers, $0''.0050 = 1/200''$.

The direction of the resultant motion is southward along the meridian $\lambda = \tan^{-1}\left(\frac{+0.0047}{+0.0010}\right) = 78^\circ$ west of Greenwich. The North Pole then appears to be shifting toward the North American Continent in the general direction of the east coast of Hudson Bay, or of Washington City, at the rate of $0''.0050$ (or in linear units, 6 inches or 15 centimeters) a year. A mean value of w would have no general interest or significance, since it depends on the particular stars in the program of observation and the proper motions assumed in calculating their declinations.

The values of w correspond to the increase in the nonperiodic portion of the Kimura or z term, as obtained by the International Lati-

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[16] Translated from Astronomische Nachrichten No. 4840–1, vol. 202 (1916), pp. 286–287.

tude Service. This increase may be seen by comparing the following mean values for z for various calendar years as derived from the various publications of the service.[16]

Year.	Mean z.	Year.	Mean z.
	"		"
1900	−0.006	1913	+0.033
1905	− .010	1915	+ .081
1910	+ .028	1917	+ .114

It is, of course, evident that the rates of change at Gaithersburg are the main support of a conclusion of this sort. Ukiah is suspected of being on unstable ground and unless the deduced motion of the pole has the same general character regardless of whether Ukiah is included or not, the conclusion rests on a questionable basis. An inspection of the values of y in Tables 8 and 9 suggests that, in order to account for the change in latitude at Gaithersburg, even when Ukiah is omitted, the values of u, v, and w must in general be similar to those in Table 10. Actual calculation confirms this view.

In order to use only the same periods of time at all stations the results in Table 10 are based on the "B" values in Tables 8 and 9, when y and z are assumed to be equal. When y and z are assumed to be unequal it is almost indifferent whether the "A" values or the "B" values are used. The results in Table 10 are, therefore, average results for the years 1900–14, inclusive, the period corresponding to the "B" values. Since Cincinnati extends only through 1915, the only considerable extension of the period covered will depend on the results at the three stations, Mizusawa, Carloforte, and Ukiah. These three are just sufficient to determine u, v, and w.

Two solutions were made with these three stations; one using the means of the "A" values of y from Tables 8 and 9, where y and z are assumed equal; the other using the mean of the "A" values of z from Tables 8 and 9.

The first result is then a mean one for the years 1900–17, inclusive. It is

$$u = +0\overset{''}{.}0007,$$
$$v = +0\overset{''}{.}0066,$$
$$w = +0\overset{''}{.}0055.$$

The second solution from the z's evidently applies to the years 1915–17 and may be considered as an extension of the results of Table 10; it is

$$u = +0\overset{''}{.}0032,$$
$$v = +0\overset{''}{.}0062,$$
$$w = +0\overset{''}{.}0160.$$

The resemblance of the two sets of values of u and v to each other and to the corresponding quantities in Table 10, suggests that the polar motion continued at about the same rate and in about the same direction during the years 1915–17 as during the years 1900–14. The values of w are different, as might be expected from the difference in the programs of stars observed. The two sets of values just given have, of course, the weakness of depending on the suspected

[16] The z does not have the same meaning here as in Tables 8 and 9.

station of Ukiah. This is unavoidable if the results are to be extended through the year 1917, and the chief value of the values just given lies in their resemblance to corresponding quantities in Table 10 that are not essentially dependent upon Ukiah.

For a revision of the polar motion deduced in this section see paragraph at end of introduction (p. 2) added while the publication was in press.

SECTION 6. THE SIGNIFICANCE OF THE POLAR MOTION.

The rate and direction of the polar motion, as just determined, agree roughly with Wanach's [17] determination. He found from the observations of 1900–1911 a motion probably not exceeding 0."0030 a year and in the general direction of Newfoundland.[18] The rate here determined is more than half as large again as that of Wanach and the direction differs by 22°, but in view of the additional observations included in the present discussion and the great difference in the method of treatment, the difference in the results for so small a quantity is not particularly surprising.[19]

Wanach remarks that the polar motion could be accounted for by a combination of oscillations of long period as satisfactorily as by a secular (progressive) change. In astronomy secular changes are, in general, merely small parts of periodic changes whose periods are very long; thus the distinction between secular and periodic motion is chiefly one of convenience. Probably, however, Wanach means to say that the polar motion could be accounted for by a combination of periodic motions whose periods might be long in comparison with a year or a 14-month period but short in comparison with the periods of the secular perturbations of astronomy, periods of perhaps from 10 to 50 years. Even with the larger motion here found and the longer period covered, we can not be sure that the suggested combination of motions of moderate period is not the true explanation. The question would then arise as to the origin of these periodic changes; the answer might be that the forces causing these changes were of meteorological origin and the periods were the rather indefinite periods, such as those of 6, 11, or 30 years, that climatologists are studying. With our present knowledge, the representation of the polar motion, as proportional to the time, and in one direction, is simply an empirical formula; extrapolation from such a formula is an uncertain process. The geophysical significance of such a motion will be touched on shortly.

It would be quite possible, however, for a polar motion of this magnitude to have been going on since the beginning of historic time without producing any perceptible change in climate. In 10,000 years the change in latitude would be at most 50 seconds, or about 1 statute mile, and this only in the regions like eastern North America and Mongolia, which are near the meridian of maximum change, the meridian being considered as a complete circle. In western Europe,

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[17] Resultate des Internationalen Breitendienstes, Vol. V, pp. 219-220.

[18] This is Wanach's form of statement. The longitude of central Newfoundland may be taken as 56° west. The details of Wanach's method for obtaining the direction and velocity of the polar motion are not given.

[19] Wanach connects the stations with one another from the start in order to obtain the coordinates of the pole and then studies the variations of these coordinates. In this investigation, each station is treated separately to obtain its rate of change of latitude, and the stations are combined only at the last in order to obtain the motion of the pole. Wanach's treatment gives full weight to Tschardjui and Cincinnati, instead of half weight or less, a difference that results in decreasing the deduced motion of the pole.

which lies at nearly right angles to this meridian, the change in latitude would be nearly zero. It is, however, precisely in western Europe that we have nearly all our long series of accurate astronomical observations. It is not improbable, therefore, that these long series would have sufficed to bring to light such a motion of the pole as the one here supposed had they been taken somewhere else, but being where they are, they have been insufficient.

This is a convenient place to give the formulas for the change in latitude of a point and the change in the difference of longitude between two points, due to the motion of the pole. Suppose that in a given interval of time the North Pole has moved toward the Equator by a small amount θ and along the meridian whose longitude is α. Let ϕ and ϕ' denote the latitudes of a place at the beginning and end of the motion, respectively, and let λ be the longitude. Then the change of latitude is given by

$$\phi' - \phi = \theta \cos (\lambda - \alpha). \tag{5}$$

Let λ_2 and λ_1 denote the longitudes of two places at the beginning of the motion, ϕ_2 and ϕ_1 the corresponding latitudes and λ'_2 and λ'_1 the longitudes at the end of the motion. Then the change in difference of longitude is given by

$$(\lambda'_2 - \lambda'_1) - (\lambda_2 - \lambda_1) = \theta\ [\tan \phi_2 \sin (\lambda_2 - \alpha) - \tan \phi_1 \sin (\lambda_1 - \alpha)]. \tag{6}$$

Equations (5) and (6) come from the differential variations of spherical triangles and apply only when θ is a small quantity and when the distances from the pole of the places concerned are very large as compared with θ. In the right-hand sides of (5) and (6) the primed quantities may be substituted for the unprimed. The left-hand side is expressed in the same unit as θ.

Prof. Wegener,[19] of Marburg, has suggested that a comparison of older and more recent observations on differences of longitude shows the difference of longitude[20] between Europe and North America to be increasing. Formula (6) shows that a motion of the pole along the meridian of 78° west would have an effect of this sort. The magnitude of the effect for a motion of 0.″0050 a year is, however, much smaller than the difference that Wegener finds, being only about one-thirtieth as much.

In order to see whether the motion of the pole from 1900 to 1917 is a continuing phenomenon or not, it would be desirable to have a careful investigation made of existing long series of latitude observations, particularly those made in America and India. Too much should not be expected from such series, as the periodic variations are large enough to mask the secular variation unless the former have been carefully studied and eliminated. The results at a single observatory, unless obtained from circumpolar stars observed at both culminations, are so dependent on the declinations and on the

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[19] Petermann's Mitteilungen, vol. 58 (1912), p. 307. This is a paper running through several numbers of Petermann, which was afterwards elaborated in book form under the title Die Entstehung der Kontinente und Ozeane, of which a second edition has recently appeared, but which the author of this investigation has not seen.

[20] The difference of longitude Washington-Paris determined by wireless in 1913-14 is 0.ˢ09 greater than the one dependent upon the trans-Atlantic cable determination of 1866 and 0.ˢ05 less than the one in 1892. (See Publications of the U. S. Naval Observatory, second series, Vol. IX, pp. E 98, E 99, and U. S. Coast and Geodetic Survey Report for 1897, App. 2, The Adjustment of the Longitude Net of the United States in Connection with that of Europe, 1866–1896.) A study of the various trans-Atlantic longitude results as given in the above reports seems to show that little information of value can be obtained from them which will throw light on the motion of the pole or the creep of the continental blocks.

values adopted for the fundamental constants of astronomy, that their value is doubtful unless discussed with extreme care and there is always the possibility of reasoning in a circle—from the latitude to the declination and from the declination back to the latitude. A strong plea for observations on circumpolar stars at both culminations is made by Prof. Chelli,[21] of Turin Observatory.

It is unfortunate that the number of observatories of the International Latitude Service should now be reduced to three. This is just enough to determine the quantities u, v, and w, but there is no check from any supernumerary station. At least four stations are essential to any satisfactory determination and the original number of six, or an even larger number, would be highly desirable.

The hypothesis of the displacement of the pole within the earth has been put forward from time to time to explain the observed facts of geology and paleontology. Often the hypothesis has assumed forms that appear decidedly extravagant to astronomers and mathematicians.[22] It is, therefore, appropriate to call attention to the physical consequences of even the comparatively moderate polar displacement of 0".0050 that has just been deduced. The subject of a possible displacemnt of the pole has been treated mathematically by several writers, and most satisfactorily perhaps as regards secular changes of the pole, by Darwin.[23]

The effect of such a progressive motion of the pole on its periodic motion is small, amounting at most to a change in the amplitude and phase of the periodic motion necessary to produce a change of about 0".001 in the position of the pole.[24] With observations of the present degree of accuracy, a quantity of this size can not be separated from the other elements that enter into the periodic motion.

Let us consider the effect on the position of the pole of a transfer of matter from one portion of the surface to the other, with especial reference to the idea that such a transfer, due to the erosion and deposition of matter by the rivers of the world, might explain the displacement of the pole. Suppose a mass m to be transferred from a point on the earth's surface, whose latitude is ϕ_1 and whose longitude is λ_1, to another point on the surface whose geographic coordinates are ϕ_2 and λ_2; let c be the earth's mean radius and C and A the maximum and minimum moments of inertia of the earth about axes through its center, and let θ be the angular displacement of the earth's axis of figure, that is, of the axis about which the moment of inertia is a maximum, in short, the displacement of the pole, due to the transfer of mass m; then θ is given by[25]

$$\theta=\frac{mc^2}{2(C-A)}\sqrt{\sin^2 2\,\phi_1+\sin^2 2\,\phi_2-2\sin 2\,\phi_1 \sin 2\,\phi_2 \cos(\lambda_2-\lambda_1)}. \quad (7)$$

But from the known ellipticity of the earth, $C-A=0.00110\ Mc^2$, approximately, M being the mass of the earth, so that we get, numerically,

$$\theta=\frac{455\ m}{M}\sqrt{\sin^2 2\,\phi_1+\sin^2 2\,\phi_2-2\sin 2\,\phi_1 \sin 2\,\phi_2 \cos(\lambda_2-\lambda_1)}. \quad (8)$$

[21] Astronomische Nachrichten, No. 4631, vol. 193 (1913), p. 405.
[22] See an article by the late Prof. Joseph Barrell of Yale in Science, vol. 40 (1914), p. 333.
[23] The influence of geological changes on the earth's axis of rotation, Philosophical Transactions, Part I, vol. 167 (1877), p. 271, or Scientific Papers, Vol. III, p. 1.
[24] See p. 103.
[25] This formula is readily derived from section 11 of Darwin's article referred to in footnote 23, above.

The above value of θ is in radians and must be multiplied by 206265 to reduce it to seconds of arc. The maximum value of the radical evidently occurs when ϕ_1, λ_1, and ϕ_2, λ_2 are antipodal points in latitude $\pm 45°$. This maximum value of the radical is then equal to 2.

Some rough estimate may be made of the amount of matter transported by the river systems of the world.[26] From Geikie's Geology (p. 589) it will be seen that it is almost certainly an overestimate to assume that the land, as a whole, suffers an average annual loss from erosion of 1/1000 foot or 0.3 millimeter; allowing 149 million square kilometers to the land area of the earth we find for the mass eroded 4.47×10^{10} cubic meters, or 1.21×10^{11} metric tons,[27] if a density of 2.7 be assumed. Let us suppose all this mass to be, originally, in the immediate vicinity of some point in latitude $+45°$ and to be transported to the antipodal point in latitude $-45°$. This is certainly a very gross exaggeration of the effect of the matter transported by rivers, for no river carries matter anything like halfway around the earth and there is certainly considerable mutual compensation of the effects of different rivers on the position of the pole. Even making, however, the extreme supposition just stated, and putting $M = 6.03 \times 10^{21}$ metric tons, we still find for θ a value of 0.″0038, a value less than the one deduced from observations. The suppositions made are so extreme that it is probable that the effect of erosion and deposition is not equal to the tenth part of the value given.

To the amount of matter eroded by the rivers, there might be added the amount of matter removed in mining operations without greatly increasing the computed displacement of the pole. From fairly recent statistics the world's output of coal amounts to about 1,200,000,000 metric tons per year, and of iron and steel combined, about a tenth of that. These two materials make up the bulk of the mineral output, so, if we put the total output of the minerals at 2 billion (2×10^9) metric tons, we shall almost certainly make an overestimate, and these 2 billion tons are but a small part of the 1.21×10^{11} or 121 billion tons estimated to be removed by the rivers of the globe.

Although the amount of matter removed by the rivers is insufficient, even under the most favorable conditions to account for the observed displacement of the pole, yet this displacement may be accounted for by a widespread redistribution of matter having a thickness of the same order of magnitude as the average layer that was assumed to have been removed by the rivers. Let us consider the approximate expression for the radius vector, r, of the earth in terms of the latitude, ϕ,

$$r = a - af \sin^2 \phi, \tag{9}$$

a being the equatorial radius and f the flattening or ellipticity. By differentiation,

$$dr = -af \sin 2\phi \, d\phi.$$

Putting $d\phi = 0.''0050$ and using known values of a and f we find for dr in millimeters

$$dr = -0.51 \sin 2\phi.$$

---

[26] For data of this sort, see, among other works of reference: Geikie, Text Book of Geology, 4th Ed., Vol. I, pp. 487–496 and 586–597. Rudzki, Physik der Erde, pp. 470–485.

[27] This is a very high estimate, equivalent to the effect of about 290 Mississippis or 1,500 Danubes.

If we reckon longitude (λ) from the meridian along which the pole is assumed to be approaching the equator and suppose that at every point in the earth there is added algebraically a layer of variable thickness Δr, in millimeters given by

$$\Delta r = -0.51 \sin 2\phi \cos \lambda, \tag{10}$$

then the ellipticity of the earth will remain unaltered, but the position of the pole of figure will shift with respect to the continents by just the 0.″0050 used in building up the equation. It is as if we took a geographic globe flattened at the poles and squeezed it so that the amount of flattening remained unchanged, but the flat places were no longer under the Arctic Ocean and the Antarctic Continent.

This simple geometrical calculation implies other transfer of matter than on the surface merely; every equipotential surface, within the earth is affected in much the same way as the outer surface, though to a diminishing extent as we approach the center. To follow out the analogy of the squeezed globe, conceived as solid, it is evident that there must be movements of matter in the interior in order to permit the globe to take on its new figure with respect to the continents painted on its surface. The deformation is evidently a plastic one and would leave no record of itself in the rising or falling of the continents with respect to sea level.

The effect on the position of the pole of a layer of matter of thickness given by equation (10), and of the density of ordinary surface rock (σ) may be computed and compared with the foregoing result. We assume that the layer is simply added algebraically to the radius and simply rests on the surface as on a rigid body without causing movement of matter in the interior. For generality, we assume a maximum thickness of h instead of 0.51 millimeter as in (10). We then have,[28] using as much of the previous notation as is necessary,

$$\theta = \frac{8\,\pi\,c^4\,h\,\sigma}{15\,(C-A)}.$$

By using $C-A=0.0011\,Mc^2$ and $M=\frac{4}{3}\,\pi\,c^3\,\rho$, where ρ denotes the mean density of the earth, we get for θ in radians

$$\theta = 364\,\frac{h}{c}\,\frac{\sigma}{\rho},$$

or

$$h = 0.00275\,c\,\theta\,\frac{\rho}{\sigma}.$$

This gives the maximum thickness of the stratum. With $\frac{\rho}{\sigma}=2.1$ and $\theta=0.''0050$ we find $h=0.89$ millimeter.

The full amount of the matter added or removed does not appear as a rising or sinking with respect to sea level. The addition or removal of matter raises or lowers the sea level. By a fundamental theorem in the theory of attraction the apparent rising or sinking with respect to sea level will be the thickness of matter added or

[28] See sec. 11 of Darwin's article referred to in the footnote on p. 39.

removed, respectively (assuming the law of thickness to be of the form (10), with the general h in place of 0.51), multiplied by $1-\frac{3}{5}\frac{\sigma}{\rho}$; that is, by about $\frac{7}{10}$.

If the earth be treated as elastic, rather than as plastic or rigid, it appears that the effect of an additional layer of given thickness is less than it would be if the earth were rigid.[29] The elastic yielding of the earth causes a rearrangement of matter, and to estimate the effect of a surface layer we must, for the earth, apply a reducing factor of about $\frac{3}{4}$ to the effect of such a layer when placed on a rigid body. The foregoing estimates have supposed an actual transfer of matter from one part of the surface to another, or internal changes equivalent to such a transfer. If the changes are merely changes of density in a relatively thin outer shell of the earth, then the effect of an elevation or depression caused by an expansion or contraction of portions of this shell is less than the effect of an equal elevation or depression caused by transfer of matter.[30]

The purpose of the foregoing calculations and remarks regarding the effect of the transfer of matter on the position of the pole is merely to give some idea of the order of magnitude of the quantities involved and to call attention to some of the elements that enter into the problem. Large and presumably widespread elevations and subsidences of the earth's surface have occurred in the past and changes are still occurring, but we know so little about their causes and laws that it would probably be difficult to connect known geologic facts with a possible motion of the pole.

SECTION 7. ADDITIONAL OBSERVATIONS AT UKIAH.

While the manuscript of this publication was being prepared for the printer recent results for Ukiah were received from Dr. H. G. Van de Sande Bakhuyzen, secretary of the International Geodetic Association. The transmitting letter is dated September 26, 1921. These results are given in Table 10a.

TABLE 10a.—*Observed latitudes at Ukiah latitude station.*

Date.	Latitude.	Date.	Latitude.	Date.	Latitude.	Date.	Latitude.
1918.		1919.		1920.		1921.	
		Jan. 20	39° 08′ 12″.25	Jan. 15	39° 08′ 12″.32	Jan. 17	39° 08′ 12″.17
		Feb. 22	12.21	Feb. 14	12.30	Feb. 10	12.26
		Mar. 10	12.26	Mar. 14	12.42	Mar. 13	12.21
Apr. 10	39° 08′ 12″.24	Apr. 4	12.12	Apr. 2	12.28	Apr. 2	12.22
Apr. 27	12.25	Apr. 28	12.02	Apr. 29	12.18		
May 27	12.25	May 28	12.01	May 23	11.97		
June 28	39° 08′ 12.38	June 27	39° 08′ 12.04	June 17	39° 08′ 11.94		
July 30	12.34	July 24	12.11	July 25	11.99		
Aug. 31	12.35	Sept. 1	12.20	Aug. 31	11.88		
Oct. 13	12.20	Oct. 14	12.22	Oct. 14	11.90		
Nov. 12	12.23	Nov. 17	12.24	Nov. 17	12.05		
Dec. 20	12.26	Dec. 20	12.31	Dec. 21	12.22		

[29] See article by Jeffreys entitled "Causes contributing to the annual variation of latitude" in Monthly Notices of the Royal Astronomical Society, vol. 76 (1910), p. 504.

[30] See sec. 23 of Darwin's paper already cited. It may be said that Laplace's law of density used by Darwin is now believed to overestimate the yielding of the matter of the earth's interior to compression.

These values were plotted as explained on page 8 and readings were taken to obtain harmonic constants from a 6-year series covering the years 1915–20, inclusive. The curve for the missing period between 1918.04, the last result for Ukiah published in the Astronomische Nachrichten, and April 10, 1918, was drawn from values computed with the harmonic constants as given by the series 1912–17. The results of the analysis corrected for the effect of one component on another, but not corrected for "slope" are:

Annual component.		14-month component.	
Amplitude R.	Epoch ζ.	Amplitude R.	Epoch ζ.
″	°	″	°
0.121	93.8	0.201	276.9

The means were taken by years and by 14-month periods and cleared as by the process described on page 73. The harmonic constants used in the clearance were those just obtained. The results are:

Mean latitudes of Ukiah.

BY CALENDAR YEARS.

Calendar year.	Seconds of mean latitude, uncleared.	Clearance.	Seconds of mean latitude, cleared.
	″	″	″
1918	12.243	−0.021	12.222
1919	12.174	+0.011	12.185
1920	12.104	+0.034	12.138

BY 14-MONTH PERIODS.

Period included.	Seconds of mean latitude, uncleared.	Clearance.	Seconds of mean latitude, cleared.
	″	″	″
1917.664–1918.848	12.226	+0.017	12.243
1918.848–1920.032	12.187	+0.007	12.194
1920.032–1921.207	12.120	−0.011	12.109

The results show that the latitude of Ukiah after a long period of increase is decreasing again, but nothing conclusive can be inferred from this comparatively short period of observations at a single station. Whether the result is due to a creep of the ground, to the declinations used, to a movement of the pole, or to abnormal refraction can not even be plausibly guessed without the observations at other stations for comparison.

Chapter III.—THE LATITUDE OF LICK OBSERVATORY.

It does not seem practicable at the present time for the U. S. Coast and Geodetic Survey to attempt any extended discussion of a possible progressive change in the latitude of Lick Observatory. In order, however, to put before the reader the facts now available there is quoted below from Prof. Lawson's article[1] the passage in which he deals with the subject. The table of latitudes has been corrected in accordance with a communication from the director of Lick Observatory, Dr. W. W. Campbell, and some remarks, chiefly extracts from letters written by him, have been added.[2]

The extract from Prof. Lawson's article reads:

Meridian Circle observarions have been maintained at Lick Observatory since 1893 by Astronomer R. H. Tucker, who has kindly supplied me with the following memorandum of the values for the latitude of the station. The results are based on observations both on circumpolar and zenith stars for an average of sixty-eight nights per year during the period from 1893 to 1918, except for the interval 1908 to 1912, when Mr. Tucker was in South America. The values tabulated are the averages by quarter years, as near as possible, and the number of nights in each quarter during which observations were made averages seventeen. The results have been corrected for the revised refractions adopted at Lick Observatory and for the variation of latitude.

Average quarterly values for the latitude of Lick Observatory based on Meridian Circle observations by Astronomer R. H. Tucker from 1893 to 1918.

Epoch.	Latitude.	Epoch.	Latitude.	Epoch.	Latitude.
1893.82	37° 20′ 25″.48	1899.12	37° 20′ 25″.80	1906.11	37° 20′ 25″.58
1894.12	.69	.32	.82	.45	.63
.33	.47	.63	.58	1907.75	.56
.51	.44	1900.38	.73	1908.33	.18
.83	.43	.89	.64	1912.18	.79
1895.11	.61	1901.46	5.94	.51	.60
.33	.57	.57	6.03	.75	.38
.63	.52	.92	5.71	.92	.78
.83	.56	1902.13	.90	1913.16	.68
1896.11	.82	.37	.76	.53	.32
.40	.72	.62	.87	1914.14	.68
.67	.43	.88	.68	.37	.67
.91	.52	1903.12	.65	.63	.52
1897.13	.41	.35	.76	.90	.59
.31	.47	.60	.88	1917.18	.34
.72	.62	.92	.25	.39	.58
.92	.67	1904.11	.48	.62	.54
1898.15	.94	.30	.76	.88	.51
.36	.65	1905.59	.26	1918.11	.55
.83	.77	.77	.42		

I have taken these values and plotted them on coordinates in figure 1, B.[3] It is apparent from an inspection of the figure that, notwithstanding the variations in the values, there is in general a fairly steady increase in the latitude from 1893 to 1903. This increase is expressed in the mean line $A-B$. It amounts to $0''.4$ in ten years, or at the rate of $0''.04$ or 1.24 meters per year.[4]

Between 1903.60 and 1903.92 there is an exceptionally large drop of $0''.63$ in the value for the latitude. In this interval occurred the earthquake of August 2, 1903, which was rather severe at the Lick Observatory. It seems not improbable that the rather large drop in the value of latitude may be due to a shift of the ground at the time of this earthquake. The values for latitude since 1903.60 clearly fall into a grouping distinct from the grouping of the values for the period preceding that date. The mean expression for the values, as indicated by the line $C-D$, is, however, not so satisfactory as the line $A-B$, owing largely to the interval of no observations

[1] "The mobility of the Coast Ranges of California, an exploitation of the elastic rebound theory:" University of California Publications, Bulletin of the Department of Geology, vol. 12, No. 7 (Jan. 11, 1921).

[2] These extracts are used with Dr. Campbell's permission.

[3] Figure 10 takes the place of figure 1, B. It contains the corrections communicated by the director of Lick Observatory.

[4] See p. 49 of this publication for revised figures obtained by least-square adjustment.

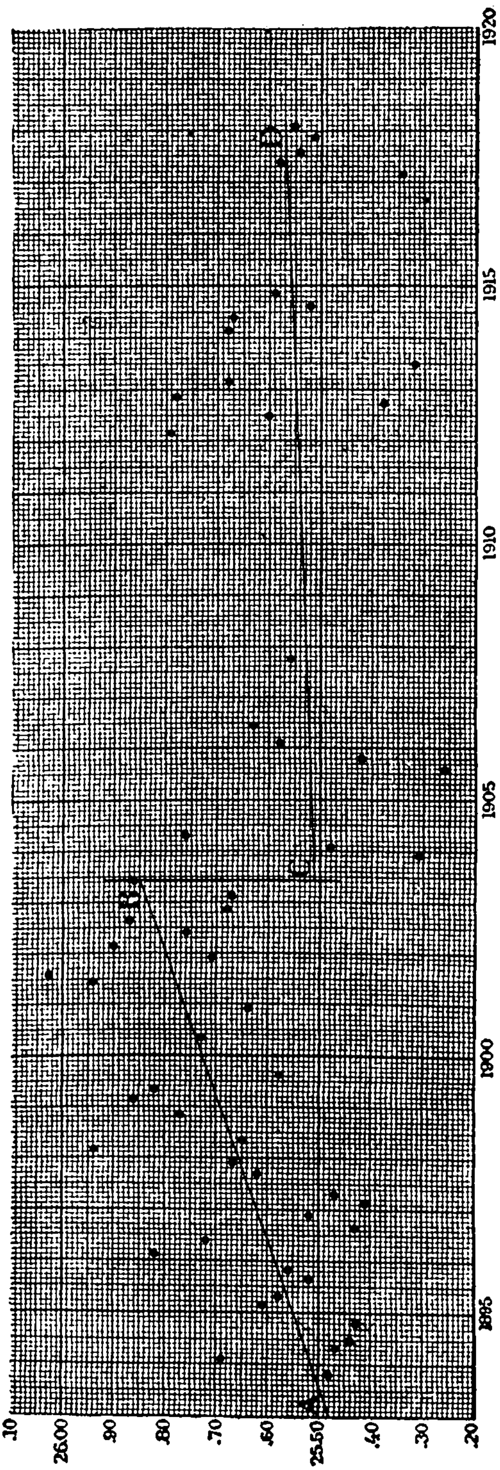

Fig. 10.—Progressive change of latitude at Lick Observatory.

from 1908 to 1912. This line rises from 1903 to 1915, but at a less rapid rate than the rise of the line *A–B*. The rate of increase of the value for latitude for this period is 0″.022 or 0.70 meter per year.[4] Beyond 1915 the data are insufficient for the determination of a mean position. It is interesting to note that there is no indication of sudden change of position at the time of the slip on the San Andreas fault in 1906.

The rate of latitude of Lick Observatory on either side of 1903.6 is much greater than that at Ukiah, and there may be doubt as to the true value of the rate, owing to the fact that the lines *A–B* and *C–D* each represent the mean position of a series of points which depart notably and irregularly from the adopted mean. There can, however, be no doubt as to the general significance of the observations, particularly between 1893 and 1903. They accord with those of Ukiah in pointing to a northerly creep of the region. The increase of latitude both at Ukiah and Mount Hamilton, taken together with the legitimate deductions that may be made, in the light of the rebound theory, from the results of the geodetic surveys in the intervening territory, seems to establish the fact of a northerly creep of the Middle Coast Ranges.

With regard to the abrupt change in latitude that seems to have occurred between 1903.60 and 1903.92, Dr. Campbell on April 29, 1921, writes as follows:

In supplying the data to Prof. Lawson, Mr. Tucker endeavored to base them, as far as practicable, upon quarterly mean values; that is, about four latitudes per annum. He was not aware that the date 1903.6 might be a critical one in Lawson's problem. and the latitudes for 1903 were grouped unfortunately. Mr. Tucker is of the opinion that the four data for the year 1903 should be regrouped for only three mean epochs. You will see the reason for this from the following tabulation of Mr. Tucker's results for that year as grouped by months:

	″
January, 8 nights, mean latitude	25. 76
February, 8 nights, mean latitude	25. 61
March, 2 nights, mean latitude	25. 57
Mean for 3 months 1903.10	25. 67
April, 6 nights, mean latitude	25. 68
May, 11 nights, mean latitude	25. 85
June, 4 nights, mean latitude	26. 16
Mean for 3 months 1903.37	25. 86
Mr. Tucker was absent on the Atlantic coast from July 16 to September 16.	
October, 2 nights, mean latitude	25. 61
November, 2 nights, mean latitude	25. 26
December, 8 nights, mean latitude	25. 25
Mean for 3 months 1903.92	25. 31

Again in the same letter he writes:

The plottings for the four data of the year 1903 should be changed to conform to the three substitute data.

Mr. Tucker's series of latitude values are unfortunately not on a homogeneous system. They are merely by-products of his extensive observations made for the purpose of determining the accurate positions of stars widely distributed over the northern sky. The bases of fundamental stars are quite varied, having been selected to meet the requirements of the successive programs upon which he was engaged, and without any thought whatsoever as to the latitude values which could be extracted as by-products. The four quoted results for 1897.72 to 1898.36, inclusive, depend upon *circumpolar stars* observed at both upper and lower culminations. All the other results are based upon a *combination* of certain so-called standard systems *and* circumpolars observed at both culminations. These systems are as follows:[5]

	″
Auwers Berliner Jahrbuch+circumpolars 1893.82–1896.40	$\phi_{11}=25.57$
Boss U. S. Northern Boundary Commission circumpolars 1896.67–1897.31 inc	$\phi_{4}=25.46$
Circumpolars exclusively 1897.72–1898.36 inc	$\phi_{4}=25.72$
Auwers 303+circumpolars 1898.83–1900.38 inc	$\phi_{5}=25.75$
Newcomb+circumpolars 1900.89–1904.30 inc	$\phi_{13}=25.74$
Auwers Berliner Jahrbuch+circumpolars 1905.59–1918.11 inc	$\phi_{21}=25.53$

[4] See. p. 49 of this publication for revised figures obtained by least-square adjustment.
[5] The subscripts of the ϕ's indicate the number of quarter-yearly values in the list on p. 44 contributing to the latitude.

Mr. Tucker is of the opinion that the variations in his tabular latitudes are due in part to differences in the fundamental systems and in the remaining part to accidental errors. My opinion, on the contrary, if I may venture it, is that Lawson has found evidence of a change in the Mount Hamilton latitudes at a time approximating that of our very severe 1903 August 2 earthquake.[5a] You will note that while the latitudes for the two Auwers Berliner Jahrbuch periods are substantially equal in the mean, yet the results increase from 1894 to 1896 and again from 1905–8 to 1912–15. The results on the Newcomb basis are on the average large, but we can not fail to note that they drop suddenly following 1903.6 and that these later values are in agreement with the 1905–8 Auwers results. In my opinion, none of the observed variations would call for serious consideration were it not for the sudden drop in the values in the latter half of 1903 and the conformity of these latter to those of the succeeding years. The subject at least merits most careful consideration.

Dr. Campbell takes up the subject in another letter, dated May 12, 1921, saying:

Perhaps I ought to repeat, in order to avoid any misunderstanding of interpretation, that my belief in a variation of the latitude of Mount Hamilton refers only to the 1903 epoch, perhaps as an accompaniment of our severe local earthquake of August 2, 1903. The observed latitudes preceding and more immediately following that date *all depend upon one and the same fundamental basis*, Newcomb's. I think you may find value in Prof. Tucker's results for individual nights of 1903, and they are as in the following table.[6]

Mount Hamilton (Lick Observatory) latitudes in 1903 by individual nights.

Date.	Observed latitude, ϕ.	$\phi-\phi_0$ correction.	Mean latitude of group, ϕ_0.
	° ′ ″		
Jan. 5	+ 37 20 24.96		
8	25.97		
10	25.48		
12	26.43		
13	25.98		
14	25.48		
15	26.25		
16	26.09		
1903.03	25.84	− 0.08	25.76
Feb. 10	25.83		
16	25.83		
17	25.59		
20	25.26		
24	26.35		
25	25.99		
26	25.57		
27	25.85		
1903.14	25.78	− 0.17	25.61
Mar. 9	25.61		
11	25.93		
1903.19	25.77	− 0.20	25.57
1903.10			25.67
	° ′ ″		
Apr. 22	+ 37 20 25.64		
23	25.93		
24	25.28		
27	25.97		
29	25.09		
30	26.45		
1903.32	25.88	− 0.20	25.68
May 1	26.17		
4	25.77		
5	25.92		
8	26.26		
11	26.44		
12	26.06		
14	25.47		
15	26.44		
25	25.98		
27	25.92		
28	25.96		
1903.37	26.04	− 0.19	25.85
June 2	26.30		
8	26.50		
10	26.29		
11	26.19		
1903.43	26.32	− 0.16	26.16
1903.37			25.80

[5a] In expressing the opinion that Prof. Lawson had found "evidence of a change" at or about the epoch of the severe earthquake, 1903.6, I have not for a moment thought that any actual change could equal the value 0."63, which is the vertical interval between Lawson's two inclined lines at that epoch. This interval was interpreted by me to be equal to a much smaller actual change plus a term of such size as could easily arise from accidental errors. It is recommended again that the reader note carefully the observed latitude values for the individual nights in 1903, comparing those obtained after 1903.6 with those obtained before that epoch.—W. W. C., April 4, 1922.

[6] For convenience in printing, this table has been slightly rearranged from the form given in the letter.

Mount Hamilton (Lick Observatory) latitudes in 1903 by individual nights—Continued.

Date.	Observed latitude, ϕ.	$\phi-\phi_o$ correction.	Mean latitude of group, ϕ_o.
	° ′ ″		
Oct. 5	+ 37 20 25.54		
6	25.48		
1903.76	25.51	+ 0.10	25.61
Nov. 10	25.33		
25	24.93		
1903.88	25.13	+ 0.13	25.26
Dec. 1	24.60		
7	25.18		
14	25.59		
21	25.08		
22	25.46		
24	25.62		
28	25.30		
29	24.80		
1903.97	25.18	+ 0.07	25.25
1903.92			25.31

The data by nights are not corrected for the variation of latitude as determined simultaneously at the international latitude stations. Mr. Tucker has computed the corrections for the several months on this account, and the corrected latitudes as set down for the several months in the last column are those quoted in my recent letter to you. You may want to apply slightly different corrections for the latitude variations. It is of interest to note that the latitudes for a considerable proportion of the nights in the quarter preceding the earthquake are above 26 seconds, and for several nights following the earthquake a number of the results are below 25 seconds.

Prof. Tucker searched for a refraction or other similar effect, which would be responsible for smaller observed latitudes in the fall months than in the first two quarters of the year, on the basis of his observations running through about twenty-five years, and as a result he has no reason to suspect that such an effect exists.

There is nothing improbable in a sudden change of latitude as the result of an earthquake. Such changes are known to have occurred in California during the earthquake of 1906.[7] The dislocation of surface strata due to that earthquake is, however, rather smaller than the change of latitude of Lick Observatory apparently accepted by Prof. Lawson as not improbable. The greatest of the dislocations of strata deduced for the earthquake of 1906 is about 6 meters or 0″.20, and most of them are considerably less, while the change of latitude of Lick Observatory late in 1903 seems to have been from 0″.40 to 0″.60. Moreover, the triangulation of the Coast and Geodetic Survey gives little evidence of such a displacement of Lick Observatory. Between 1887 and the time of the resurvey, shortly after the earthquake, the latitude of Lick Observatory appears from the surveys to be unchanged, while the longitude has changed but 0″.005 or 0.12 meter. Even this change is considered doubtful by Hayford and Baldwin.[8]

The main object of this report is not, however, the discussion of sudden changes of latitude due to the dislocation of surface strata, nor of the geological questions involved in a gradual creep of the surface strata; the object is rather the examination of the astro-

[7] Coast and Geodetic Survey Report for 1907, Appendix 3, Earth Movements in California Earthquake of 1906, by Hayford and Baldwin; also Report for 1910, Appendix 5, Triangulation in California, by Duvall and Baldwin. Other references are given in Prof. Lawson's paper.

[8] See references in the preceding footnote; p. 76 of the Report for 1907, and p. 195 of the Report for 1910. See also addendum on p. 108.

nomical evidence for or against a progressive change of latitude. If the increase in the latitudes of Gaithersburg, Cincinnati, and Ukiah deduced in the preceding chapter is due mainly to a secular shifting of the pole, the latitude of Lick Observatory should show a similar increase practically identical with that of Ukiah. In the extract quoted at the beginning of the chapter Prof. Lawson presents the evidence for such an increase even larger than the one at Ukiah.

Instead of drawing the lines *AB* and *CD* by eye we may make a least-squares adjustment, using observation equations of the form previously adopted, namely,

$$\Delta\phi = x + yt,$$

in which $\Delta\phi$ is the difference between the observed latitude and some convenient but arbitrary initial latitude, x is the adjusted latitude at some arbitrary epoch, t the time reckoned from the epoch, and y the adjusted rate of change of latitude. In making these adjustments the observations of 1903 were regrouped as suggested by Dr. Campbell (see extract on p. 46), and the latitude for 1908.33 was omitted in accordance with a further suggestion from him. The values of y deduced from the adjustment are—

For the period 1893.82 to 1903.37, $y = +0''.0361 \pm 0''.0052$ per year.
And for the period 1903.92 to 1918.11, $y = +0''.0040 \pm 0''.0045$ per year.

These lines *AB* and *CD* are shown with these rates of change or slopes in figure 10, which takes the place of Prof. Lawson's figure 1, *B*.

These rates of increase are not very different from those found by Prof. Lawson in the passage quoted and are of the same order of magnitude as the rate for Ukiah. Unfortunately, however, calculations of this sort are nearly meaningless unless they cover a long period, and this is so because, as Dr. Campbell states on page 46, the observations are not based on a homogeneous system of declinations. If the period covered were long, the progressive change of latitude would then presumably be too great to be masked by changes in the declination system, but this is hardly true of the Lick observations. It may be noted, however, that the 11 quarter-yearly means from 1893.82 to 1896.40 and the 21 means from 1905.59 to 1918.11 are both on the Berliner Jahrbuch system and both give practically the same latitude, the mean seconds of the earlier value being $25''.57$ and of the later ones $25''.53$ or $25''.55$ when the suspected value for 1908.33 is omitted.

If we accept a sudden decrease of the latitude due to the earthquake in the autumn of 1903, then the fact that the latitudes have become equal again must mean that the sudden decrease has been compensated by a gradual increase, which would, on Prof. Lawson's theory, be due to a creep of the surface strata, or might, on the tentative hypothesis in the preceding chapter, be due to a shifting of the pole. In round numbers, this rate of gradual increase would be about $0''.02$ a year, provided we assume a sudden decrease of $0''.4$ in 1903; the rate, of course, would be subject to all uncertainties due to errors in the assumed sudden decrease, observation errors, and errors due to incomplete identity of the declination systems which have been taken as absolutely identical. Perhaps all that may be safely asserted is that it would be difficult to disprove the existence at Lick Observatory of a progressive rate of increase of latitude of the same order of magnitude as the rate for Ukiah.

It is questionable whether it would be worth while to try further refinements by reducing the Lick Observatory results as nearly as possible to a common system. Dr. Campbell writes on May 16, 1921:

I do not think it is practicable to determine a set of systematic corrections which will reduce the observed latitudes based upon the various fundamental systems to a homogeneous system. Prof. Tucker's work as published seems to show the following differences in the declination systems:

Publications, Lick Observatory, VI, p. 13; Berliner Jahrbuch—Boss, U. S. Northern Boundary Commission, = +0″.02.
Publications, Lick Observatory, VI, p. 148; Berliner Jahrbuch—Auwers 303 = −0″.34 at −13° dec.
Lick Observatory Bulletin, X, p. 43; Berliner Jahrbuch—Newcomb = −0″.16 at equator = +0″.08 at +25° dec.
Lick Observatory Bulletin, X, p. 42; Berliner Jahrbuch—Boss Preliminary General Catalogue = +0″.11 at equator.

Mr. Tucker's result for the quantity (Auwers new Berliner Jahrbuch system minus old Berliner Jahrbuch system) equals −0″.02 at the Equator for 110 stars observed. The two declination systems are sufficiently identical for present purposes.

The observed latitudes obtained by giving equal weights to the circumpolars and fundamental stars can be reduced to *approximate* homogeneity—that is, to the *early Berliner Jahrbuch system*—by applying *one-half* the differences as quoted above to them.

The Lick results for individual nights through May, 1904, have been published and will be found in volumes IV, VI, and X of the publications of the Lick Observatory.

The difficulties arising from the declinations and proper motions of the stars suggest that the two most practicable ways of studying the secular variation of latitude, where the changes involved are necessarily very small, are (1) a plan substantially the same as that of the International Latitude Service, in which all observatories have the same latitude within a few seconds and use the same program of stars and the same methods of observing and (2) the systematic, long-continued observation of circumpolar stars at both culminations. It is to be hoped that the work of the International Latitude Service will be extended, rather than restricted, and that the observatories will be maintained without interruption for long periods. If it should appear that the creep of surface strata is a common occurrence it will be necessary to select stations where, from geological considerations, little or no such creep is to be expected. At worst, if this were impossible, it might be necessary to greatly multiply the number of stations in the hope that, in the long run, the northward creep of some stations would nearly neutralize the southward creep of others. The plan of observing circumpolar stars at both culminations does not require international cooperation, as the other plan does. Any observatory with a good meridian circle can do the work. It would be necessary to observe on every clear day in order to get a good hold on the periodic, or approximately periodic, portion of the variation of latitude. The annual term in particular is almost certain to be peculiar to the station in question. Part of the annual term is no doubt due to the seasonal changes in refraction. Effects of this sort might be expected to be large, since one culmination would almost always be observed by daylight when the other culmination occurred at night. Very probably the annual term for one circumpolar would be different from the term of another circumpolar culminating at different altitudes, so each star would have to be analyzed separately. Since available circumpolar stars are not numerous, it might be desirable to take circum-meridian observations at both culminations in order to increase the number of pointings.

Chapter IV.—ADDITIONAL RESULTS OF THE DISCUSSION.

The results that form the principal object of this publication have already been given in preceding chapters. It was practicable to obtain also, with comparatively little extra labor, certain other results regarding the variation of latitude, and these are set forth in this chapter. Their chief interest is that, in general, they confirm results obtained by other methods of discussion or from observations covering a shorter time. A brief summary of this chapter and of all preceding chapters will be found in Chapter V, page 67.

SECTION 1. THE HARMONIC CONSTANTS.

TABLE 11.—*Harmonic constants, final values.*

[Epochs of 14-month component reduced to 1900.00.]

Station and longitude.	Years of series inclusive.	Slope in seconds per year. m	Harmonic constants. Annual component. Amplitude. R	Annual component. Epoch. ζ	14-month component. Amplitude. R	14-month component. Epoch. ζ
		″	″	°	″	°
Mizusawa, λ = −141° 08′	1900-05	−0.0067	0.105	14.7	0.135	178.6
	1900-06	− .0067	.106	14.9	.135	177.5
	1906-11	.0000	.132	10.4	.213	163.2
	1906-12	.0000	.123	9.2	.226	165.3
	1910-15	+ .0083	.142	18.3	.214	176.4
	1911-17	+ .0091	.113	14.4	.178	179.4
	1912-17	+ .0093	.101	8.7	.170	184.2
Tschardjui, λ = −63° 29′	1900-05	.0000	.101	340.0	.132	103.0
	1900-06	.0000	.105	335.9	.123	102.3
	1906-11	+ .0233	.130	325.3	.217	97.4
	1906-12	+ .0229	.128	326.7	.226	97.9
	1908-14	+ .0186	.132	323.8	.231	104.7
	1909-14	+ .0267	.144	322.8	.232	106.8
Carloforte, λ = −8° 19′	1900-05	− .0040	.098	275.7	.127	39.0
	1900-06	− .0040	.102	278.0	.129	35.9
	1906-11	+ .0100	.118	277.3	.209	32.1
	1906-12	+ .0100	.106	276.9	.208	36.0
	1910-15	+ .0150	.117	269.3	.218	52.0
	1911-17	+ .0157	.093	274.4	.189	57.2
	1912-17	+ .0187	.079	273.4	.188	62.1
Gaithersburg, λ = +77° 12′	1900-05	.0000	.023	196.6	.143	321.4
	1900-06	.0000	.028	205.4	.144	318.6
	1903-08	+ .0117	.010	256.2	.158	309.5
	1903-09	+ .0107	.012	163.5	.177	310.5
	1906-11	+ .0117	.058	182.4	.222	304.2
	1906-12	+ .0120	.051	178.5	.220	307.0
	1908-14	+ .0140	.057	191.4	.217	316.2
	1909-14	+ .0150	.072	159.2	.220	318.7
Cincinnati, λ = +84° 25′	1900-05	.0000	.009	219.4	.157	317.1
	1900-06	.0000	.022	217.2	.154	311.3
	1903-08	+ .0067	.027	261.7	.169	298.8
	1906-11	+ .0125	.057	199.3	.205	296.8
	1906-12	+ .0154	.054	196.4	.201	298.3
	1908-13	+ .0183	.052	171.9	.199	304.9
	1910-15	+ .0292	.099	160.2	.217	320.9
Ukiah, λ = +123° 13′	1900-05	+ .0142	.042	84.5	.126	276.6
	1900-06	+ .0109	.038	91.2	.130	273.5
	1906-11	+ .0133	.059	75.9	.217	254.9
	1906-12	+ .0121	.057	68.4	.216	258.0
	1910-15	+ .0183	.087	95.4	.211	278.4
	1911-17	+ .0171	.062	99.3	.207	282.8
	1912-17	+ .0192	.069	98.3	.210	285.0

The harmonic constants in Table 3 are not the final values, though they are evidently amply accurate for the purposes for which they are used. The fact that there is in general a progressive increase in the latitude of a station affects the harmonic constants deduced in the ordinary way, and these require a correction, as explained in Chapter VI, section 3, page 80. The constants of Table 11 are those of Table 3 corrected in this way and are to be considered as the definitive values of these constants resulting from this investigation. The quantity m, in the notation of Chapter VI, section 3, which was used in deducing the harmonic constants of Table 11 from those of Table 3, is also given. The quantity m is given for the year as unit of time and is the slope of the line of mean latitudes or annual rate of increase of mean latitude. It was obtained separately for each series by a graphic adjustment of the data plotted in figures 4*a* to 9*a* and 4*b* to 9*b*. A straight line was drawn by eye that fitted the plotted points of the six or seven years of the series as nearly as possible and the slope of this line computed; this gave the value of m for the annual component. This value was multiplied by 1.184, the ratio of the periods, to obtain the value of m for the 14-month component. The values of m should resemble in a general way the values of y or z in Tables 8 and 9. On account of the character of the observations the resemblance is far from close in many cases.

SECTION 2. THE ANNUAL COMPONENT OF THE MOTION OF THE POLE AND THE KIMURA TERM.

It is of interest to determine how far the annual variations are due to a motion of the pole and how far to causes peculiar to each station. For this purpose we may subject the constants to the adjustment described in section 4 of Chapter VI (p. 82), and then study the residuals of the separate stations. Since the harmonic constants are valid only as average values for a given period, only those series should be used that cover identical periods of time for all stations concerned.

The adjusted values of the constants of the north polar motion, as deduced from all available series, are given in Table 12. For the present the discussion applies only to the portion of the table above the double horizontal line; the portion below will be treated later. This table gives the constants for two methods of representing the motion. For the first method there are given the quantities a, b, α, and β, in the notation of Section 4 of Chapter VI. The annual component along the meridian of Greenwich is represented by a term $a \cos(\kappa t-\alpha)$, where t is the time reckoned from the beginning of the year. The quantity κ depends on the unit of time used, but for the annual component $\kappa t=360°$, when t is put equal to the number of time units in a year. The annual component along the meridian of 90° west of Greenwich is similarly represented by a term of the form $b \cos(\kappa t-\beta)$. The quantities α and β are the values of κt at the time of the maximum excursion of the pole away from its mean position toward the equator along the meridians of Greenwich and 90° west, respectively. For example, the value of α being on the average somewhat greater than 240°, the maximum southward excursion of the pole along the meridian of Greenwich occurs after 240/30 or 8 months have elapsed; that is, in September. The maximum southward excursion along the ninetieth meridian west occurs after $\beta/30$ months have elapsed; that is, in June. The annual motion of the North Pole about its mean position is thus seen to be from west to east.

As is well known, a harmonic motion of the kind just specified means that the pole describes an ellipse. The elements of this elliptic motion are given in the right-hand portion of the table. The columns for the lengths of the semiaxes and for the eccentricity are self-explanatory. The direction, θ, of the major axis gives the longitude of the meridian along which the equatorward excursion has its maximum numerical value, A. In longitudes $\theta \pm 90°$ the equatorward excursion has its minimum numerical value, B. The epoch, μ, gives the time of year at which the maximum excursion takes place, $\mu/30$ being the number of months elapsed from the beginning of the year to this maximum. When the ellipse is nearly circular the values of θ and μ are not very accurately determined, and small changes in a, b, α, and β will make them vary widely. The quantity $\theta + \mu$, however, remains nearly constant and nearly equal to α, approaching equality with α as the ellipse approaches a circle. An example of this is the series for 1912–17. This is an exceptional series, for the directions of the major and minor axes are nearly interchanged as compared with their directions for the other series. The value of $\theta + \mu$ is, however, almost the same as for the other series.

TABLE 12.—*Harmonic constants of the annual motion of the pole.*

Years of series inclusive.	Harmonic constants for meridian of—				Elements of the elliptic motion.				
	Greenwich.		90° west.		Semiaxes.		Eccentricity.	For major axis.	
	Amplitude.	Epoch.	Amplitude.	Epoch.	Major.	Minor.		Direction.	Epoch.
	a	α	b	β	A	B	E	θ	μ
	″	°	″	°	″	″		°	°
1900–05 [a]	0.075	254.3	0.047	168.3	0.075	0.047	0.78	+ 4.1	251.7
1900–06 [a]	.079	256.2	.052	172.1	.079	.052	.76	+ 6.8	251.8
1906–11 [a]	.099	248.1	.076	164.9	.100	.075	.66	+ 12.1	238.9
1906–12 [a]	.090	247.0	.072	163.2	.091	.070	.63	+ 12.7	237.1
1910–15 [b]	.119	240.0	.099	153.2	.119	.098	.56	+ 8.4	233.0
1911–17 [c]	.087	240.6	.082	155.1	.089	.080	.43	+ 25.1	217.7
1912–17 [c]	.076	238.3	.083	145.8	.083	.076	.42	+103.5	133.5
1900–11 [d]	.087	250.8	.062	166.1	.087	.061	.71	+ 7.5	245.5
1900–17 [e]	.087	238.8	.074	152.2	.088	.074	.54	+ 10.4	230.0
1900–11 [f]	.088	243.7	.076	165.3	.092	.072	.63	+ 27.4	221.7
1900–17 [g]	.086	243.0	.076	152.3	.080	.076	.47	− 4.8	247.8

[a] 6 stations used.
[b] 4 stations used.
[c] 3 stations used.
[d] 6 stations used. Mean harmonic constants for series 1900–05, and 1906–12.
[e] 3 stations used. Mean harmonic constants for series 1900–05, 1906–12, and 1912–17.
[f] 6 stations used. Mean harmonic constants for series 1900–05, and 1906–12, with the Kimura term included in the adjustment.
[g] 3 stations used. Mean harmonic constants for series 1900–05, 1906–12, and 1912–17, with the Kimura term included in the adjustment.

The elliptic elements in Table 12 are far from representing the annual components of the several stations, and the outstanding residuals have enough interest to be given in detail. This is done in

Table 13, which contains the differences between the observed and the adjusted values of R cos ζ and R sin ζ, the residuals being taken in the sense "observed" minus "adjusted." Only the residuals for the series in Table 12 above the double line are given in Table 13. The adjusted value of R cos ζ is given by $R \cos \zeta = m \cos \lambda + p \sin \lambda$, the notation being that of page 87, and the values of m and p being from the equations on page 90. The "observed" value of R cos ζ and R sin ζ are readily found from the harmonic constants of Table 11. Similarly the adjusted value of R sin ζ is given by: $R \sin \zeta = n \cos \lambda + q \sin \lambda$.

For brevity the residual in R cos ζ, with sign as noted above, will be here denoted by v_1 and the residual in R sin ζ by v_2.

TABLE 13.—*Residuals of the individual stations for the elliptic elements of the annual motion given in previous table.*

[Units in the fourth decimal of seconds.]

Years of series inclusive.	Mizusawa.		Tschardjui.		Carloforte.		Gaithersburg.		Cincinnati.		Ukiah.	
	v_1	v_2	v_1	v_2	v_1	v_2	v_1	v_2	v_1	v_2	v_1	v_2
1900–05	+563	−239	+626	+ 65	+232	−245	+279	+ 3	+414	− 79	+315	− 60
1900–06	+556	−278	+579	− 22	+253	−244	+290	−20	+353	−131	+321	− 96
1906–11	+550	−352	+574	−152	+408	−231	+224	−14	+233	−296	+557	−100
1906–12	+510	−316	+618	−150	+375	−207	+236	− 6	+201	−278	+591	− 96
1910–15	+332	− 73	......	......	+444	− 91	......	......	− 2	− 6	+328	− 62
1911–17	+293	− 97	......	......	+388	−128	......	......	......	......	+287	− 95
1912–17	+260	− 60	......	......	+344	− 79	......	......	......	......	+255	− 59
1900–17	+338	−144	......	......	+450	−190	......	......	......	......	+331	−141

The pairs of residuals are seen to have a general resemblance to one another at all stations. The residuals for a given station would naturally be expected to vary according to the stations used; the table, however, does not immediately reveal an effect of this sort, for the latter is not separated from the changes in residuals due to the variation in the harmonic constants from one period to another. The effect could be brought out, however, for the first four series in the table by using less than the 6 stations and determining the elements of the elliptic motion with the resulting residuals for the new elements.

The Kimura term, or the z term in the discussion of the motion of the pole, is a term common to all stations.[1] This term consists of several parts, one of which has been found to have an annual period; in so far as the values of v_1 and v_2 for a given series remain constant in passing from station to station, their constant values may be used in forming the expression for this annual portion of the Kimura term. It is seen from the table that the assumption of a Kimura term constant from station to station, even when only a single period is considered, is only roughly approximate. The effect represented by the Kimura term is now supposed to be chiefly due to refraction,[2] and there is no reason for supposing such an effect to be constant at all stations, except in so far as the climate, the local topography, and the structure of the observatory [3] are alike at all stations.

[1] The variation of the latitude of a station from its initial value is written $x \cos \lambda + y \sin \lambda + z$; here x and y are the rectangular coordinates of the pole (in seconds) and z is the Kimura term.

[2] Resultate, Vol. V, p. 189.

[3] The refraction to which the Kimura term is attributed is probably for the most part room refraction; that is, refraction between the air within the observatory and the air outside. The assumption that the displacement of the apparent zenith implied in the Kimura effect is due entirely to the nonhorizontality on a large scale of the surfaces of equal air density, leads to quite improbable barometric gradients.

Table 13 is then presented for its possible use in studying, for the individual stations, the annual portions of the latitude variation not due to the motion of the pole rather than for the purpose of extracting from it a Kimura term common to all stations. If we wish such a term, we may take as representative the data on the last line, which represents the mean of 18 years at the three stations which have observations over the entire period. The mean v_1 is $+373$ and the mean v_2 is -158. This indicates that the mean annual portion of the Kimura term for the 18 years and the three stations may be written in the form

$$\left.\begin{array}{ll} & z = +0''.0373 \cos \kappa t - 0''.0158 \sin \kappa t, \\ \text{or} & z = 0''.0405 \cdot \cos (\kappa t - 337^\circ.0). \end{array}\right\} \quad (1)$$

Here κt has the same meaning as on page 52; that is, if the time t be reckoned in days from the beginning of the year, then, for the degree as unit, $\kappa = 360/365.25$.

For comparison with this value we have Wanach's determination [4] for the years 1900–11, inclusive, determined from all six stations by quite another method than the one here used. For the annual part Wanach gets a result equivalent to

$$\left.\begin{array}{ll} & z = 0''.047 \cos (\kappa t - 336^\circ.8), \\ \text{or} & z = +0''.0432 \cos \kappa t - 0''.0185 \sin \kappa t. \end{array}\right\} \quad (2)$$

A result more properly comparable with (2) may be had by taking the mean values of v_1 and v_2 from the first and third lines of Table 13, thus covering exactly the same stations and the same period as Wanach. The mean v_1 is $+415$ and the mean v_2 is -117 so that the Kimura term in this case is given by

$$\left.\begin{array}{ll} & z = +0''.0415 \cos \kappa t - 0''.0117 \sin \kappa t, \\ \text{or} & z = +0''.0431 \cos (\kappa t - 344^\circ.3). \end{array}\right\} \quad (3)$$

Expression (3) is almost identical with that used by Dyson [5] which reads

$$\left.\begin{array}{ll} & z = +0''.041 \cos \kappa t - 0''.011 \sin \kappa t, \\ \text{or} & z = 0''.0424 \cos (\kappa t - 344^\circ.5). \end{array}\right\} \quad (4)$$

The source of Dyson's expression is not given; it presumably covers the years 1900–11. The agreement between (2) and (3) is entirely satisfactory when the difference between the methods of treatment is considered, and the agreement between (2) or (3) and (1) may be cousidered good in view of the difference in the period covered and in the stations used.

We might determine from the adjustment in addition to the constants of the annual motion of the pole the constants of the annual portion of the Kimura term assumed to be the same at all stations. The formulas for this case are given on page —. This has been done for two of the more comprehensive series and the results are given below the double line in Table 12. The Kimura terms deduced from the adjustment resemble the expressions (1) and (3). Corresponding to (3), we have for the series 1900–11

$$\left.\begin{array}{ll} & z = +0''.0448 \cos \kappa t - 0''.0153 \sin \kappa t, \\ \text{or} & z = 0''.0473 \cos (\kappa t - 341^\circ.1), \end{array}\right\} \quad (5)$$

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[4] Resultate, Vol. V, p. 188.
[5] Monthly Notices, Royal Astronomical Society, Vol. 78 (1918), p. 457.

and corresponding to (1) we have for the series 1900–17,

$$\left.\begin{aligned} z &= +0.''0381 \cos \kappa t - 0.''0162 \sin \kappa t, \\ \text{or} \quad z &= 0.''0414 \cos (\kappa t - 337.^\circ 0). \end{aligned}\right\} \quad (6)$$

The values of the harmonic constants below the double line differ from the corresponding ones above it chiefly in giving a larger oscillation along the ninetieth meridian (the y axis) and an annual path of the pole more nearly circular. It is probable that these values are to be preferred to the values above the horizontal line since the existence of the Kimura term is unquestionable and its approximate constancy from station to station is fairly well established. It seems more logical, then, to include the Kimura in the adjustment from the beginning; this process avoids giving undue weight in the series 1900–11 to the region around the two neighboring stations, Gaithersburg and Cincinnati.

At any rate the constants below the horizontal line for the series 1900–11 agree better with other determinations for the same period. For comparison we have two different determinations by Wanach[6] and one by Dyson.[7] The results are as follows:

Harmonic constants.

Designation of constant.	Wanach.		Dyson.
	(1)	(2)	
a	0.''0841	0.''0847	0.''089
b	0.''0759	0.''0760	0.''079
α	255.°2	245.°0	242.°0
β	165.°2	164.°0	155.°0

The agreement between the results for 1900–11 below the line with the results of Dyson and Wanach is as good as the agreement of the latter among themselves; the chief discrepancy is in the value of β.

The results for the two series below the line are presumed to represent average conditions better than any of the other results in Table 12; they are, therefore, used in deducing the motion of the undisturbed pole of inertia in section 5 of Chapter VI (p. 104). One conclusion to be drawn from this discussion of the annual motion is that the elements of the annual path of the pole depend considerably on the method of treatment used, and that it is very desirable to have a sufficient number of stations to get a good hold on the Kimura term. The 3 stations now maintained are just sufficient to determine the constants of the annual polar motion and the annual portion of the Kimura term, the latter being assumed to be the same at all 3 stations. There is no station to furnish a check on this assumption, a condition of affairs which it is hoped will soon be remedied; the original 6 stations were none too many.

The semiannual portion of the variation of latitude has been included with the annual as being a mere harmonic of it. The same harmonic analysis that gives the annual terms gives the semiannual

[6] Resultate, Vol. V, pp. 211 and 217.
[7] Monthly Notices of the Royal Astronomical Society, vol. 78, 1918, p. 456.

terms also. Material for forming these terms is given in Table 14. Its form is somewhat different from that of other tables of harmonic constants. Only the s's and c's are given, not the amplitudes and epochs (R's and ζ's). This is because the amplitudes and epochs for the same station differ widely for different series, as indeed the s's and c's do also, but in such a case it is not correct to take the arithmetic mean of the amplitudes and epochs for different series to get a mean amplitude and a mean epoch for the entire period, while such a taking of means is entirely proper for the s's and c's. When a mean s and a mean c have been found, the corresponding R and ζ may be found in the usual way. (See p. 74.)

The quantities in the table are not the immediate results of the harmonic analysis, but have been corrected for "slope" (see pp. 51 and 80) in the same manner as the quantities in Table 11, and the effect of the 14-month component has been eliminated. The 7-month component, being nearer in period to the semiannual component than the 14-month, would have a much larger effect if it existed, but as its existence is quite doubtful (see p. 61) no attempt has been made to correct for it.

TABLE 14.—*Harmonic constants for the semiannual portion of the variations of latitude.*

Station and longitude.	Years of series inclusive.	s_2	c_2	Station and longitude.	Years of series inclusive.	s_2	c_2
		″	″			″	″
Mizusawa, Japan,	1900–05	−0.004	0.000	Gaithersburg, Maryland,	1900–05	−0.006	0.000
λ = −141° 08′.	1900–06	− .003	.000	λ = +77° 12′.	1900–06	− .008	− .004
	1906–11	+ .004	− .012		1903–08	+ .002	+ .001
	1906–12	+ .004	− .013		1903–09	+ .004	+ .001
	1910–15	+ .012	− .019		1906–11	+ .003	− .012
	1911–17	+ .017	− .018		1906–12	+ .007	− .008
	1912–17	+ .014	− .020		1908–14	+ .011	− .011
					1909–14	+ .013	− .011
Tschardjui, Turkestan,	1900–05	+ .008	+ .005				
λ = −63° 29′.	1900–06	+ .007	+ .006	Cincinnati, Ohio,	1903–05	+ .001	− .003
	1906–11	− .009	− .002	λ = +84° 25′.	1900–06	− .004	− .002
	1906–12	+ .001	+ .004		1903–08	+ .001	− .001
	1908–14	− .006	− .005		1906–11	− .002	− .007
	1909–14	− .007	− .003		1906–12	+ .001	− .002
					1908–13	+ .008	− .005
Carloforte, Sardinia,	1900–05	− .003	− .004		1910–15	+ .013	− .010
λ = −8° 19′.	1900–06	− .003	− .006				
	1906–11	+ .001	− .006	Ukiah, California,	1900–05	+ .006	− .007
	1906–12	+ .004	− .005	λ = +123° 13′.	1900–06	+ .004	− .008
	1910–15	+ .005	− .006		1906–11	.000	− .010
	1911–17	+ .012	− .005		1906–12	+ .001	− .008
	1912–17	+ .018	− .003		1910–15	+ .012	− .002
					1911–17	+ .008	− .008
					1912–17	+ .010	− .008

The values of s_2 and c_2 in Table 14 are evidently small and their significance is therefore likely to be obscured by accidental errors of one sort or another. There appears to be in general a numerical increase in these quantities between the first and the last series for each station. The values of s_2 and c_2 for the later series at different stations somewhat resemble one another and do not definitely reverse in sign on opposite sides of the pole, as they would if the semiannual term were due chiefly to the motion of the pole.

As the results in this chapter are incidental to the main purpose of this report, it has not seemed worth while to undertake any very elaborate discussion of the semiannual component. Elements of the elliptic motion might be computed from Table 14, for the same series of years as are shown in Table 12, but the only series actually considered are two representative ones; the first is the mean result, for all stations, for the years 1900–11, inclusive; the second is the mean result, for three stations, for the years 1900–17, inclusive. From the values of a, b, α, and β as found from the formulas on pages 82 and 86, we may write for the semiannual terms in the motion of the North Pole along the axes previously specified (p. 33) from the series 1900–11

$$\left.\begin{array}{l}\text{motion along } x \text{ axis} = 0.''0016 \cos(2\kappa t - 311^\circ.6),\\ \text{motion along } y \text{ axis} = 0.''0038 \cos(2\kappa t - 174^\circ.0),\end{array}\right\} \quad (7)$$

from the series 1900–17

$$\left.\begin{array}{l}\text{motion along } x \text{ axis} = 0.''0046 \cos(2\kappa t - 351^\circ.2),\\ \text{motion along } y \text{ axis} = 0.''0010 \cos(2\kappa t - 45^\circ.4).\end{array}\right\} \quad (8)$$

In these equations, κ and t have the meanings already specified for the annual component (p. 52). A result that should be comparable with (7) is that found by Wanach,[6] which covers also the years 1900–11 and is equivalent to

$$\left.\begin{array}{l}\text{motion along } x \text{ axis} = 0.''\,0032 \cos(2\kappa t - 23^\circ),\\ \text{motion along } y \text{ axis} = 0.''\,0031 \cos(2\kappa t - 175^\circ).\end{array}\right\} \quad (9)$$

Equations (7) and (9) agree satisfactorily, difference in methods considered, for the motion along the y axis, but rather poorly for the motion along the x axis. The residuals of (7) and (8) for the separate stations in the sense "observed" minus "adjusted" are given in Table 15.

TABLE 15.—*Residuals for the semiannual element of polar motion.*

Station.	Residual by formula (7)— In c_2. v_1	Residual by formula (7)— In s_2. v_2	Residual by formula (8)— In c_2. v_1	Residual by formula (8)— In s_2. v_2
Mizusawa	−73	−11	−65	+44
Tschardjui	−22	+ 6	....	....
Carloforte	−68	+ 3	−88	+59
Gaithersburg	−22	−15	....	....
Cincinnati	−16	− 5	....	....
Ukiah	−46	+20	−64	+43
Mean	−41	+ 0	−72	+49

The portion of the semiannual variation of latitude peculiar to each station as given by Table 15 is seen to exceed, in general, the portion due to the semiannual motion of the pole as given by formulas (7) or (8).

Just as in Table 13, the similarity of the residuals in Table 15 for the different stations suggests that a common value of all the residuals should be represented in the Kimura term; this part of the term would be

$$v_1 \cos 2\kappa t + v_2 \sin 2\kappa t, \qquad (10)$$

[6] Resultate, Vol. V, p. 217.

the mean values of v_1 and v_2 being found as in Table 13. Such a term might, like the annual portion of the Kimura term, be chiefly due to some peculiarity of the seasonal climatic cycle common to all stations, or it might be due to periodic errors in the declinations used. The alteration with the time of the values of s_2 and c_2, to which attention has been called and which is reflected in the difference between the two sets of residuals in Table 15, suggest periodic errors in the declination as a more probable cause, for the star program necessarily changes with the time, and periodic errors may not have been so completely eliminated from the later star places as from the earlier. The magnitude of such period errors, as judged by the systematic corrections necessary to reduce from one star catalogue to another,[7] is in general considerably larger than would be given by an expression like (8) with v_1 and v_2 of the order of magnitude of the values in Table 15. No investigation, however, of such periodic errors in the declinations used by the International Latitude Service has here been attempted.

It does not seem practicable to formulate in a few words the results of this investigation in so far as they concern the semiannual term, this term being small and likely to be obscured by accidental errors. The numerical results are presented in Tables 14 and 15 and in formulas (7), (8), and (10), which are given for what they may be worth.

Formulas (7) and (8) do not agree at all well with each other, and the discrepancy would lead one to doubt the reality of a semiannual term in the polar motion given by either one of them. Wanach[8] discusses the two halves of the series 1900–11 separately (by a method quite different from the one here used) and finds, as regards the semiannual term, a passable agreement between the two halves. He thence infers that the semiannual term is real and approximately of the value found for the series 1900–11. To be sure, the result for the 24-year period, 1890.5–1914.5, does not agree well with the other result, but he attributes the discordance to the nature of the material available before 1900.

The change with time in the semiannual terms for the individual stations seems pretty well established by an inspection of Table 14, also the fact that, particularly for the latter part of the period 1900–17, there is a semiannual portion in the Kimura term. The mean value of the semiannual portion may be represented by expression (10), with the mean values of v_1 and v_2 from Table 15, or as follows:

$$\text{Semi-annual portion of Kimura term} = +0''.0072 \cos 2\kappa t + 0''.0049 \sin 2\kappa t$$
$$= 0''.0087 \cos (2\kappa t - 145^\circ.8).$$

SECTION 3. THE 14-MONTH COMPONENT OF THE MOTION OF THE POLE.

From the harmonic constants of the 14-month component of the separate stations we may deduce by a least-squares adjustment the harmonic constants for so much of the variation as is due to the motion of the pole. This has been done and the result is set forth in Table 16. This table has the same form as Table 12, except that the probable error of θ (indicated by the $\pm$ sign) is shown in connection with θ. This matter will be considered later.

[7] See for example Boss's Preliminary General Catalogue of 6188 Stars, etc., Washington (Carnegie Institution), 1910, Appendix III.
[8] Resultate, Vol. V, pp. 218–219.

TABLE 16.—*Harmonic constants for the 14-monthly motion of the Pole.*

[For notation see pp. 82-86. Epochs reduced to 1900.00.]

Years of series inclusive.	Harmonic constants for meridian of—				Elements of elliptic motion.					
	Greenwich.		90° W.		Semi-axes.		Eccentricity.	For major axis.		
	Amplitude.	Epoch.	Amplitude.	Epoch.	Major.	Minor.		Direction.	Epoch.	
	a	α	b	β	A	B	E	θ	μ	$\theta+\mu$
	″	°	″	°	″	″		° °	°	°
1900-05 [a]	0.131	32.7	0.141	310.6	0.146	0.125	0.52	59.4 ± 4.0	337.4	36.8
1900-06 [a]	.131	30.1	.139	307.7	.145	.125	.50	57.3 ± 4.8	336.7	34.0
1906-11 [a]	.206	23.3	.217	293.4	.217	.206	.31	89.6 ± 18.5	293.7	23.8
1906-12 [a]	.210	25.9	.218	295.6	.219	.210	.28	94.2 ± 22.2	291.6	25.8
1910-15 [b]	.210	40.6	.219	312.2	.220	.209	.32	74.6 ± 16.3	326.8	41.4
1911-17 [c]	.179	47.1	.208	310.0	.212	.175	.57	110.1 ± 2.7	293.2	43.3
1912-17 [c]	.179	52.8	.205	312.0	.213	.169	.61	116.6 ± 1.5	290.4	47.0
1900-11 [d]	.173	29.8	.176	300.2	.177	.173	.21	80.5	309.5	30.0
1900-17 [e]	.170	35.5	.179	303.0	.180	.168	.36	110.3	283.9	34.2

[a] 6 stations used.
[b] 4 stations used.
[c] 3 stations used.
[d] 6 stations used. Mean harmonic constants of separate stations found from series for 1900-05 and 1906-11 and elements of elliptic motion determined from mean constants.
[e] 3 stations used. Mean harmonic constants of separate stations found from series for 1900-05, 1906-11 and 1912-17 and elements of elliptic motion determined from mean constants.

For comparison with these results there is available the results deduced by Dyson and Wanach.[9] Wanach's result is deduced on the assumption of circular motion. The radius and epoch for a motion of this sort are given under a and α, respectively, and these should be approximately equal to the quantities $\frac{1}{2}(a+b)$ and $\frac{1}{2}(\alpha+\beta-270°)$ of the elliptic motion for a corresponding period.

Harmonic constants.

Designation of constant.	Wanach, 1900-11.		Dyson.	
	(1)	(2)	1900-05	1906-11
a	0.″1738	0.″167	0.″127	0.″206
b			0.″131	0.″213
α	26°.1	26°.0	31°.5	25°.4
β			309°.3	294°.2

Dyson's results were translated into the notation of this report by the use of his own value for the free period, 432.2 days. Part of the slight difference between the epochs for corresponding periods may be due to the difference in the free periods assumed, Wanach's value being 432.8 days and for this report 432.5 days. The results obtained in this report agree satisfactorily with the results of Dyson and Wanach for corresponding periods of time. The ellipses deduced from Dyson's values of a, b, α, and β, will be found to be of the same general character and to have their axes in the same general direction as those for like periods in Table 16.

It is of interest in this connection to mention a verification of the a and α for the series 1912-17 coming from outside the International

[9] See references in footnotes 6 and 7 on p. 56.

Latitude Service. Observations at Greenwich with the Cookson floating zenith telescope give a curve of latitude which, when analyzed, yields the following expression for the 14-month term for 1912–17,

$$0''.191 \cos (\kappa t - 49^\circ.9).$$

The amplitude differs only 0″.012 from the corresponding *a* in Table 16 and the epoch differs 2°.9, a satisfactory agreement for both. The Greenwich results may be introduced into the determination of the 14-monthly polar motion, but, as might be expected, their introduction does not change substantially the figures in Table 16.

Since the latitude of a place depends both on the direction of the earth's axis and on the direction of the plumb line at the place, it is conceivable that the plumb line of each station may have an individual oscillation of its own in a 14-month period, thus causing variation in latitude independent of the polar motion. The observations here discussed, however, lend little or no support to this idea. An examination was made of the residuals from the adjustment that furnished the data for Table 16; that is, of the residuals from the 14-month components of polar motion analagous to those that furnish the Kimura term for the annual variation. The residuals were found to be much smaller than for the annual component and to vary greatly from one series to another, giving evidence of being mainly due to accidental errors. The results are, therefore, not given in detail. The magnitude of the residuals was such that the probable error of $s = R \sin \zeta$ or of $c = R \cos \zeta$, as obtained from the adjustment of the harmonic constants of the 14-month component, may be estimated at $\pm 0''.006$ or $\pm 0''.007$.

There is one reason for the existence of a 14-month oscillation peculiar to each station that may present itself to one considering the subject. The motion of the pole changes the centrifugal force over the earth's surface, and the ocean endeavors to conform to the resulting changes in the equipotential surface, thus producing a 14-monthly latitude-variation tide which has been investigated and detected. If the earth were covered with water, this tide would have the same character at all stations in the same latitude; but the earth, not being so covered, the plumb lines of all such stations are not equally affected by the gravitational action of the tidal load. A little calculation will show, however, that the difference of latitude due to an effect of this sort can not exceed 0″.0001 or 0″.0002 and may therefore be neglected.[10]

The harmonic analysis of the 14-month component gives readily enough the data for a possible term in the latitude variation with a 7-month period. The theory of the variation of latitude does not indicate the existence of such a term in the motion of the pole, and since, as regards the local periodic deflections of the plumb line, we have neglected the 14-month term we should naturally neglect with greater reason the 7-month term, which would be merely a harmonic

[10] The latitude of a place is subject to an indirect influence, due to the effect of the displacement of the pole on the direction of the plumb line. The displacement of the pole causes equal changes in the geocentric latitudes of all points in the same meridian, but does not cause equal changes in the geographic latitudes, which are the quantities directly measured. From one point of view this slight inequality in the change of geographic latitude along the same meridian may be considered as due to the effect on the plumb line of the swellings and depressions of the earth tides produced by the displacement of the pole and the attendant changes in the centrifugal force. A correction for this fact would be necessary only if the motion of the pole were to be deduced from two sets of latitude stations situated on widely separated parallels of latitude; the amount of the correction necessary to reduce stations to latitude 45° as a standard is of the order of magnitude of 0″.0003 and therefore unimportant with observations of the present degree of accuracy.

of the 14-month term. The 7-month terms deduced from the analysis give every indication, in spite of all refinements of computation, of being due merely to accidental errors. The results of the computation of the 7-month terms are therefore not given here.

The mathematical theory of the variation of latitude (see p. 93) assumes that small changes occur in the position of the earth's axis of figure; that is, the axis about which the moment of inertia is greatest.[11] Changes of this sort of a roughly periodic character would occur because of the unsymmetrical annual distribution of snowfall, barometric pressure, etc. These periodic changes cause what may be called a forced periodic oscillation of the pole of figure, the period being a year. The pole of rotation does not in general coincide with the pole of figure, but is subject to a forced oscillation in an orbit of its own, different from the orbit of the pole of figure, but having the same period. The pole of rotation will have in addition what is called a free oscillation in a period determined, if the earth were a rigid body, merely by its moments of inertia and the angular velocity of its rotation, or determined for our actual earth by the foregoing quantities and, in addition, the mean elastic moduli of the earth. This free period is the 14-month period, or more definitely 432.5 days.

Furthermore, theory shows that if the forced oscillation of the pole of figure were strictly periodic, the exact period or periods being unimportant provided none of them coincides too nearly with the free period, and if the two principal equatorial moments of inertia were equal, then, whatever the character of the forced oscillation, simple or complex, the free oscillation would always consist of a uniform circular motion of the pole of rotation. If, on the other hand, the principal equatorial moments of inertia were unequal, the forced oscillation being still strictly periodic, the free oscillation would take place in an ellipse with its major axis lying in the meridian of that equatorial axis about which the principal moment of inertia is the larger; that is, in the meridian of the shorter equatorial axis of the geoid considered as an ellipsoid of three unequal axes.[12]

With a view to detecting a possible inequality in the equatorial moments of inertia the assumption of uniform circular motion was avoided in making the adjustment to determine the elements of the polar motion given in Table 16. By such a process, of course, the motion, even if exactly circular and uniform, would come out elliptic owing to the presence of accidental errors.[13] In such a case, however, there would be no evidence of even approximate constancy in the direction of the major axis of the ellipse, represented by θ in Table 16. The variability of θ in this table, and the size of some of the probable errors,[14] show the presence of considerable accidental

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[11] This axis may also be called *the* axis of inertia, and a pole of this axis may be termed indifferently either the pole of figure or the pole of inertia.

[12] For the relation between moments of inertia, inequalities in the radii of the geoid, and inequalities in gravity on the surface of the geoid, see Helmert, Höhere Geodäsie, Vol. II, chap. 2.

[13] Departures from perfect periodicity in the free and forced oscillations are necessarily treated as accidental errors.

[14] The larger probable errors should not be taken as anything more than rude approximations; their size is due to the small eccentricity of the ellipse, which necessarily renders uncertain the location of its major axis, rather than to the greater inaccuracy of the underlying observations. For these larger errors the assumption made in deriving the formulas, namely, that changes in θ may be treated as differentials, no longer gives a good approximation for a change in θ of the size of the larger probable errors. In connection with all these probable errors, large and small, it should be remembered that they include neither the errors of the assumption that the variation of latitude is representable by harmonic terms of the kind here used nor the errors of the conventions by which the fictitious observation equations were introduced. (See p. 87.)

error, nevertheless there seems to be a tendency for the major axes to cluster around a direction not far from 90° west of Greenwich.

For comparison with this result there is Helmert's determination of the same thing by a very different method,[15] namely, from gravity observations. Of these observations Helmert made various adjustments differing in the data used and the weights assigned. The determinations of the longitude of the equatorial axis with the larger principal moment of inertia (the shorter axis of the geoid) varied correspondingly from 94° W. to 122° W. His adopted result was 107° W. with a probable error of $\pm 4°$. The results from the latitude variation are seen to be affected with a greater uncertainty than those from gravity observations, but are not inconsistent with the latter, confirming the general character of Helmert's result as far as it concerns the position of the shorter axis of the geoid. A rough confirmation of this sort is far from useless, for the character of Helmert's result is rather surprising, since any great inequality in the equatorial moments of inertia or equatorial axes seems inconsistent with our generally accepted notions about the condition of hydrostatic equilibrium prevailing within the earth, or, in other words, with the theory of isostasy. Furthermore, Helmert's whole method might be questioned on account of systematic errors in the gravity observations arising from the topography and the surface geology in the vicinity of the gravity stations and because the gravity observations are at present confined to one-fourth only of the earth's surface, that is, to the land.

The difference between the two principal moments of inertia and the corresponding difference in the equatorial semiaxes are quite as important to know as the longitude of the shorter axis. Helmert's various adjustments yield results for the difference between the semiaxes varying from 154 meters to 510 meters. His adopted result is 230 meters. The corresponding difference between A and B, the principal moments of inertia about axes in the plane of the equator is given by

$$B-A=0.000024\ Ma^2,$$

where M is the mass of the earth and a the mean equatorial radius. Calling C the moment of inertia about the polar axis, we find $(B-A)/(C-\frac{1}{2}(A+B))=1/46$, a quantity which enters into the determination of the form of the ellipse of polar motion. Schweydar[16] has computed the form of the ellipse of the polar motion corresponding to Helmert's adopted result; he finds an ellipticity of 0.016, that is an eccentricity of 0.18, which is smaller than any eccentricity in Table 16, though not far from the smallest. The result from the variation of latitude as regards the difference between the equatorial semiaxes of the geoid seems again to be a confirmation of Helmert's result in a general way, though one suggesting a rather greater difference than the one that Helmert finally adopted.

In regard to the difference between the equatorial semiaxes two facts should be noted, which point in opposite directions. First, the difference between the equatorial semiaxes varies as the square of the eccentricity of the ellipse of polar motion, so that

[15] Neue Formeln für den Verlauf der Schwerkraft im Meeresniveau beim Festlande. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, 1915, p. 676.
[16] Astronomische Nachrichten, No. 4855, Vol. 203 (1916), p. 101.

an eccentricity like 0.36 in Table 16, which is just twice Schweydar's value deduced from Helmert, implies a difference between the semiaxes of the earth of four times that found by Helmert, or 920 meters. Second, there seems to be a tendency as series are made longer for the eccentricities to decrease, indicating that the larger eccentricities are at least partly due to accidental errors. Of course, if all eccentricity were due to error of this sort, we should expect the directions of the major axes to be scattered all over a semicircle, but this is not the case. This tendency toward eccentricities diminishing as the length of series increases may be noted in the series for 1900–11 which gives the smallest eccentricity in the table, smaller than either of its component parts,[17] the series for 1900–05 and 1906–11. The series 1900–17 shows the same tendency, yielding an eccentricity less than the eccentricities of two of its three component parts,[18] the series for 1900–05, 1906–11, and 1912–17.

By balancing these two opposite tendencies we are led to see in the figures of Table 16 a rough confirmation of Helmert's general result, which is all that the data now available will yield. Perhaps, on the whole, the latitude observations indicate a difference in the equatorial semiaxes rather greater than Helmert's adopted result, say 300 meters or even more, rather than 200. These figures and the figure 90° west longitude for the direction of the axis of the ellipse of free polar motion are both subject to a correction for the unsymmetrical distribution of land and water and its effect on the yielding of the ocean waters to the centrifugal forces arising from the displacement of the pole. (See p. 101.) No definitive results are therefore stated until this correction has been further investigated. It appears to be rather small. If the number of latitude stations could be maintained at six or more over a considerable period, it should be possible to get a result for the figure of the earth having as great an apparent degree of accuracy as the gravity observations give, and free from the suspicion of systematic error due to the fact that gravity observations, being at present confined to land, cover only one-fourth of the earth's surface and are much affected by local topography and geology.

SECTION 4. COMPONENTS OF LONGER PERIOD.

The annual climatic cycle appears to be the fundamental cause of the forced annual oscillation of the pole, though there are difficulties in obtaining an accurate evaluation of all the effects involved.[19] Other climatic cycles are known or suspected and it is natural to look for their effects on the motion of the pole. An extended study of the subject would be quite beyond the scope of this report. All that is here attempted is to utilize a period of observation available by extending slightly some work done by Wanach.[20] Attention was

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[17] This result is not unreasonable, but arises partly from the difference in direction of the major axes. A simple extreme case will illustrate. Suppose two equal concentric ellipses of small eccentricities with major axes at right angles to one another, and let a curve be drawn with radius vector equal to the mean of the two radii vectores. Neglecting the fourth powers of the eccentricity, the new curve will be a circle, i. e., an ellipse of eccentricity zero.

[18] The eccentricities, from the first two component parts, as set down in the same table, come from six stations, not from three, and are therefore not strictly comparable with the series 1900–17.

[19] Schweydar: Zur Erklärung der Bewegung der Rotationspole der Erde; Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1919, p. 357. Jeffreys: Causes contributory to the annual variation of latitude; Monthly Notices of the Royal Astronomical Society, Vol. 76 (1916), p. 409.

[20] Resultate, Vol. V, p. 217.

drawn by Angenheister [21] to periods of 3 and 6 years which he found in the heights of the barometer at a chain of 9 stations encircling the globe. He gave also some estimates to show that these inequalities in barometric pressure might produce sensible effects in the motion of the pole. There is a period of about 3 years in the behavior of sunspots with which the 3-year barometric inequality is presumably connected.

It is not difficult to utilize the monthly readings of the latitude, already used in obtaining the annual component, for the purpose of obtaining terms in the latitude whose period is some multiple of a year. To study a possible 6-year term the readings of three consecutive months were grouped together and the mean treated as a single reading; this gives 24 readings in a 6-year period, and at Mizusawa, Carloforte, and Ukiah, the 18 years of observation gave just three such periods. The second harmonic of the 6-year term gives a 3-year and the third harmonic a 2-year term, which was also derived, although no 2-year period was suggested by Angenheister. The augmenting factors for the amplitudes and the reduction of the epochs to the beginning of the year 1900, both being made necessary by the use of 3-month groups as single observations, are easily found. (See p. 79).

From the harmonic constants of the 3 stations the corresponding motion of the pole was then deduced. Let x_1, x_2, and x_3, denote the resolved portion of the polar motion on the x axis (Meridian of Greenwich) due to the 6-year, 3-year and 2-year terms, respectively; the subscripts denote the harmonic order of the term with the 6-year term as fundamental; let y_1, y_2, and y_3 denote the resolving portions of polar motion of the same periods on the y axis; for these terms the following values were deduced with which are shown Wanach's values for comparison.

Period.	This investigation.	Wanach.
6 years	$x_1 = 0''.0031 \cos(\kappa t/6 - 80°)$	$= 0''.0223 \cos(\kappa t/6 - 106°)$.
	$y_1 = 0''.0121 \cos(\kappa t/6 - 272°)$	$= 0''.0083 \cos(\kappa t/6 - 268°)$.
3 years	$x_2 = 0''.0107 \cos(\kappa t/3 - 225°)$	$= 0''.0038 \cos(\kappa t/3 - 125°)$.
	$y_2 = 0''.0146 \cos(\kappa t/3 - 41°)$	$= 0''.0170 \cos(\kappa t/3 - 9°)$.
2 years	$x_3 = 0''.0147 \cos(\kappa t/2 - 204°)$	$= 0''.0084 \cos(\kappa t/2 - 49°)$.
	$y_3 = 0''.0136 \cos(\kappa t/2 - 284°)$	$= 0''.0053 \cos(\kappa t/2 - 7°)$.

The κ in these equations is, of course, the same as in the expressions for the annual portion of the Kimura term on page 55. Wanach's values are from observations covering the years 1900–11, inclusive, with all 6 stations. The other values depend on 3 stations but cover the longer period 1900–17, inclusive. There is some resemblance in the two sets of values for the 3-year and the 6-year terms that has survived the difference in method of treatment, in the stations used, and in the period covered. Since, furthermore, there appears to be a priori reason for expecting such terms, we may consider their existence as fairly probable, but how accurately the polar motion may be given by either

[21] Über die dreijährige Luftdruckschwankung und ihren Zusammenhang mit Polschwankungen; Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Math.-phys. Klasse, 1914, p. 1. The paper is described as a preliminary communication, but nothing further on the subject from Angenheister has come to the author's attention.

set of expressions is another question. The 2-year terms seem to be merely the result of manipulating figures. The 3-year terms in the expressions for the polar motion reproduce better the terms of like period at the several stations from which the former were derived than do the terms in the polar motion with periods of 2 or 6 years. The mean residual (without regard to sign) in the *s*'s and *c*'s of the 3 stations is 0.″0058 for the 6-year terms, 0.″0027 for the 3-year terms, and 0.″0088 for the 2-year terms.

The summation for the 6-year component afforded a partial check on the numerical work. The sixth harmonic is, of course, the annual component. The annual term so obtained, when all corrections have been applied, should be equal to the mean result of the three consecutive 6-year series already obtained, namely, 1900–05, 1906–11, and 1912–17. In taking the mean of the three it is more accurate to average the *c*'s and *s*'s than the amplitudes and epochs. The results of this check were entirely satisfactory; the epochs agreed within a fraction of a degree and the amplitudes within a few units of the fourth decimal of a second, that being the last decimal place retained in the computations.

The fifth harmonic of the 6-year component comes out very large; it is explained by the fact that the period of this harmonic, 1.20 years, is so nearly that of the 14-month term, or 1.184 years, so that the two periods do not separate much in the course of 18 years. The fifth harmonic can, in fact, be made to furnish an approximate value of the 14-month term, thus affording a rough check.

Chapter V.—SUMMARY AND CONCLUSIONS.

The rate of apparent increase in the latitude of Ukiah found by Prof. Lawson, 0.″0094 per year, is reduced to 0.″0081 per year by the application of a more accurate method to the same material as Prof. Lawson used. Neither rate is to be further increased on account of any supposed change in the zero from which changes in latitude were reckoned.

To make sure that the apparent increase is not due to the star places used, the other stations of the International Latitude Service were examined for possible changes of latitude similar to those at Ukiah. The definitive latitudes of the International Latitude Service from 1900.00 on were used so far as available and then the provisional latitudes through the year 1917. The details of the methods used are given in Chapter II, pages 8 to 30. The results for all stations are given in Tables 8 and 9. These tables indicate that the increase in latitude at Ukiah is in no way exceptional, that all stations except Mizusawa show, on the whole, apparent increases in latitude throughout the period covered, and that toward the latter part of that period Mizusawa also shows an increase. The average rate of increase of Gaithersburg usually exceeds the average rate of Ukiah over like periods of time. The rate of increase of latitude at Ukiah varies from 0.″0063 to 0.″0260 a year according to the method used and the period covered. The rate at Gaithersburg varies from 0.″0083 to 0.″0218 a year. The larger values just mentioned for the two stations are too uncertain to be used as a basis for any trustworthy conclusion. The more reliable rates are not far from the rate of 0.″0081 already found. (For particulars, see Tables 8 and 9.)

A general apparent increase in latitude at all stations might mean an approach of these stations toward the pole, but a much more probable explanation is an increasing error in the star places due to erroneous values of the proper motions, an error that would affect all stations alike, since the star program is alike at all stations. An inspection of Tables 8 and 9 shows a comparatively large increase at American stations and a small increase or an actual decrease at Asiatic and European stations, which are on the opposite side of the pole from America. This suggests that superposed on the purely apparent change of latitude due to star-places there may be a change of latitude due to a displacement of the North Pole toward the American continent. A least-squares adjustment on this hypothesis of the rates of change given in Tables 8 and 9 gives different but fairly consistent results according to the data and weights used. These results are set forth in Table 10, page 34. An arbitrary mean of these results was adopted as one definitive result of this investigation, namely a progressive shifting of the North Pole at the rate of 0.″0050 (equal to 15 centimeters or 6 inches) per year in magnitude and directed toward the Equator along the meridian of 78° west of Greenwich. This is a mean result for the years 1900–14, inclusive; the observations may be interpreted as meaning that a polar motion of the same general character continues through 1917. A revision of the calculations was undertaken while this publication was in press and was completed too late for the results to be thoroughly incorporated in it. The revision led to the complete rejection of Tschardjui in the determination of the polar displacement. The effect of the rejection is to increase the annual rate of displacement of the North Pole for the period 1900–1917 to about 0."0062. The direction of the displacement is a few degrees nearer to the meridian of 90° west than the 78° just mentioned, the direction apparently varying somewhat during the period tested. For reference to the article containing the revised calculation see the last paragraph of the introduction, page 2.

In regions situated on meridians nearly at right angles to the meridian of 78° west the shifting of the pole would have practically no effect on the latitude. One such region is evidently western Europe, which is just the region where the longest series of accurate observations have been made. These observations would then be of little use in deciding whether or not a polar displacement of the general nature of the one here deduced has been going on for some time in the past and might therefore be expected to continue for some time in the future. The shifting of the pole during the years in question may be due to a combination of periodic terms having periods of a few years or a few decades, terms whose existence has not yet been established but which might correspond to meteorological cycles of one sort or another. Very recent observations at Ukiah might indicate that this shifting of the pole is not continuing and might even be reversing itself, but the evidence of Ukiah alone on this point is not conclusive.

This shifting of the pole toward the American continent combined with a change in the average proper motion of the stars used of about the same magnitude (see values of *w* in Table 10) appears to be a fairly satisfactory explanation of the apparent increase in latitude at Ukiah, and at all other stations of the International Latitude Service during the period in question, without invoking the effects of a creep of surface strata. No attempt is made to pronounce on Prof. Lawson's hypothesis in general; the question in this investigation is simply the interpretation of the astronomical evidence with special reference to Ukiah. The hypothesis of a surface creep at Ukiah is not absolutely excluded, but if a creep exists, its effect appears to be subordinate to the effects of polar motion and of errors in the assumed proper motions of the stars. A brief discussion is given of the geophysical questions involved in a possible progressive shifting of the pole (pp. 39 to 42).

The passage in Prof. Lawson's article dealing with the latitude of Lick Observatory is reproduced on page 44. A rate of change of latitude of Lick Observatory before and after September, 1903, is deduced by least-squares adjustments from the data used by Prof. Lawson. Rates of increase as found from these adjustments resemble those deduced by Prof. Lawson by a graphic process, but the value of the entire result is rendered doubtful by a lack of homogeneity of the declinations used. There is some evidence in the latitudes themselves of a sudden decrease in the latitude of Lick Observatory in September, 1903, about the time of a severe local earthquake. Some decrease of this general nature is accepted as fairly plausible by the director, Dr. Campbell, but is doubted by Mr. Tucker, the astronomer who made the latitude observations. Triangulation executed both before and after 1903 gives no clear evidence of a shift in the position of Lick Observatory. If this sudden decrease be accepted, there is then evidence, based solely on stars belonging to a single declination system, that the latitude of Lick Observatory may have increased progressively at about the rate due to the motion of the pole given above. The lack of homogeneity of the declinations for any extended period renders any very definite conclusion on the subject rather difficult.

The harmonic constants derived in the course of the work to represent the variation of latitude at the various stations are corrected for all known sources of error and the definitive results are given in Tables 11 and 14 (pp. 51 and 57).

The portion of the variation of latitude having an annual period is composed of the motion of the pole with an annual period and to various other effects not due to the motion of the pole and to a certain extent different from station to station. These other effects are lumped together as the annual part of the Kimura term. The annual portion of the motion of the pole, as deduced from an adjustment of the harmonic constants of the several stations, varies according to the way in which the Kimura term is treated. The results for the annual motion considered to be on the whole the most satisfactory are given in the two lower lines of Table 12, on page 53. They agree satisfactorily with results deduced by other methods by Dyson and Wanach. Expressions for the annual portion of the Kimura term itself are given by equations (1), (3), (5), and (6), (pp. 55 and 56), agreeing satisfactorily with Wanach's and Dyson's results, equations (2) and (4).

Harmonic constants for the semiannual term are given in Table 14 (p. 57). Attention is called to an increase in the amplitude of this term toward the end of the period treated. It is suggested that some of the semiannual effect may be part of the Kimura term and may be due to systematic errors in the declinations having a period of 12 hours of right ascension. The annual motion of the pole is shown in figures 14 and 15.

The motion of the pole in its "free" period of 432.5 days is deduced from the harmonic constants for this term at the several stations. There is no evidence from the observations of any effect of this period in the Kimura term and no a priori reason for an effect of this sort large enough to be considered. Harmonic constants for the motion of the pole in the free period are given in Table 16 (p. 60). The results of Dyson and Wanach deduced by other methods are given for comparison; they agree satisfactorily as far as they go with the results in Table 16.

The chief point of interest in Table 16 is the direction of the major axis of the ellipse of polar motion. The results vary considerably and are affected by large probable errors, but a direction not far from the meridian of 90° west of Greenwich is pretty clearly indicated. This longitude is the longitude of the axis of the larger principal equatorial moment of inertia, if there is any perceptible difference in the principal equatorial moments. The larger principal moment corresponds to the shorter axis of the geoid considered as an ellipsoid of three unequal axes. The eccentricity of the ellipse of polar motion affords a means for deriving the difference between the two equatorial moments of inertia or between the corresponding equatorial radii of the geoid. The eccentricity of Table 16 vary considerably but seem to indicate, on the whole, a difference between the two equatorial radii of the order of magnitude of 200 or 300 meters.[1]

[1] These results are subject to a correction not yet precisely evaluated but relatively rather small for the mobility of the ocean waters. (See pp. 99 and 101.) For this reason the results are stated only tentatively and in round numbers.

These results (radius of the equatorial ellipse in longitude 90° W., 200 or 300 meters shorter than the radius in longitude 0° or 180°) agree roughly with the results of Helmert from gravity observations (shortest radius at 107° W., difference in radii of 230 meters). The determination of these quantities from the latitude observations is necessarily rough, and an accurate result would require a long series of observations, but even the rough result is of value as lending plausibility to Helmert's conclusions because the latter appear to conflict with our ideas about conditions in the interior of the earth and to be perhaps open to suspicion on account of systematic topographic or geologic effects in the gravity observations.

The observations were examined with a view of detecting possible terms in the motion of the pole having periods of 3 or 6 years, there being some a priori reasons for expecting terms of this sort. The results are obtained from three stations over an 18-year interval. They are compared with results obtained by a different method by Wanach from a 12-year interval but based on all 6 stations. The two results of the two discussions are compared on page 65. Some little resemblance between the two has survived the difference in the stations used, in the method, and in the period covered. This resemblance, combined with a priori considerations, appears to be just about sufficient to render fairly probable the existence of terms of this sort, but the values of their amplitudes and epochs are far from certain.

The motion of the undisturbed pole of figure as deduced from the annual motion of the pole of rotation is given on pages 105 and 106, figures 14 and 15. The amplitude of the motion of the undisturbed pole is much smaller than that of the pole of rotation, and its motion less accurately determined.

It appears to the writer, as a result of this investigation, that it would be desirable to repeat some of the calculations when the definitive latitudes become available for the years after 1912.000.

It also appears unfortunate that the number of latitude stations should have been reduced to three and that one of these should be in a region where the ground is supposed to be unstable. Three stations are just enough to determine the harmonic constants of the annual motion of the pole with the annual part of the Kimura term included, and additional stations to serve as a check are highly desirable.

Three stations are just enough also to determine a progressive shift of the pole combined with a constant correction to the average proper motion in declination (the w of Chap. II, p. 33). Again, additional stations to serve as a check are imperative if the results are to be considered as really certain.

The complications introduced by the declinations in delicate researches on the motion of the pole are such that it is desirable to be independent of declinations if possible, even at the expense of the inaccuracy of daylight observations. It, therefore, appears desirable to determine latitude from observations on circumpolar stars at both culminations. The observations should be made systematically and as frequently as possible, and the results for each star at each observatory should be discussed separately. In this way it is not necessary to have a chain of latitude observatories all on the same parallel; any well-equipped and well-staffed observatory outside of the Tropical Zone would be able to contribute results of value in the study of the variation of latitude.

Chapter VI.—MISCELLANEOUS MATHEMATICAL DEVELOPMENTS.

This chapter contains various mathematical developments regarding the details of the calculations previously referred to. The mathematical notation of each section of the chapter is independent of that of the other sections.

SECTION 1. SUMS OF CERTAIN TRIGONOMETRIC SERIES—CLEARANCE FORMULAS.

The following formulas for the summation of a series of sines or cosines of angles in arithmetical progression are continually needed in the developments of the harmonic analysis. The proofs will be found in almost any work on analytical trigonometry.

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\sin(\alpha+3\beta)\cdots\cdot+\sin[\alpha+(n-1)\beta]$$
$$=\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}. \tag{1}$$

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cos(\alpha+3\beta)\cdots\cdot+\cos[\alpha+(n-1)\beta]$$
$$=\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}. \tag{2}$$

When $n\beta$ equals 2π or a multiple of 2π the sum of the terms of the left-hand members of (1) and (2) is evidently zero.

Let us multiply equation (1) by $\frac{1}{2}$, then in (1) replace α by $\alpha+\beta$, multiply the equation after the replacement by $\frac{1}{2}$ and add to the first result. Every term in each series has a term in the other exactly like it except the first term of the first series and the last one of the second series; by combining like terms we find

$$\begin{aligned}&\tfrac{1}{2}\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\sin(\alpha+3\beta)\cdots\cdot+\sin[\alpha+(n-1)\beta]\\&\quad+\tfrac{1}{2}\sin(\alpha+n\beta)\\&\quad=\tfrac{1}{2}\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}\\&\quad+\tfrac{1}{2}\sin\left(\alpha+\frac{n+1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2},\end{aligned}$$

or by obvious transformations of the right-hand side

$$\begin{aligned}&\tfrac{1}{2}\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)\cdots\cdot+\sin[\alpha+(n-1)\beta]\\&\quad+\tfrac{1}{2}\sin(\alpha+n\beta)\\&\quad=\sin\left(\alpha+\frac{n}{2}\beta\right)\sin\frac{n\beta}{2}\cot\frac{\beta}{2}.\end{aligned} \tag{3}$$

In a similar way from (2)

$$\begin{aligned}&\tfrac{1}{2}\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)\ldots\ldots+\cos[\alpha+(n-1)\beta]\\&\quad+\tfrac{1}{2}\cos(\alpha+n\beta)\\&\quad=\cos\left(\alpha+\frac{n}{2}\beta\right)\sin\frac{n\beta}{2}\cot\frac{\beta}{2}.\end{aligned} \tag{4}$$

Formulas (3) and (4) apply to the case frequent in practice where a set of ordinates is to be summed with only half weight assigned to the ordinates at the beginning and end.

On page 5 formula (4) is applied to an expression of the form $A \cos(\kappa t-\alpha)$. Fifteen ordinates are used, or $n=14$. It is assumed that $\kappa t=2\pi$ when $t=432.5$ days, this being taken as the length of the free period of the latitude variation. Then dividing this period into 14 parts, corresponding to $t=0$, $\frac{432.5}{14}$, $2\times\frac{432.5}{14}$, etc., days, we find

$$\tfrac{1}{2} A \cos(0-\alpha)+A \cos\left(\frac{2\pi}{14}-\alpha\right)+A \cos\left(2\times\frac{2\pi}{14}-\alpha\right)\cdots\cdot$$
$$+A\cos\left(13\times\frac{2\pi}{14}-\alpha\right)+\tfrac{1}{2} A\cos\left(14\times\frac{2\pi}{14}-\alpha\right)=0.$$

Again we use the same value of κ, but instead of extending the ordinates over 432.5 days we extend them over exactly 14 months or rather 14/12 of a Julian year, that is, 426.125 days. Then putting for brevity

$$\gamma=\frac{426.125}{432.5}\times\frac{2\pi}{14}=0.140751\pi=25^\circ\!.3353$$

we find

$$\tfrac{1}{2} A\cos(0-\alpha)+A\cos(\gamma-\alpha)+A\cos(2\gamma-\alpha)\cdot\cdot\cdot\cdot+A\cos(13\gamma-\alpha)$$
$$+\tfrac{1}{2} A\cos(14\gamma-\alpha)=A\cos(7\gamma-\alpha)\sin 7\gamma\cot\frac{\gamma}{2}.$$

The numerical value of $\sin 7\gamma\cot\frac{\gamma}{2}$ comes out 0.206; therefore, since $\cos(7\gamma-\alpha)$ is never numerically greater than unity, the mean of the fifteen ordinates (end ordinates half weight) cannot exceed $\frac{0.206A}{14}$ $=0.015\,A$, as stated on page 5.

The formulas on pages 14 and 17 for clearing the mean latitude over a given period from the effect of one component are easily derived from formula (4). Let us consider first the formula on page 14.

Let us represent the 14-month component whose effect is to be removed from the mean over a year by $R\cos(\kappa t-\zeta')$ where ζ' is the ζ when the time is reckoned from the beginning of the year. If the time is reckoned in months $\kappa=25^\circ\!.33526$, and t takes on the values 0, 1, 2, 3, . . . 11, 12. We have therefore in (4),

$$\alpha=-\zeta',\ \beta=25^\circ\!.33526,\ n=12,$$

and the mean of the 13 terms (first and last half weight) is

$$\tfrac{1}{12} R\cos(6\kappa-\zeta')\sin 6\kappa\cot\tfrac{1}{2}\kappa. \tag{5}$$

The product $\frac{1}{12}\sin 6\kappa\cot\frac{1}{2}\kappa$ is independent of the harmonic constants of the 14-month component; its value is 0.174, of which the logarithm is 9.2405 − 10. To connect the ζ' with the ζ reduced to 1900.00, the latter being indicated by the absence of a prime, we have for the change of phase in one year $12\kappa=304^\circ\!.023$. Therefore the phase falls behind in one year by $360^\circ-304^\circ\!.023=55^\circ\!.977$, which is the amount by which the ζ reckoned from the beginning of any year exceeds the ζ reckoned from the beginning of the previous year, or

$$\zeta'=\zeta+55^\circ\!.977\,n, \tag{6}$$

where n is the number of years elapsed since 1900.00. Since $6\,\kappa = 152^\circ.0$, expression (5) becomes $0.174\,R\cos(152^\circ.0-\zeta-55^\circ.977n)$. This expression is contained in the mean as taken; to clear the mean it must be subtracted, or the angle may be increased by 180° and the resulting expression added. This process gives for the quantity to be added algebraically to clear the mean over a year from the effect of the 14-month component the expression:

$$+0.174\,R\cos\,(332^\circ.0-\zeta-55^\circ.977\,n). \tag{7}$$

A similar process is used for deriving the expression to clear the mean over a 14-month (432.5 day) period from the effects of the annual component. The result, when 13 values enter into the mean, with half weight to the first and last, is

$$+0.142\,R\cos\,(213^\circ.1-\zeta+360^\circ f). \tag{8}$$

In (8), R and ζ are the amplitude and epoch of the annual component; f is the interval expressed as a fraction of a year from the beginning of the year to the beginning of the series; that is, to the time of the first value of the set of which the mean is taken.

SECTION 2. HARMONIC ANALYSIS—ELIMINATION OF THE EFFECT OF ONE COMPONENT ON ANOTHER

It may be well at this point to call attention to the two-fold use of the word "component." The context will show which use is meant. There is the ordinary use according to which we speak of the component of any vector quantity in a given direction and there is another use, very frequent in this report, borrowed from the theory of the harmonic analysis of the tides; in this sense a component is simply one of the periodic terms into which the mathematical expression for a periodic phenomenon may be separated. One component is distinguished from another by its period, and a component of given period may comprise all parts of the mathematical expression having that period or its submultiples but not multiples of the period.[1] In practice, however, these periodic terms or components are limited to harmonic terms, and, therefore, are not separable into terms of shorter period.

In this report, in conformity with the practice adopted in the harmonic analysis of tides, these components are assumed to be of the general form $R\cos\,(at-\zeta)$. The quantities R, a, and ζ, are constants; t is the time. R is called the amplitude of the component, and is always taken positive; its physical dimensions are the same as those of the quantity represented. The quantity a is called the speed and is related to the period P of the component by the equation $aP=2\pi$ or 360° according as radians or degrees are used in the computation. The speed of one component may be a simple multiple of another, or the two speeds may be practically incommensurable. The quantity ζ is called the epoch and it evidently depends on the origin from which the time t is reckoned. ζ is the interval from the origin of time to the first maximum of the term $R\cos\,(at-\zeta)$, the interval being of course in degrees or radians of which there are respectively 360° or 2π to a

[1] For example, an annual component has a term with a period of 1 year and may have terms, if the latter are not especially excluded, with periods of 6 months, 4 months, 3 months, etc. A term with a period of two or three years would not be a part of the annual component.

period. The quantity $360° - \zeta$ represents the phase at the origin of time of the periodic variation represented.

The quantities R and ζ are called the harmonic constants of the particular component in question. In this report all ζ's as finally given are reduced to the beginning of the year 1900 as origin of time, regardless of whether the observations from which they are deduced include the year 1900 or not; the application of this reduction to 1900.00 assumes of course that the phenomenon represented repeats itself indefinitely. The expression $R\cos(at-\zeta)$ may be expanded into $R\cos\zeta\cos at + R\sin\zeta\sin at = c\cos at + s\sin at$, where $c = R\cos\zeta$ and $s = R\sin\zeta$. The quantities c and s are also sometimes called harmonic constants.

The relations among these four quantities R, ζ, c, and s, may be represented graphically. Take rectangular axes OC and OS (fig. 11). If c and s be taken as the rectangular coordinates of a point P, then R and ζ will be its polar coordinates, the pole being O and the initial

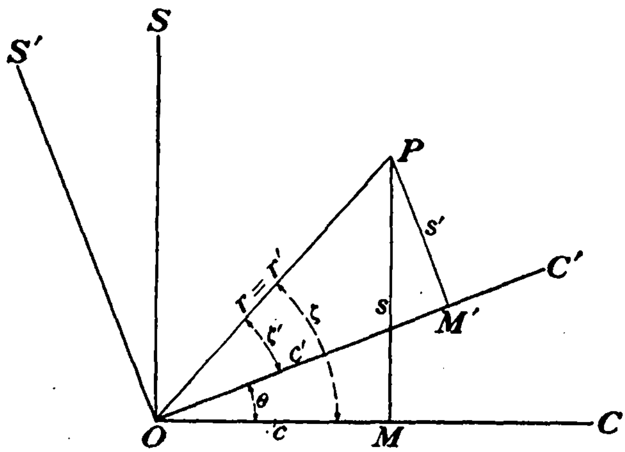

FIG. 11.—Relation among various sets of harmonic constants.
The r's in the figure represent the R of the text.

line OC. A change in the origin time changes the ζ by an amount proportional to the interval between the old and new origins, but leaves the R unchanged. If the quantities reckoned from a new origin of time be given primed letters to distinguish the corresponding quantities reckoned from the old origin and written without primes, we have for θ the difference between the s's,

$$\theta = \zeta - \zeta' = a(t-t') = a\tau, \tag{1}$$

We have also,

$$\begin{aligned} c' = R\cos\zeta' &= R\cos(\zeta-\theta) \\ &= R\cos\zeta\cos\theta + R\sin\zeta\sin\theta \\ &= c\cos\theta + s\sin\theta. \end{aligned} \tag{2}$$

Similarly,

$$s' = -c\sin\theta + s\cos\theta. \tag{3}$$

Other writers sometimes assume as standard forms for the harmonic terms, $R\cos(at+\eta)$, $R\sin(at-\eta)$, $R\sin(at+\eta)$. The R's are of course the same in all four forms. Wanach uses the last-named form in the Resultate des Internationalen Breitendienstes, volume V. The relation between the ζ used in this report and Wanach's η, is

$$\zeta+\eta=90° \text{ or } 450°. \tag{4}$$

Let us now suppose for simplicity that the effect to be analyzed has two components only; that is, we are given a number of values y_0, y_1, y_2, y_3, etc., of the expression

$$\left.\begin{aligned} y&=A \cos (mt-\zeta_1)+B \cos (nt-\zeta_2) \\ &=c_m \cos mt+s_m \sin mt+c_n \cos nt+s_n \sin nt. \end{aligned}\right\} \tag{5}$$

The speeds m and n are known. The object is to determine the values of the amplitudes A and B and the epoch's ζ_1, and ζ_2, or else the c's and s's, from which the former set of quantities may be deduced. In theory four values of y would suffice to determine the four quantities required, but in practical applications of the harmonic analysis the y's are so affected by accidental errors that a determination from four values of y would be of little use, particularly if the corresponding times were near together. In practice either the values of y are given at uniform intervals of time or else a curve is available from which may be read the value of y at any time. The latter case is the one treated here.

To find c_m and s_m (or A and ζ_1) it is convenient to take a uniform interval between the values of y equal to some submultiple of the period of the first term. For definiteness we shall carry the reasoning through with the number most frequently used for the purpose in this report, namely, 12; that is, we take curve readings at intervals of one-twelfth of the period of the first term. For this interval

$$mt=360°/12=30°.$$

We denote these readings in order by y_0, y_1, y_2, etc., and arrange them in 12 groups as follows:

Group number.	Value of mt.	Readings.	Mean value.	
0	$0°+j(360°)$	$y_0, y_{12}, y_{24}, y_{36}, \dots$	Y_0	(6)
1	$30°+j(360°)$	$y_1, y_{13}, y_{25}, y_{37}, \dots$	Y_1	
2	$60°+j(360°)$	$y_2, y_{14}, y_{26}, y_{38}, \dots$	Y_2	
3	$90°+j(360°)$	$y_3, y_{15}, y_{27}, y_{39}, \dots$	Y_3	
...	...	...	...	
11	$330°+j(360°)$	$y_{11}, y_{23}, y_{35}, y_{47}, \dots$	Y_{11}	

j is an integer equal to zero for the y's in the first subcolumn in the column of readings, to unity for those in the next, to two for the next, and so on. It is easy to see that the values of y in each group, in so far as they depend only on the first term $A \cos (mt-\zeta)$, are all alike because the values of $(mt-\zeta)$ differ by 360° and the term repeats itself. Although within a group the first term repeats, the second term does not do so; but if we suppose the group to cover a long period of time, the second term will be now positive, now negative, and its mean value will be nearly zero. How nearly zero will be considered presently; for the present we simply neglect the mean value of the second term in the several groups, and suppose the mean value of the second term in the several groups to be what any one of its members would be if the first term were the only one present. Using the second form of (5) and considering for simplicity the mean value of a group to be that of its first member we get,

$$\left.\begin{aligned} Y_0 &= c_m \cos 0^\circ + s_m \sin 0^\circ, \\ Y_1 &= c_m \cos 30^\circ + s_m \sin 30^\circ, \\ Y_2 &= c_m \cos 60^\circ + s_m \sin 60^\circ, \\ &\cdots\cdots\cdots\cdots\cdots \\ Y_{11} &= c_m \cos 330^\circ + s_m \sin 330^\circ. \end{aligned}\right\} \quad (7)$$

These are 12 equations for only two unknowns c_m and s_m; we treat them as observation equations in the method of least squares. The normal equation for c_m is evidently

$$[\cos^2 (k\ 30^\circ)]\ c_m + [\cos (k\ 30^\circ) \sin (k\ 30^\circ)]\ s_m = [y_k\ (\cos k\ 30^\circ)]. \quad (8)$$

Square brackets indicate summation with respect to k, which takes on integral values from 0 to 11, inclusive. By writing $\cos^2 (k\ 30^\circ) = \frac{1}{2} + \frac{1}{2} \cos (k\ 60^\circ)$ and $\sin (k\ 30^\circ) \cos (k\ 30^\circ) = \frac{1}{2} \sin (k\ 60^\circ)$ and applying formula (2) on page 71 we see that the coefficient of c_m is simply 6 and the coefficient of s_m is zero. We have then

$$6\ c_m = [Y_k \cos (k\ 30^\circ)]. \quad (9)$$

In a similar fashion the normal equation for s_m simplifies into

$$6\ s_m = [Y_k \sin (k\ 30^\circ)]. \quad (10)$$

where k as before is successively 0, 1, 2, ... 11.

It may be convenient to suppose a constant term h added to the expression (5) for y. It will evidently affect all values of Y_k alike in (6) and have no effect on the correctness of formulas (9) and (10), since $[h \cos (k\ 30^\circ)]$ and $[h \sin (k\ 30^\circ)]$ are clearly zero. Formulas (9) and (10) are the essence of the harmonic analysis and the forms for the analysis are designed simply to furnish convenient forms for computing the indicated summations in (9) and (10) and in similar equations for terms whose periods are submultiples of those of the first term and for deducing the R's and ζ's from the s's and c's.

In taking up the effect of the omitted second term it is convenient to make a slight change in the form of equations (9) and (10). Consider the different values of the y's in Y_0. According to formula (9) they are all multiplied by $\cos 0^\circ$; it is more convenient, however, to consider y_0 as being multiplied by $\cos 0^\circ$, y_{12} by $\cos 360^\circ$, y_{24} by $\cos 2 \times 360^\circ$, etc. Similarly for the multipliers of the y's in Y_1 we may use successively $\cos 30^\circ$, $\cos (30^\circ + 360^\circ)$, $\cos (30^\circ + 2 \times 360^\circ)$ etc. Let i represent the subscript of any y in the array (6), and suppose the curve-readings used cover p periods, so that the greatest value of i is $12\ p - 1$. Then it is easy to see that (9) and (10) may be written

$$\left.\begin{aligned} 6\ c_m &= \frac{1}{p}\ [y_i \cos\ (i \times 30^\circ)], \\ 6\ s_m &= \frac{1}{p}\ [y_i \sin\ (i \times 30^\circ)]. \end{aligned}\right\} \quad (11)$$

The values of i in the summations range from 0 to $12\ p - 1$. Equations (11) do not give the true value of c_m and s_m but the values of certain approximations to them; let us denote these approximations as computed from (11) by c_m' and s_m' and reserve the symbols without the prime for the true values. From the second form of (5), by writing x for $\frac{n}{m}$ 30°

$$\begin{aligned}y_i &= c_m \cos (i\ 30°) + s_m \sin (i\ 30°) + c_n \cos (ix) \cos (i\ 30°) \\ &\quad + s_n \sin (ix), \\ y_i \cos (i\ 30°) &= c_m \cos^2 (i\ 30°) + s_m \sin (i\ 30°) \cos (i\ 30°) \\ &\quad + c_n \cos (ix) \cos (i\ 30°) + s_n \sin (ix) \cos (i\ 30°) \qquad (12) \\ &= \tfrac{1}{2} c_m + \tfrac{1}{2} c_m \cos (i\ 60°) + \tfrac{1}{2} s_m \sin (i\ 60°) + \tfrac{1}{2} c_n \cos i\ (30° + x) \\ &\quad + \tfrac{1}{2} c_n \cos i\ (30° - x) + \tfrac{1}{2} s_n \sin i\ (30° + x) - \tfrac{1}{2} s_n \sin i\ (30° - x). \qquad (13)\end{aligned}$$

By performing the summations indicated in (11) using formulas (1) and (2), page 71, we see that the terms in sin (i 60°) and cos (i 60°) give a zero result in the sum and that we get

$$\begin{aligned}6\ c_m' = 6\ c_m + \frac{1}{2p}\Bigg\{ & c_n \cos \Big[(6\ p-1)\ (30°+x)\Big] \frac{\sin\ [6\ p\ (30°+x)]}{\sin \frac{1}{2}\ (30°+x)} \\ & + c_n \cos \Big[(6\ p-1)\ (30°-x)\Big] \frac{\sin\ [6\ p\ (30°-x)]}{\sin \frac{1}{2}\ (30°-x)} \\ & + s_n \sin \Big[(6\ p-1)\ (30°+x)\Big] \frac{\sin\ [6\ p\ (30°+x)]}{\sin \frac{1}{2}\ (30°+x)} \\ & - s_n \sin \Big[(6\ p-1)\ (30°-x)\Big] \frac{\sin\ [6\ p\ (30°-x)]}{\sin \frac{1}{2}\ (30°-x)} \Bigg\}. \qquad (14)\end{aligned}$$

For brevity let us call the angles $(6p-1)(30°+x)$ and $(6p-1)\ (30°-x)$, θ and ϕ, respectively, and the factors

$$\frac{\sin [6p(30°+x)]}{\sin \frac{1}{2}\ (30°+x)} \text{ and } \frac{\sin [6p(30°-x)]}{\sin \frac{1}{2}\ (30°-x)},\ F \text{ and } f, \text{ respectively.}$$

Then (14) may be written

$$6\ c_m' = 6\ c_m + \frac{1}{2p}\Big[c_n\ (F \cos \theta + f \cos \varphi)\ + s_n\ (F \sin \theta - f \sin \varphi)\Big]. \qquad (15)$$

A similar calculation gives

$$6\ s_m' = 6\ s_m + \frac{1}{2p}\Big[c_n\ (F \sin \theta + f \sin \varphi) + s_n\ (f \cos \varphi - F \cos \theta)\Big]. \qquad (16)$$

The quantities within the brackets in (15) and (16) are always finite and as p increases we have more and more nearly

$$\left.\begin{aligned}6\ c_m' &= 6\ c_m, \\ \text{and} \qquad 6\ s_m' &= 6\ s_m,\end{aligned}\right\} \qquad (17)$$

as was assumed without strict proof at the beginning of this section. Even if p is small, equations (17) may still give a good approximation. If the time covered by the $12p$ readings, which by hypothesis correspond just p periods of the first term of (5), also corresponds approximately to an integral number of periods of the second term then the angles ϕ and θ are nearly 0° or 180° and their sines are nearly zero. The quantity F is in the cases that arise in practice much smaller than f, the factor sin $\frac{1}{2}$ (30° + x) in the denominator being larger than sin $\frac{1}{2}$ (30° − x). In the tidal work the series are sufficiently long so that terms in F are dropped entirely. With $\sin\phi=0$ and terms in F dropped the approximation (17) still holds good.

Even when these conditions do not hold, equations (15) and (16) are still useful. Let us evaluate known portions for a case needed in the variation of latitude, namely, when the first term of (5) represents the annual component and the second term the 14-month component and when $p=7$. The value of x is 25°.33526. We find

$$\theta=136^\circ.41,$$
$$\phi=166^\circ.41,$$
$$F=0.591,$$
$$f=6.740.$$

If the 14-month component had a period of precisely 14 months (14/12 years) instead of 432.5 days, the 7 years would represent exactly 6 periods of 14 months, the value of ϕ would have been 180° and terms in $\sin\phi$, instead of being merely small, would have been zero. With these values using the subscript a instead of m for the annual component and the subscript b instead of n for the 14-month component, we get from (15) and (16)

$$\left.\begin{aligned} 6c_a+0.4374\ c_b-0.0840\ s_b&=6c_a', \\ 6s_a+0.1422\ c_b+0.4985\ s_b&=6s_a'. \end{aligned}\right\}\quad(18)$$

The right-hand sides of (18) represent the results of the ordinary uncorrected means of the groups shown in array (6).

Let c_m and s_m now refer to the 14-month component and c_n and s_n to the annual, and let us suppose curve-readings taken at intervals of one-twelfth of the 14-month period, the first reading, y_o, coinciding with the first reading for the annual component. If we make $p=6$, we are covering an interval of 6×432.5 days, or a little over 7 years. We find in this way, remembering that we must use b for m and a for n in the subscripts of (15) and (16)

$$\left.\begin{aligned} 6c_b+0.5852\ c_a+0.1429\ s_a&=6c_b', \\ 6s_b-0.1668\ c_a+0.4886\ s_a&=6s_b'. \end{aligned}\right\}\quad(19)$$

The right-hand sides of (19) are known from the uncorrected results of the usual process of forming groups and means and analyzing them for the 14-month component. Equations (18) and (19) give four linear equations for deducing the true values of c_a, c_b, s_a, and s_b from their uncorrected values c_a', c_b', s_a', and s_b'. The solution of them gives

$$\left.\begin{aligned} c_a&=+0.1679(6c_a')+0.0001(6s_a')-0.0122(6c_b')+0.0023(6s_b'), \\ s_a&=+0.0000(6c_a')+0.1679(6s_a')-0.0040(6c_b')-0.0139(6s_b'), \\ c_b&=-0.0164(6c_a')-0.0040(6s_a')+0.1680(6c_b')+0.0001(6s_b'), \\ s_b&=+0.0047(6c_a')-0.0137(6s_a')-0.0000(6c_b')+0.1679(6s_b'). \end{aligned}\right\}\quad(20)$$

These are equations used in clearing one component from the effect of another for a so-called 7-year series, which means that between the first and the last reading for the annual component there are 6 years, 11 months (6.917 years) and between the first and the last reading for the 14-month component 7.005 years. The results are written in terms of $6c_a'$, $6c_b'$, etc., rather than in terms of plain c_a', c_b', because the former quantities appear on the forms for harmonic analysis, and the form adopted saves a division by 6.

The approximate equality obtained in this way between the periods of time covered in the two sets of readings is desirable in view of the fact that the so-called harmonic constants are, in fact, somewhat variable and that the values obtained for them should be considered merely as mean values for the periods in question. An equality holding to about the same degree of approximation as the one just considered is 6 years = five 14-month periods. The interval between the first and last curve readings is 5 years 11 months (5.917 years) for the annual component and $4\frac{11}{12}\times 432.5$ days = 5.821 years for the 14-month component. This is the so-called 6-year series. The equations for deducing the corrected values of c_a, c_b, etc., from their uncorrected values are found for the 6-year series to be

$$\left.\begin{aligned} c_a &= +0.1676(6c_a') - 0.0001(6s_a') + 0.0106(6c_b') + 0.0032(6s_b'), \\ s_a &= +0.0000(6c_a') + 0.1677(6s_a') - 0.0027(6c_b') + 0.0126(6s_b'), \\ c_b &= +0.0146(6c_a') - 0.0051(6s_a') + 0.1677(6c_b') - 0.0001(6s_b'), \\ s_b &= +0.0033(6c_a') + 0.0128(6s_a') + 0.0000(6c_b') + 0.1677(6s_b'). \end{aligned}\right\} \quad (21)$$

Equations (20) and (21) are the formulas used in passing from columns 3 and 4 of Table 3, page 10, to columns 5 and 6.

In ordinary tidal work with short-period components the process of eliminating effects of components, other than the one analyzed, corresponds approximately to setting $F=0$ and using on the left-hand side of (18) the uncorrected values of c_b and s_b instead of the true values c_b and s_b. This process is not sufficiently accurate in the discussion of a year's observations on the long-period components and a process is used due to Darwin [2] that resembles the one here described down to the derivation of equations (18) and (19). The equations, however, contain 10 unknowns, the s's and c's of the 5 long-period tides. It is suggested that these equations be solved by successive approximations, since one coefficient in each is much larger than the others. It would be entirely feasible to give a general solution of these equations in terms of the known quantities, which are analogous to $6c_a'$, $6s_a'$, $6c_b'$, $6s_b'$, as was done for equations (20). The amount of work in calculating such a general solution would be considerable, which is perhaps the reason why Darwin does not give such a solution. However, the amount of work in solving the equations by successive approximation is considerable also, and has to be repeated for each year of observation discussed, while with a general solution available the computation is relatively brief.

In the analysis to obtain possible 6-year, 3-year, and 2-year terms the 6-year period was divided into 24 parts instead of 12. The corresponding changes in formulas (9) and (10) are easily made.

In treating the 6-year component the mean of three readings a month apart was taken and the result used as a single reading. The necessary correction may be deduced as follows: For any component let at_1 be the value of at for the middle reading. Let readings be taken at intervals τ before and after the time t_1. Then the three values of the term $R \cos (at-\zeta)$ are:

$$R \cos [a(t_1-\tau)-\zeta],\ R \cos [at_1-\zeta],\ R \cos [a(t_1+\tau)-\zeta]$$

By combining the first and third the mean may be written

$$R \cos [at_1-\zeta]\left[\frac{1+2 \cos a\tau}{3}\right],$$

[2] The Harmonic Analysis of Tidal Observations, sec. 10, Scientific Papers, Vol. I, of British Association for the Advancement of Science, report for 1883.

which differs from the true quantity desired, namely, the middle reading, by the factor $\frac{1+2\cos a\tau}{3}$, which for small values of τ is nearly unity. This factor reduces all the values of the y's that enter into the harmonic analysis in the same ratio; the epoch ζ is therefore unaffected, but the amplitude R is reduced from its true value in the ratio $(1+2\cos a\tau):3$. The amplitude deduced from computations where a mean of three readings is treated as a single reading must be multiplied by $\frac{3}{1+2\cos a\tau}$. This quantity may be called an *augmenting factor*, which is the name applied in the harmonic analysis of tides where factors of this same general nature are used. With a 6-year component and a month's interval $a\tau=5°$. For a 3-year component $a\tau=10°$, and for a 2-year component $a\tau=15°$. The common logarithms of the augmenting factors for the 6-year, 3-year, and 2-year components are, respectively, 0.0011, 0.0044, and 0.0100.

SECTION 3. THE CORRECTION FOR "SLOPE" IN THE HARMONIC ANALYSIS.

The expression for a function by a Fourier series takes the form

$$\left.\begin{aligned} f(x) = \tfrac{1}{2}c_0 + c_1 \cos x + c_2 \cos 2x \cdots \\ + s_1 \sin x + s_2 \sin 2x \cdots \end{aligned}\right\} \quad (1)$$

where $$c_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,$$

and $$s_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx. \quad (2)$$

No provision is made in (1) for a term kx proportional to x. Such a term, if present, would mean that the sine curves of the various harmonics instead of being laid off upward or downward from the level base line $y=\frac{1}{2}c_0$, according to the usual graphic representation, would be laid off from an inclined base line having the slope k. In discussing the variation of latitude there was found, besides the expected periodic terms, an apparent progressive change of latitude, roughly proportional to the time and represented graphically by a line inclined to the axis of abscissas (the time-axis); that is, there was found something analagous to the term kx, which we have supposed to be introduced into (1).

A great variety of functions in no way periodic can be represented by a Fourier series within the range from $x=0$ to $x=2\pi$, but if we suppose the function represented to be in fact essentially periodic except for a term kx and if we apply the ordinary formulas (2) to determine the coefficients in the expansion of such a function, neglecting the presence of the term kx, then the coefficients of the periodic terms will be falsified by this neglect. These coefficients, in addition to their proper values, will contain quantities which may be found for any harmonic term of order n by putting kx for $f(x)$ in (2). Denoting by γ_n and σ_n the additional quantities introduced into c_n and s_n we have

$$\left.\begin{aligned} \gamma_n &= \frac{k}{\pi}\int_0^{2\pi} x\cos nx\,dx = 0, \\ \sigma_n &= \frac{k}{\pi}\int_0^{2\pi} x\sin nx\,dx = -\frac{2k}{n}. \end{aligned}\right\} \quad (3)$$

These are the quantities already in the coefficients as determined by the ordinary process; therefore, by reversing the signs we have as corrections to c_n and s_n the quantities $+0$ and $+\frac{2k}{n}$, respectively.

If $f(x)$ is given only by its numerical values at uniform intervals from 0 to 2π, the coefficients are then deduced by the methods of the harmonic analysis, the integrations in (2) being replaced by finite summations. It can be shown without much difficulty by the methods of the calculus of Finite Differences, or simply by using the numerical values of kx in their appropriate places in any of the usual forms for harmonic analysis and by working the form through, that when the period is divided into 12 parts we have in place of (3)

$$\left.\begin{aligned}\gamma_n &= -\frac{\pi k}{6},\\ \sigma_n &= -\frac{\pi k \cot(15^\circ \times n)}{6}.\end{aligned}\right\}\quad(4)$$

For a period divided into 24 parts

$$\left.\begin{aligned}\gamma_n &= -\frac{\pi k}{12},\\ \sigma_n &= -\frac{\pi k \cot(7\tfrac{1}{2}^\circ \times n)}{12}.\end{aligned}\right\}\quad(5)$$

It is easily seen that for small values of n the value of σ_n from (4) or (5) is not very different from that given by (3). The difference in the corresponding values of γ is somewhat larger. It is explained by the well-known fact that at points of discontinuity of $f(x)$, say for $x=x_1$, the Fourier series obtained from (2) gives the mean of the two values of $f(x_1)$ or more precisely gives

$$\lim_{\epsilon=0} \tfrac{1}{2}\Big[f(x_1-\epsilon)+f(x_1+\epsilon)\Big].$$

Such discontinuities really occur at $x=0$ and $x=2\pi$. Evidently, as far as the series (1) goes, $f(0)=f(2\pi)$, but the value of kx is zero in one case and 2π in the other. The series (1) therefore gives $\frac{1}{2}(0+2\pi)k=\pi k$ in both cases. The values of γ_n in (4) and (5) were obtained with the first ordinate, or $f(0)$, taken equal to zero; if instead we take the first ordinate as πk and work through the form for harmonic analysis, we get $\gamma_n=0$, whatever be the number of parts into which the period is divided. This is merely to explain the difference between the values of γ_n as obtained from integration in (2) and from summations in (4) or (5). In the actual practice of correcting the results of harmonic analysis for a constant slope of the base line the quantity γ_n would have a value as given by (4) or (5), or whatever the value might be for the number of ordinates used, and would not be zero.

It is convenient to use instead of the slope k the difference between the ordinates of the inclined base at the beginning and the end of a period; denote this quantity by m. Then we have $m=2\pi k$ and from (4) and (5) we get the following table of corrections. For comparison, the values from equation (3) are also given; the integration in (3) corresponds to an infinite number of ordinates.

Correction to Fourier coefficients for inclination of base line.

[The quantity m is to be taken positive if the base line rises as x increases, and negative if the base line falls.]

Correction to—	12 ordinates.	24 ordinates.	Infinite number of ordinates.	Correction to—	12 ordinates.	24 ordinates.	Infinite number of ordinates.
s_1	$+0.311m$	$+0.316m$	$+0.318m$	c_1	$+0.083m$	$+0.012m$	0
s_2	$+.144m$	$+.155m$	$+.159m$	c_2	$+.083m$	$+.012m$	0
s_3	$+.083m$	$+.101m$	$+.106m$	c_3	$+.083m$	$+.012m$	0
s_4	$+.018m$	$+.072m$	$+.080m$	c_4	$+.083m$	$+.012m$	0
s_5	$+.022m$	$+.054m$	$+.064m$	c_5	$+.083m$	$+.012m$	0
s_6	....	$+.042m$	$+.053m$	c_6	$+.083m$	$+.012m$	0

SECTION 4. FORMULAS FOR THE ELEMENTS DEFINING THE MOTION OF THE POLE.

Throughout this section, except in the three concluding paragraphs on page 93, it will be assumed that the variation of latitude is due solely to the motion of the pole of rotation and furthermore that this motion is expressible in terms of harmonic constants of the ordinary sort. How far these assumptions are justified has been briefly considered in Chapter IV.

The assumed harmonic motion of the North Pole is referred to coordinate axes passing through a mean position of the pole of rotation. In conformity with the practice of the International Latitude Service, the positive direction of the x axis is taken southward along the meridian of Greenwich and the positive direction of the y axis southward along the meridian of 90° west of Greenwich. The harmonic motion for any given period is specified by x and y, the components along the corresponding axes; these components are taken in the form

$$\left.\begin{aligned} x &= a \cos(\kappa t - \alpha), \\ y &= b \cos(\kappa t - \beta). \end{aligned}\right\} \quad (1)$$

The time, t, is reckoned from some convenient origin; the quantity κ is called the speed and is related to the period, P, of the harmonic oscillation by the relation

$$\kappa = \frac{360^\circ}{P},$$

or

$$\kappa = \frac{2\pi}{P},$$

according as angles are reckoned in degrees or radians. The angles α and β are the intervals from the assumed origin of time to the first maxima of the respective component oscillations. These intervals are reckoned in parts of the period, which is then considered as 360° or as 2π according as α and β are in degrees or radians.

The equations (1) considered as parametric equations in terms of t define the curve in which the pole moves only as far as oscillations of the period in question are concerned. As is well known, the curve is an ellipse oblique, in general, to the coordinate axes. The equation of this ellipse and formulas for the directions of its principal axes and the times when the pole is at the end of such axes, are derived without difficulty by the ordinary processes of analytic geometry.

The formulas that present themselves in this way are not well adapted to logarithmic calculation and other formulas will be derived that seem rather more convenient in this respect.[3]

The distance of the pole from the origin at any time is $\Delta\phi = \sqrt{x^2+y^2}$ and the maximum numerical value of $\Delta\phi$ corresponds to a position at the ends of the major axis of the ellipse, the minimum numerical value to a position at the ends of the minor axis. From (1)

$$\Delta\phi^2 = x^2+y^2 = (a^2 \cos^2 \alpha + b \cos^2 \beta) \cos^2 \kappa t + (a^2 \sin^2 \alpha + b^2 \sin^2 \beta) \sin^2 \kappa t \\ + 2(a^2 \cos \alpha \sin \alpha + b^2 \cos \beta \sin \beta) \cos \kappa t \sin \kappa t.$$

By expressing functions of κt in terms of $2\kappa t$ with a corresponding transformation for α and β we get

$$\Delta\phi^2 = \tfrac{1}{2}(a^2+b^2) + \tfrac{1}{2}(a^2 \cos 2\alpha + b^2 \cos 2\beta) \cos 2\kappa t + \tfrac{1}{2}(a^2 \sin 2\alpha \\ + b^2 \sin 2\beta) \sin 2\kappa t.$$

This may be written in the form

$$\Delta\phi^2 = \tfrac{1}{2}(a^2+b^2) + \tfrac{1}{2}c^2 \cos 2\,(\kappa t - \mu) \qquad (2)$$

by assuming

$$\left.\begin{aligned} a^2 \cos 2\alpha + b^2 \cos 2\beta &= c^2 \cos 2\mu \\ a^2 \sin 2\alpha + b^2 \sin 2\beta &= c^2 \sin 2\mu. \end{aligned}\right\} \qquad (3)$$

When $\kappa t = \mu$ or $\kappa t = \mu + 180°$ the value of $\Delta\phi^2$ is evidently a maximum; when $\kappa t = \mu \pm 90°$ the value of $\Delta\phi^2$ is a minimum. Since μ is determined from 2μ, both μ and $\mu \pm 180°$ are included in the values of μ less than 360° that are given by (3).

To adapt (3) to logarithmic calculation let us form values of $c^2\cos[2\mu - (\alpha+\beta)]$ and $c^2\sin[2\mu - (\alpha+\beta)]$. By expanding cos $[2\mu - (\alpha+\beta)]$ and using (3) there results,

$$c^2 \cos [2\mu - (\alpha+\beta)] = a^2 \cos [2\alpha - (\alpha+\beta)] + b^2 \cos [2\beta - (\alpha+\beta)] \\ = (a^2+b^2) \cos (\alpha - \beta). \qquad (4)$$

In a similar manner

$$c^2 \sin [2\mu - (\alpha+\beta)] = (a^2 - b^2) \sin (\alpha - \beta). \qquad (5)$$

By dividing (5) by (4)

$$\tan [2\mu - (\alpha+\beta)] = \frac{a^2 - b^2}{a^2+b^2} \tan (\alpha - \beta). \qquad (6)$$

Assume the following equations for determining the auxiliary quantities d and γ:

$$\left.\begin{aligned} d \cos \gamma &= a, \\ d \sin \gamma &= b, \end{aligned}\right\} \qquad (7)$$

where γ may be taken in the first quadrant since a and b are positive. Then (6) may be written

$$\tan [2\mu - (\alpha+\beta)] = \cos 2\gamma \tan (\alpha - \beta). \qquad (8)$$

Equation (8) taken alone determines four possible values of μ less than 360°, not only the μ and the $\mu \pm 180°$ found from (3) but also $\mu \pm 90°$. To exclude the latter we note that by (4) $2\mu - (\alpha+\beta)$ must be so taken that its cosine has the same sign as cos $(\alpha - \beta)$.

[3] These formulas were proposed by the author as a problem in the American Mathematical Monthly for January, 1921, p. 30.

By this criterion we exclude the values of μ corresponding to $\mu \pm 90°$ in (3) and get only the values of μ corresponding to maximum values of $\Delta\phi^2$.

If we denote by θ the angle which the resultant displacement makes with the x axis we have

$$\Delta\phi \cos\theta = x,$$

$$\Delta\phi \sin\theta = y,$$

or

$$\left.\begin{aligned} \Delta\phi^2 \sin 2\theta &= 2xy, \\ \Delta\phi^2 \cos 2\theta &= x^2 - y^2. \end{aligned}\right\} \tag{9}$$

If in the expressions (1) for x and y we put $\kappa t = \mu$ and use the resulting values of x and y in (9), we get values of $\Delta\phi^2$ and 2θ corresponding to maximum values of $\Delta\phi$; that is, to the ends of the major axes. In this way we find

$$2xy = 2ab \cos(\kappa t - \alpha) \cos(\kappa t - \beta) = ab\{\cos[2\kappa t - (\alpha - \beta)] + \cos(\alpha - \beta)\},$$

or from (4) on putting $\kappa t = \mu$,

$$\Delta\phi^2 \sin 2\theta = \frac{ab}{c^2}(a^2 + b^2 + c^2)\cos(\alpha - \beta). \tag{10}$$

Again we have for the second equation of (9)

$$x^2 - y^2 = a^2\cos^2(\kappa t - \alpha) - b^2\cos^2(\kappa t - \beta) = \tfrac{1}{2}(a^2 - b^2) + \tfrac{1}{2}(a^2\cos 2\alpha - b^2\cos 2\beta)\cos 2\kappa t + \tfrac{1}{2}(a^2\sin 2\alpha - b^2\sin 2\beta)\sin 2\kappa t,$$

or by putting $\kappa t = \mu$ and using (4),

$$\begin{aligned} \Delta\phi^2 \cos 2\theta &= \frac{1}{2}(a^2 - b^2) + \frac{1}{2}\,\frac{a^4\cos^2 2\alpha - b^4\cos^2 2\beta}{c^2} \\ &\quad + \frac{1}{2}\,\frac{a^4\sin^2 2\alpha - b^4\sin^2 2\beta}{c^2} \\ &= \frac{1}{2}\left(\frac{a^2 - b^2}{c^2}\right)(a^2 + b^2 + c^2). \end{aligned} \tag{11}$$

From (10) and (11)

$$\tan 2\theta = \frac{2ab}{a^2 - b^2}\cos(\alpha - \beta). \tag{12}$$

The use of the auxiliary γ again simplifies the calculation and (12) may be written

$$\tan 2\theta = \tan 2\gamma \cos(\alpha - \beta). \tag{13}$$

It will be found that (13) by itself gives four values of θ less than 360°, two corresponding to the ends of the major axis and two to the minor axis. The criterion for distinguishing those for the major axis is found in (10). Evidently, since ab, c^2, $a^2 + b^2 + c^2$ and $\Delta\phi^2$ are all positive, for a maximum, 2θ must be taken so that $\sin 2\theta$ has the same sign as $\cos(\alpha - \beta)$.

The value of $\Delta\phi$ from (2) when $\kappa t = \mu$ gives the semi-major axis of the ellipse, or

$$A^2 = \Delta\phi^2_{\max} = \tfrac{1}{2}(a^2 + b^2 + c^2). \tag{14}$$

From (3) by squaring and adding

$$c^4 = a^4 + b^4 + 2a^2b^2 \cos 2(\alpha - \beta)$$
$$= (a^2 + b^2)^2 - 4a^2b^2 \sin^2 (\alpha - \beta)$$
$$= (a^2 + b^2)^2 \left[1 - \left(\frac{2\,ab}{a^2 + b^2} \sin (\alpha - \beta)\right)^2\right]. \tag{15}$$

By using again the auxiliary angle γ, (15) may be written

$$c^4 = (a^2 + b^2)^2 [1 - \sin^2 2\gamma \sin^2 (\alpha - \beta)]. \tag{16}$$

We may introduce the further auxiliary δ defined by

$$\sin \delta = |\sin 2\gamma \sin (\alpha - \beta)|.$$

Taking δ in the first quadrant (16) becomes

$$c^2 = d^2 \cos \delta, \tag{17}$$

since by (7) $d^2 = a^2 + b^2$.
Then (14) becomes

$$A = \Delta\phi_{\max} = d \cos \tfrac{1}{2}\delta. \tag{18}$$

For the semi-minor axis $B = \Delta\phi_{\min}$ it is easy to prove

$$B = d \sin \tfrac{1}{2}\delta. \tag{19}$$

The eccentricity of the ellipse (E) is found from

$$\frac{A^2 - B^2}{A^2} = E^2 = \frac{\cos \delta}{\cos^2 \frac{1}{2}\delta}. \tag{20}$$

The values of A, B, and θ may also be found by eliminating t from equations (1), which gives an equation of the second degree, which is easily shown to represent an ellipse and to which the standard methods may be applied for obtaining its dimensions and position; that is, the quantities A, B, and θ. It is interesting to note that the relations between 2γ, 2θ, $(\alpha - \beta)$, δ, and $2\mu - (\alpha + \beta)$ are the same as if these quantities formed the parts of a right spherical triangle, the hypothenuse being 2γ and the sides inclosing the right angle, 2θ and δ, the angles opposite these sides being, respectively, $90° - 2\mu + \alpha + \beta$ and $\alpha - \beta$. This fact may be used to establish additional relations among the above quantities.

The direction in which the pole rotates about the origin may also be deduced from the values of α and β.

If $180° < \alpha - \beta < 360°$, the rotation is positive for the axes used, i. e., from east to west. If $0 < \alpha - \beta < 180°$, the rotation is negative.

Recapitulating the formulas and criteria we have: Motion of pole is given by

$$\left.\begin{aligned} x &= a \cos (\kappa t - \alpha) \text{ for motion along } x \text{ axis,} \\ y &= b \cos (\kappa t - \beta) \text{ for motion along } y \text{ axis,} \end{aligned}\right\} \tag{I}$$

a and b always positive.
Compute the auxiliaries d and γ by

$$\left.\begin{aligned} d \cos \gamma &= a, \\ d \sin \gamma &= b, \end{aligned}\right\} \tag{II}$$

d being always positive.

The time of maximum displacement (position at end of major axis) occurs when $\kappa t = \mu$, μ being given by

$$\tan\,[2\mu - (\alpha + \beta)] = \cos 2\gamma \tan\,(\alpha - \beta). \qquad \text{(III)}$$

Take $2\mu - (\alpha + \beta)$ in such a quadrant that $\cos\,[2\mu - (\alpha + \beta)]$ has the same sign as $\cos\,(\alpha - \beta)$.

The angle which the direction of maximum displacement makes with x axis, or θ, is given by

$$\tan 2\theta = \tan 2\gamma \cos\,(\alpha - \beta). \qquad \text{(IV)}$$

Take 2θ in such a quadrant that $\sin 2\theta$ has the same sign as $\cos(\alpha - \beta)$.

Compute the auxiliary angle δ by

$$\sin \delta = |\, \sin 2\gamma \sin\,(\alpha - \beta) \,|\,. \qquad \text{(V)}$$

Take δ in the first quadrant regardless of the algebraic sign of the right-hand side of (V).

The semi-major and semi-minor axes A and B of the ellipse and its eccentricity E are given by

$$\left.\begin{aligned} A &= d \cos \tfrac{1}{2}\delta, \\ B &= d \sin \tfrac{1}{2}\delta, \\ E^2 &= \frac{\cos \delta}{\cos^2 \frac{1}{2}\delta}. \end{aligned}\right\} \text{(VI)}$$

The point whose motion is given by (I) describes the ellipse in a positive direction (from the positive direction of the x axis towards the positive direction of the y axis) if

$$180° < \alpha - \beta < 360°$$

or in a negative direction if

$$0 < \alpha - \beta < 180°.$$

There will be two values of μ and two of θ from (III) and (IV) each differing from the other member of the pair by 180°, even after the criteria have been applied. Let us take for definiteness that value of θ which lies on the positive side of the x axis and affect all quantities corresponding to it by a subscript zero, and the times and directions corresponding to the other maxima and the minima by the subscripts 1, 2, 3, according to the order of time in which they occur. Then the value of μ_0 corresponding to θ_0 is chosen by the test that $\cos\,(\mu_0 - \beta)$ is positive. We then have the following correspondences between the μ's and the θ's as they occur in order of time, according as the direction of rotation is positive or negative.

Positive rotation $180° < \alpha - \beta < 360°$		Negative rotation [a] $0 < \alpha - \beta < 180°$	
μ_0	θ_0 Max.	μ_0	θ_0 Max.
$\mu_1 = \mu_0 + 90°$	$\theta_1 = \theta_0 + 90°$ Min.	$\mu_1 = \mu_0 + 90°$	$\theta_1 = \theta_0 - 90°$ Min.
$\mu_2 = \mu_0 + 180°$	$\theta_2 = \theta_0 \pm 180°$ Max.	$\mu_2 = \mu_0 + 180°$	$\theta_2 = \theta_0 \pm 180°$ Max.
$\mu_3 = \mu_0 + 270°$	$\theta_3 = \theta_0 + 270°$ Min.	$\mu_3 = \mu_0 + 270°$	$\theta_3 = \theta_0 + 90°$ Min.

[a] The direction in which the pole of rotation actually moves about the origin is negative for the axes here used. The formulas were originally devised for computations with the harmonic constants of tidal currents and for these the rotation may have either direction.

The most convenient method of obtaining the elements of the ellipse from the harmonic constants of the latitude variation at the individual stations seems to be the indirect one of finding first the adjusted harmonic constants of the variation of latitude along the coördinate axes, that is, by finding the quantities a, b, α and β of equations (1).

The observations at a station give data for obtaining that part of the variation of latitude with the period $P=\frac{360^\circ}{\kappa}$ in the form

$$\begin{aligned}\Delta\phi &= R\cos(\kappa t-\zeta)\\ &= R\cos\zeta\cos\kappa t + R\sin\zeta\sin\kappa t.\end{aligned} \tag{21}$$

But we also have by resolving the values of x and y along the meridan of the station (longitude λ, west longitudes positive)

$$\Delta\phi = x\cos\lambda + y\sin\lambda. \tag{22}$$

The x and y are those of equations (1) and on substituting their values in (18) we get

$$\begin{aligned}\Delta\phi &= a\cos(\kappa t-\alpha)\cos\lambda + b\cos(\kappa t-\beta)\sin\lambda,\\ &= (a\cos\alpha\cos\lambda + b\cos\beta\sin\lambda)\cos\kappa t + (a\sin\alpha\cos\lambda\\ &\quad + b\sin\beta\sin\lambda)\sin\kappa t.\end{aligned} \tag{23}$$

Let us use the abbreviations

$$\left.\begin{aligned} a\cos\alpha &= m,\\ a\sin\alpha &= n,\\ b\cos\beta &= p,\\ b\sin\beta &= q.\end{aligned}\right\} \tag{24}$$

By substituting these expressions in (23) and comparing with the second form of (21) we get for any one station,

$$\left.\begin{aligned} m\cos\lambda + p\sin\lambda &= R\cos\zeta = c,\\ n\cos\lambda + q\sin\lambda &= R\sin\zeta = s.\end{aligned}\right\} \tag{25}$$

Two stations at which R and ζ have been found give four equations, which are just sufficient to determine m, n, p, and q. From these the quantities a, b, α, and β are found by (24) and the elements of the ellipse by equations (I) to (VI).

To treat the case arising when the harmonic constants of more than two stations are available some criterion must be found for testing the accuracy with which the adopted elements of the elliptic motion represent the harmonic constants at the several stations. If we use the letters m, n, p, and q, to signify the adopted values of those quantities and denote by ϵ_t the error at any time in the representation of the latitude of a station due to the difference between its harmonic constants as found directly and as deduced from m, n, p, and q, then ϵ_t is given by the formula,

$$\epsilon_t = (m\cos\lambda + p\sin\lambda - c)\cos\kappa t + (n\cos\lambda + q\sin\lambda - s)\sin\kappa t. \tag{26}$$

The error ϵ_t assumes in the course of a period (as κt varies from 0 to 2π) all values between $-\epsilon_m$ and $+\epsilon_m$, where ϵ_m is the maximum numerical value of ϵ_t and is given by the equation

$$\epsilon_m^2 = (m\cos\lambda + p\sin\lambda - c)^2 + (n\cos\lambda + q\sin\lambda - s)^2. \tag{27}$$

The frequency of the errors of ϵ_t of a given size at a particular station does not vary at all according to the Gaussian law, large errors near the limits $\pm\epsilon_m$ being relatively more frequent than the smaller ones, but we may assume with some degree of plausibility that when the values of the ϵ_m's at a number of different stations are compared, the relative frequency of ϵ_m's of different sizes will vary according to the Gaussian law, except of course as regards sign, the ϵ_m's being essentially positive. The mean square of ϵ_t or ϵ_a^2 may be found by taking the mean value of ϵ_t^2 over a period, or

$$\epsilon_a^2 = \frac{1}{2\pi}\int_0^{2\pi/\kappa} \epsilon_t^2 dt, \tag{28}$$

t being in radians and the value of κ being taken accordingly. It is easy to show that

$$\epsilon_a^2 = \tfrac{1}{2}\,\epsilon_m^2. \tag{29}$$

In obtaining the adjusted values of the elements of the polar motion that fit most nearly to the harmonic constants of the several stations, the criterion used will be

$$[\epsilon_a^2] = \text{minimum}.$$

The square brackets indicate summation, as is usual in the theory of least squares, and the summation extends over all stations used in the adjustment. From (27) and (29) the quantity S to be minimized may be written

$$S = [\epsilon_a^2] = \tfrac{1}{2}\{[(m\cos\lambda + p\sin\lambda - c)^2 + (n\cos\lambda + q\sin\lambda - s)^2]\} \tag{30}$$

In order to have a minimum value m, n, p, and q must be so taken as to satisfy the equations

$$\frac{\partial S}{\partial m} = 0,$$

$$\frac{\partial S}{\partial p} = 0,$$

$$\frac{\partial S}{\partial n} = 0,$$

$$\frac{\partial S}{\partial q} = 0.$$

or,

$$\left.\begin{aligned} m\,[\cos^2\lambda] + p\,[\sin\lambda\cos\lambda] &= [c\cos\lambda],\\ m\,[\sin\lambda\cos\lambda] + p\,[\sin^2\lambda] &= [c\sin\lambda],\\ n\,[\cos^2\lambda] + q\,[\sin\lambda\cos\lambda] &= [s\cos\lambda],\\ n\,[\sin\lambda\cos\lambda] + q\,[\sin^2\lambda] &= [s\sin\lambda]. \end{aligned}\right\} \tag{31}$$

Square brackets indicate summation extending over all the stations used.

Equations (31) would be the normal equations that would arise if we set out from observation equations of the form

$$\left.\begin{aligned} m\cos\lambda + p\sin\lambda - c &= 0,\\ n\cos\lambda + q\sin\lambda - s &= 0, \end{aligned}\right\} \tag{32}$$

but these equations are not observation equations in the strict sense of the term. If we call the residuals of equations (32), v_1 and v_2 respectively, that is, if v_1 and v_2 are the results of substituting in the left-hand sides of (32) the values of m, n, p, and q, deduced from (32), then v_1 and v_2 have no particular significance separately, as their values depend on the epoch from which t is reckoned, but the sum of their squares is significant, for evidently

$$v_1^2+v_2^2=\epsilon_m^2=2\epsilon_a^2. \tag{33}$$

In estimating the probable errors of m, n, p, and q, as derived from (32), or the probable error of a function of these quantities, the following conventions will be adopted, which are a natural extension of the convention by which the minimum value of $[\epsilon_a^2]$ was taken as the criterion for obtaining values of m, n, p, and q. The probable errors of m, n, p, and q are computed just as if equations (32) were proper observation equations, but the mean square error of a single observation is not made to depend on v_1 alone or v_2 alone, but the following equation is used:

$$(\text{mean-square error})^2=\frac{[\epsilon_a^2]}{n-4}=\frac{[v_1^2+v_2^2]}{2n-8}, \tag{34}$$

n being the number of "observation equations," which equals twice the number of stations; the number of unknowns is of course four. The probable error, the Gaussian law being assumed, is the mean-square error multiplied by 0.6745.

The difference between the harmonic constants at a station obtained from the least-squares adjustment, and those coming directly from observations, may be considered as representing simply errors of observation or as representing a variation of the apparent latitude (having the period in question) peculiar to the station, and not explainable by the motion of the pole. The existence of such individual variations of latitude with yearly period is well established. In so far as the residuals at all stations are alike their common value may be considered as the representation of the periodic annual portion of the Kimura term by harmonic constants. (See also p. 61.)

The normal equations (31) fall into two groups of two equations each, the groups being identical in form but involving different unknowns.

With the longitudes of the 6 stations of the International Latitude Service the normal equations become numerically

$$\left.\begin{aligned}+2.1433\,m-0.1995\,p&=A,\\-0.1995\,m+3.8567\,p&=B,\end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned}+2.1433\,n-0.1995\,q&=C,\\-0.1995\,n+3.8567\,q&=D,\end{aligned}\right\} \tag{36}$$

where for compactness the following abbreviations are used:

$$\left.\begin{aligned}A&=[R\cos\zeta\cos\lambda]=[c\cos\lambda],\\B&=[R\cos\zeta\sin\lambda]=[c\sin\lambda],\\C&=[R\sin\zeta\cos\lambda]=[s\cos\lambda],\\D&=[R\sin\zeta\sin\lambda]=[s\sin\lambda].\end{aligned}\right\} \tag{37}$$

The A, of course, has no connection with the same letter used to denote the semi-major axis in equations (14), (18), (20), or elsewhere.

The solution of (35) and (36) gives

$$\left.\begin{aligned} m &= +0.4688\ A+0.0243\ B, \\ p &= +0.0243\ A+0.2605\ B, \end{aligned}\right\} \quad (38)$$

$$\left.\begin{aligned} n &= +0.4688\ C+0.0243\ D, \\ q &= +0.0243\ C+0.2605\ D. \end{aligned}\right\} \quad (39)$$

When only the three stations Mizusawa, Carloforte, and Ukiah, are used with equal weights equation (31) becomes numerically

$$\left.\begin{aligned} +1.8854\ m-0.1128\ p &= A, \\ -0.1128\ m+1.1146\ p &= B, \end{aligned}\right\} \quad (40)$$

$$\left.\begin{aligned} +1.8854\ n-0.1128\ q &= C, \\ -0.1128\ n+1.1146\ q &= D. \end{aligned}\right\} \quad (41)$$

The solution of (40) and (41) gives

$$\left.\begin{aligned} m &= +0.5336\ A+0.0540\ B, \\ p &= +0.0540\ A+0.9026\ B, \end{aligned}\right\} \quad (42)$$

$$\left.\begin{aligned} n &= +0.5336\ C+0.0540\ D, \\ q &= +0.0540\ C+0.9026\ D. \end{aligned}\right\} \quad (43)$$

By substituting the values of the individual λ's in equations (38) to (43) the values of m, p, etc., may readily be expressed in terms of the s's and c's of the individual stations.

If it be assumed in advance that the eccentricity of the ellipse is zero—that is, that the pole moves uniformly in a circle—the adjustment is extremely simple. It has been noted (p. 10) that for motion of this sort the quantity $\zeta+\lambda$ should be constant for all stations,[4] and the amplitude, R, is clearly constant also. The process, then, is to take the arithmetic mean of the values of R at the several stations as the adjusted value of R and the mean of the several values of $\zeta+\lambda$ as the adjusted value of $\zeta+\lambda$, and from it deduce the adjusted values of the ζ's. If there is considerable range in the values of R and $\zeta+\lambda$, and if in spite of this the assumption of circular motion is made, it is rather better to form the values of $R\cos(\zeta+\lambda)$ and $R\sin(\zeta+\lambda)$ at the several stations, to use the arithmetic means of these quantities as their adjusted values and from these adjusted values to deduce the adjusted values of R and $\zeta+\lambda$. It is not difficult to see that this latter process is the one that results from specializing for the circle the discussion just given for the more general case of the ellipse.

It is not very difficult to obtain, on the assumption already made, the formulas for the probable errors of the elements of the ellipse deduced by formulas (I) to (VI) (pp. 85 and 86) from the adjusted values of m, n, p, and q. Since, however, the element of especial interest is the angle θ that gives the direction of the major axis of the ellipse (see p. 86) we shall confine ourselves to deducing formulas for the probable error of this quantity.

For this purpose we need the quantities $\frac{\partial\theta}{\partial m}$, $\frac{\partial\theta}{\partial n}$, $\frac{\partial\theta}{\partial p}$, and $\frac{\partial\theta}{\partial q}$. From (IV) by expanding $\tan 2\gamma$ and $\cos(\alpha-\beta)$ and expressing the result in terms of m, n, p, and q, by means of (II) and (20) we readily find

$$\tan 2\theta=\frac{2\,(mp+nq)}{m^2+n^2-p^2-q^2}=\frac{N}{D}, \quad (44)$$

[4] This applies to the negative rotation of the annual or 14-month components; for positive rotation $\zeta-\lambda$ is constant.

where N and D stand for the numerator and denominator of the preceding fraction. This D has no connection with the D previously used for $[s \sin \lambda]$. By differentiating both sides with respect to m

$$2\ (1+\tan^2 2\theta)\frac{\partial \theta}{\partial m}=\frac{D\dfrac{\partial N}{\partial m}-N\dfrac{\partial D}{\partial m}}{D^2},$$

or substituting in the left-hand side for $\tan 2\theta$ and simplifying,

$$\frac{\partial \theta}{\partial m}=\frac{D\dfrac{\partial N}{\partial m}-N\dfrac{\partial D}{\partial m}}{2\ (N^2+D^2)},$$

with similar expressions for $\dfrac{\partial \theta}{\partial p}$, etc.

The quantity $N^2 + D^2$ may be expressed in terms of the auxiliary quantities already used and turns out to be equal to $d^4 \cos^2 \delta$. We have $\dfrac{\partial N}{\partial m}=2p$ and $\dfrac{\partial D}{\partial m}=2m$ and restoring for convenience the values of the auxiliaries previously used, so that

$$N=2\ ab \cos\ (\alpha-\beta) \text{ and } D=a^2-b^2,$$

we get

$$\frac{\partial \theta}{\partial m}=\frac{(a^2-b^2)\ p-2\ ab\ m \cos\ (\alpha-\beta)}{d^4 \cos^2 \delta}$$

$$=\frac{(a^2-b^2)}{d^4 \cos^2 \delta}\left[p-\frac{2\ ab \cos\ (\alpha-\beta)\ m}{a^2-b^2}\right],$$

or by (7) and (12)

$$\frac{\partial \theta}{\partial m}=\frac{\cos 2\gamma}{d^2 \cos^2 \delta}[p-m \tan 2\theta]$$

$$=\frac{\cos 2\gamma}{d^2 \cos^2 \delta \cos 2\ \theta}[p \cos 2\theta-m \sin 2\theta]. \qquad (45)$$

$$=K\ (p \cos 2\theta-m \sin 2\theta), \qquad (46)$$

where

$$K=\frac{\cos 2\gamma}{d^2 \cos^2 \delta \cos 2\ \theta}=\frac{1}{d^2 \cos \delta}. \qquad (47)$$

The second form of K may be derived by considering the spherical triangle mentioned on page 85.

In a similar way we find

$$\frac{\partial \theta}{\partial n}=K\ (q \cos 2\theta-n \sin 2\theta), \qquad (48)$$

$$\frac{\partial \theta}{\partial p}=K\ (m \cos 2\theta+p \sin 2\theta) \qquad (49)$$

$$\frac{\partial \theta}{\partial q}=K\ (n \cos 2\theta+q \sin 2\theta). \qquad (50)$$

The independent quantities of which θ is a function are not ultimately m, n, p, and q, but the values of $R \cos \zeta=c$ and $R \sin \zeta=s$ in equations (32). The quantities c and s are treated as the independent observed quantities in the observation equations.

The formulas applying to such a case are given in almost any comprehensive work on the method of least squares.[6] In applying the formulas, however, the probable error of one of the so-called observation equations (32) must be taken as 0.6745 times the mean-square error from equation (34). If we call this probable error ϵ and denote by ϵ_θ the probable error of the adjusted value of θ, we find in this way

$$\epsilon_\theta^2=\epsilon^2\left\{\left(\frac{\partial\theta}{\partial m}\right)^2 Q_{11}+2\left(\frac{\partial\theta}{\partial m}\right)\left(\frac{\partial\theta}{\partial p}\right)Q_{12}+\left(\frac{\partial\theta}{\partial p}\right)^2 Q_{22}\right.$$
$$\left.+\left(\frac{\partial\theta}{\partial n}\right)^2 Q_{11}+2\left(\frac{\partial\theta}{\partial n}\right)\left(\frac{\partial\theta}{\partial q}\right)Q_{12}+\left(\frac{\partial\theta}{\partial q}\right)^2 Q_{22}\right\}. \quad (51)$$

If ϵ_θ is to be in degrees instead of in radians, the factor $(180°/\pi)^2$ must be introduced into the right-hand side of (51). The quantities Q_{11}, Q_{12}, and Q_{22} are the coefficients of A and B in the solution of the normal equations. Thus, when all six stations are used with equal weight, the Q's from equations (38) or (39) are as follows:

$$\left.\begin{aligned}Q_{11}&=+0.4688,\\ Q_{12}&=+0.0243,\\ Q_{22}&=+0.2605.\end{aligned}\right\} \quad (52)$$

When only the 3 stations, Mizusawa, Carloforte, and Ukiah, are used with equal weight the Q's from (42) and (43) are

$$\left.\begin{aligned}Q_{11}&=+0.5336,\\ Q_{12}&=+0.0540,\\ Q_{22}&=+0.9026.\end{aligned}\right\} \quad (53)$$

The probable error of θ should not be affected by the axes chosen; that is, if the x axis were chosen in another meridian than that of Greenwich and if the longitudes of m, n, p, and q, and of the Q's were computed for the new axes, these new values used in equations (46) to (51) should give the same value of ϵ_θ. This fact may be verified by direct transformation of no great intrinsic difficulty but of considerable length; in fact ϵ^2 is unchanged by transformation, and so is each line of (51) within the braces. This fact enables us to express ϵ_θ in terms of quantities depending on the dimensions of the adjusted ellipse itself, and not on the axes to which it is referred. The result, after introducing the factor $(180/\pi)^2$ so as to give ϵ_θ in degrees, is

$$\epsilon_\theta^2=\left(\frac{180}{\pi}\right)^2\frac{\epsilon^2}{2d^2\cos^2\delta}\left\{Q_{11}+Q_{22}-\cos\delta\,[(Q_{11}-Q_{22})\cos 2\theta+2Q_{12}\sin 2\theta]\right\}. \quad (54)$$

The square brackets here have no special significance as a sign of summation but are merely the usual symbol of aggregation. The quantities d and δ are independent of the particular axes used, as is seen from equations (18) to (20), for it is easy to show that $d^2=A^2+B^2$ and $d^2\cos\delta=A^2-B^2$. The quantity within the braces is numerically invariant for a rotation of the axes through a given angle; in fact, the portions $[Q_{11}+Q_{22}]$ and $[(Q_{11}-Q_{22})\cos 2\theta+2Q_{12}\sin 2\theta]$ will be found to be separately invariant.

[6] For example, Wright and Hayford's Adjustment of Observations (2nd edition) p. 137, or Helmert's Die Ausgleichungsrechnung nach der Methode der Kleinsten Quadrate (2d ed.) p. 180.

If we accept the hypothesis of a Kimura term that is constant at all stations, the natural procedure is then to combine the expression for the annual portion of the Kimura term with the expression for the variation of latitude due to the motion of the pole. Let the annual portion of the Kimura term be represented by $h \cos \kappa t + l \sin \kappa t$ and added to the right-hand side of equation (23), it being understood that only the annual part of the apparent variation of latitude is under consideration. By a process similar to that on pages 87 to 89 we find, instead of the four "normal" equations (31), six equations

$$\left.\begin{aligned}
&m\,[\cos^2\lambda]+p\,[\sin\lambda\cos\lambda]+h\,[\cos\lambda]=[c\cos\lambda]=A,\\
&m\,[\sin\lambda\cos\lambda]+p\,[\sin^2\lambda]+h\,[\sin\lambda]=[c\sin\lambda]=B,\\
&m\,[\cos\lambda]+p\,[\sin\lambda]+n\,h=[c]=E,\\
&n\,[\cos^2\lambda]+q\,[\sin\lambda\cos\lambda]+l\,[\cos\lambda]=[s\cos\lambda]=C,\\
&n\,[\sin\lambda\cos\lambda]+q\,[\sin^2\lambda]+\ l\,[\sin\lambda]=[s\sin\lambda]=D,\\
&n\,[\cos\lambda]+q\,[\sin\lambda]+n\,l=[s]=F.
\end{aligned}\right\}\quad(55)$$

In (55) n is the number of stations and the square brackets indicate summation, as in the theory of least squares.

These equations may be solved in terms of A, B, E, etc. For all 6 stations the solution, which corresponds to (38) and (39) is

$$\left.\begin{aligned}
m&=+0.4782\,A+0.0369\,B-0.0412\,E,\\
p&=+0.0369\,A+0.2776\,B-0.0554\,E,\\
h&=-0.0412\,A-0.0554\,B+0.1801\,E,
\end{aligned}\right\}\quad(56)$$

with similar equations for n, q, and l, namely,

$$\left.\begin{aligned}
n&=+0.4782\,C+0.0369\,D-0.0412\,F,\\
q&=+0.0369\,C+0.2776\,D-0.0554\,F,\\
l&=-0.0412\,C-0.0554\,D+0.1801\,F.
\end{aligned}\right\}\quad(57)$$

When only 3 stations are used there are just sufficient conditions to determine the unknowns, but the least-squares type of solution may be retained for uniformity. For the stations Mizusawa, Carloforte, and Ukiah, we have corresponding to (40) and (41)

$$\left.\begin{aligned}
m&=+0.5442\,A+0.0516\,B+0.0600\,E,\\
p&=+0.0516\,A+0.9032\,B-0.0136\,E,\\
h&=+0.0600\,A-0.0136\,B+0.3404\,E,
\end{aligned}\right\}\quad(58)$$

$$\left.\begin{aligned}
n&=+0.5442\,C+0.0516\,D+0.0600\,F,\\
q&=+0.0516\,C+0.9032\,D-0.0136\,F,\\
l&=+0.0600\,C-0.0136\,D+0.3404\,F.
\end{aligned}\right\}\quad(59)$$

The values of m, n, p, q, h, and l in equations (56) to (59) may readily be expressed, if desired, in terms of the individual s's and c's.

SECTION 5. THE MATHEMATICAL THEORY OF THE VARIATION OF LATITUDE.

An exhaustive treatment of the theory of the variation of latitude is not attempted here, but merely an outline, chiefly of the aspects most important in the present discussion. A number of references to the literature of the subject are given and these will supply the omitted proofs of the various statements here made.

The simplest mathematical problem that has sufficient resemblance to natural conditions to merit discussion is that of a rigid body set rotating but subject to the action of no forces. This problem was first adequately treated by Euler;[6] a more geometrical treatment was given by Poinsot.[7] The subject has aroused the interest of many mathematicians and will be found fully treated in standard works on theoretical mechanics.[8] It was to this theory that astronomers looked for a possible periodic variation in latitude, and it was still appealed to after the existence of such a variation had been established by observation but before the nature of the variation was as well understood as it is now. Moreover, the simplified form of the equations of this theory can be made to apply to the actual case of a nonrigid body merely by changing the numerical values of some of the constants. For present purposes the rotation may be sufficiently specified by referring it to rectangular axes fixed in the body but not fixed in space. The question of the position in space requires a separate treatment. Let the principal moments of inertia of the body be A, B, and C in ascending order of magnitude, and let the axes of x, y, and z coincide respectively with the axes of A, B, and C, and let p, q, and r denote respectively components of the rotation about these axes. For the case of no forces Euler's equations are:

$$\left.\begin{aligned} C\frac{dr}{dt}+(B-A)\,pq&=0,\\ A\frac{dp}{dt}+(C-B)\,qr&=0,\\ B\frac{dq}{dt}+(A-C)\,rp&=0.\end{aligned}\right\}\quad(1)$$

The general solution of these equations can be expressed in terms of Jacobi's elliptic functions, sinam u, cosam u, etc., but for any conditions resembling those of nature the modulus is so very small that the elliptic functions may be replaced by the ordinary trigonometric functions. It is more convenient to make the requisite simplifying assumptions at the start and this we shall do, although we thereby lose the advantages of greater generality and of ease in estimating the effect of omitted terms.

In applying equations (1) to the earth the northerly direction of the z axis will be treated as positive. The z axis or axis about which the moment of inertia is C, the greatest of the principal moments, is the so-called axis of figure, determined by the form of the earth, or rather by its moments of inertia; it is to be distinguished from the continually shifting axis of rotation, the direction of which is given by the instantaneous value of the resultant of the component rotations p, q, and r. Since the latitudes vary but little, the axis of rotation must never depart greatly from the axis of figure; that is, the component rotation r about the axis of figure must represent in magnitude nearly the entire rotation of the earth and p and q must be small[9] in comparison with r. The axes of x and y are in the

[6] Theoria Motus Corporum Solidorum seu Rigidorum. Greifswald, 1765; second ed. Greifswald, 1790, Chaps. XII and XIII. A possible application of the mathematics to the variability of terrestrial latitudes, at that time of course unknown to observation, is mentioned in Chap. XII, Prob. 74, Scholium 3 (p. 296 of the edition of 1790).

[7] Théorie nouvelle de la rotation des corps. Paris, 1851, particularly part 2, pp. 65–122.

[8] For example, Routh's Advanced Rigid Dynamics, fifth ed., p. 88. See also Helmert; Höhere Geodäsie, Vol. II, p. 390, or Appel, Traité de Mécanique Rationelle, 3rd Ed. Vol. II, p. 162.

[9] The ratios $p:r$ or $q:r$ would be quantities of the order of magnitude of $\Delta\phi$ in radians, $\Delta\phi$ being the variation in latitude from the mean, that is, quantities of the order of one part in a million.

equator of figure, the x axis being the axis about which the moment of inertia is the less or along which the radius of the geoid is the greater. (See fig. 12.) It is convenient to make the rotation about the z axis, which is the principal portion of the earth's rotation, positive; that is, from the positive direction of x toward the positive direction of y. If we take the axes of x and y accordingly, and compare these axes with the axes resulting from adding a north-pointing z axis to the x and y axes of the International Latitude Service (p. 33), we see that the two sets of axes can not be made to coincide,

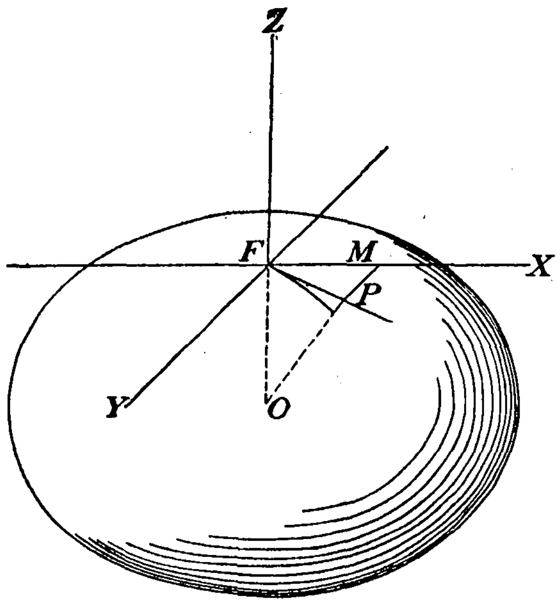

FIG. 12.—Coordinate axes for geometrical treatment of the motion of the North Pole and for the dynamical problem of the variation of latitude.

Axes are of same nature as those of International Latitude Service except that the x axis is in the meridian plane of least equatorial moment of inertia instead of in meridian plane of Greenwich. In the dynamic problem the axes of x and y are parallel to those in figure with origin at the center of the earth and with the positive direction of the y axis reversed as compared with the figure.

F = pole of inertia.
FM in angular units $= x = p/\omega$.
MP in angular units $= y = q/\omega$.

but that if the positive axes x and z coincide in the two sets, the axes of y are oppositely directed. This fact must be remembered in passing from the axes here used for the problem in mechanics to the axes of the International Latitude Service.

Since p and q are so small we neglect their product in the first of equations (1). This neglect is further justified by the fact that $B-A$ is small for the earth as compared with $C-B$ or $C-A$ in the other two equations of the set. If $A=B$, the process of dropping the term in pq is of course rigorous. From the first of the equations (1) we find

$$C\frac{dr}{dt}=0,$$

or

$$r=\text{const.}=\omega=\text{angular velocity of earth's rotation.} \qquad (2)$$

Substitute from (2) in the second of (1) and differentiate with respect to t, the time; the result may be written

$$\frac{d^2p}{dt^2}+\left(\frac{C-B}{A}\right)\omega\frac{dq}{dt}=0. \tag{3}$$

Substitute for $\frac{dq}{dt}$ in (3) from the third of (1) and we get

$$\frac{d^2p}{dt^2}+\frac{(C-A)\ (C-B)}{AB}\omega^2\ p=0.$$

or

$$\frac{d^2p}{dt^2}+k^2p=0 \tag{4}$$

where k^2 is written for $\frac{(C-A)\ (C-B)}{AB}\omega^2$, a quantity necessarily positive. The general solution of (4) is

$$p=b\cos(kt+\beta), \tag{5}$$

where b and β are constants of integration, b being restricted to small values on account of the assumptions made. The corresponding value of q comes from the second of (1) and is

$$q=\frac{-A}{(C-B)\ \omega}\frac{dp}{dt},$$

or

$$q=\sqrt{\frac{(C-A)\ A}{(C-B)\ B}}\,b\sin(kt+\beta). \tag{6}$$

The direction cosines of the resultant axis of rotation are in general $\frac{p}{\sqrt{p^2+q^2+r^2}}$, $\frac{q}{\sqrt{p^2+q^2+r^2}}$, $\frac{r}{\sqrt{p^2+q^2+r^2}}$, or since the squares of p and q are to be neglected the direction cosines are $\frac{p}{\omega}$, $\frac{q}{\omega}$, 1. To state the matter otherwise, the quantities $\frac{p}{\omega}$, $\frac{q}{\omega}$ are the sines of angular distances from the pole of rotation to the planes zy and zx, and being small may be taken as the angular distances themselves in radians. When expressed in seconds of arc and projected orthogonally on a plane tangent to the earth at the pole of figure, $\frac{p}{\omega}$, $\frac{q}{\omega}$ are the rectangular components of the variation of latitude of the same sort as are dealt with in section 4, though referred to a different set of rectangular axes, in that the x axis in section 4 is in the meridian plane of Greenwich and not, as here, in the meridian plane of the axis of the smaller principal equatorial moment of inertia. (See fig. 12.)

From equation (5) or (6) the period of the variation of latitude is seen to be $2\pi/k$. If we write out the value of k in full and take the sidereal day as unit of time, so that $\omega=2\pi$, we get for the general case

$$\text{Period in sidereal days}=\sqrt{\frac{AB}{(C-A)\ (C-B)}}. \tag{7}$$

When $A=B$, we get

$$\text{Period in sidereal days}=\frac{A}{C-A}. \tag{8}$$

According to De Sitter's[10] values the period from (8) is 304.1 sidereal days or 303.3 mean solar days; that is, about 10 months. A table of periods deduced from (7) is given below. This corresponds to the free period of the variation of latitude, a period dependent only on the properties of the body itself and not on any external forces. For a rigid earth the more nearly spherical it is the longer would be the free period. The difference between this free period and the observed free period, 14 months (432.5 mean solar days), is dealt with on page 98.

If we consider the motion of the pole as given in rectangular coordinates, $x=p/\omega$, $y=q/\omega$, p and q being taken from (5) and (6), we see that the motion of the pole is of the harmonic elliptic variety treated in section 4. This means that in its free motion, *the pole of rotation describes in the body an ellipse around the pole of figure.* The semiaxes of the ellipse, which are along the x and y axes, are, respectively, b and $b\sqrt{\frac{(C-A)\,A}{(C-B)\,B}}$. It is easy to see that the second quantity is larger than the first. The longer axis of the ellipse of polar motion is in the meridian of the y axis, that is, in the meridian of the axis of the larger of the two principal equatorial moments of inertia or the meridian of the shorter equatorial axis of the geoid. This is the same condition as prevails in the more general case referred to later in this discussion and also on pages 62–64.

For definiteness let us assume the quantity $C-\frac{1}{2}(A+B)$ to be fixed and compute the ratio of the axes of the latitude variation ellipse and the period of the variation for different values of the ratio $\frac{B-A}{C-\frac{1}{2}(A+B)}$ (called f for simplicity).

The results are shown in the table that follows:

$\frac{B-A}{C-\frac{1}{2}(A+B)}$ or f.	Ratio of Axes $\sqrt{\frac{(C-A)\,A}{(C-B)\,B}}$.	Period.
		Mean Solar Days.
0	1.000	303.3
1/100	1.005	303.3
1/50	1.010	303.3
1/40	1.011	303.3
1/25	1.020	303.3
1/10	1.051	303.7
1/5	1.105	304.8
1/4	1.133	305.7
1/3	1.183	307.0
2/5	1.224	309.5
1/2	1.290	313.2
3/5	1.361	317.9
2/3	1.413	321.7
3/4	1.481	327.2
4/5	1.526	330.9
1	1.729	350.2
6/5	1.996	379.1
7/5	2.375	424.7
8/5	2.902	505.5
17/10	3.502	575.7
9/5	4.346	695.7

[10] On Isostasy, the Moments of Inertia and the Compression of the Earth. Koninklijke Akademie van Wetenschappen te Amsterdam. Proceedings of the meeting of Apr. 23, 1915 (Vol. XVII).

The fourth line in the table, argument 1/46, corresponds to the value of $\frac{B-A}{C-\frac{1}{2}(A+B)}$ deduced from constants of Helmert's adopted gravity formula already referred to (p. 63). The corresponding ratio of the axes, 1.011, gives an ellipticity of 0.010, or an eccentricity of 0.15, which is less than the ellipticity deduced by Schweydar for a nonrigid earth, but the effect of an inequality in the equatorial moments is of the same general nature for a rigid earth as for a nonrigid earth.

The observed free period of latitude variation of about 14 months was very puzzling to the early students of the variation of latitude, who were confidently expecting a period of about 10 months, and could see no plausible reason for any other period. To be sure, an inequality in the two principal equatorial moments, A and B, would cause a lengthening of the period but this inequality must be very pronounced indeed to produce any appreciable effect on the period, as may be seen from an inspection of the last column of the table. A period of 432 days falls between the arguments $f=7/5$ and $f=8/5$; any such inequality in the moments of inertia would make itself felt very emphatically in geodetic operations, both in the values of gravity and in the deflections of the plumb line.[11] The corresponding ellipse of polar motion comes out much elongated instead of the almost circular ellipse actually observed. The difference between 10 months and 14 is then not due in any appreciable degree to the inequality between the equatorial moments of inertia. The now generally accepted explanation of the lengthening of the period was given by Newcomb,[12] and the matter may be presented very clearly in his own words:

> Mr. Chandler's discovery gives rise to the question whether there can be any defect in the theory which assigns 306 days as the time of rotation. The object of this paper is to point out that there is such a defect, namely, the failure to take account of the elasticity of the earth itself and of the mobility of the ocean.
>
> The mathematical theory of the rotation of a solid body, on which conclusions hitherto received have been based, presupposes that the body is absolutely rigid. As the earth and ocean are not absolutely rigid, we have to inquire whether their flexibility appreciably affects the conclusions. That it does can be shown very simply from the following consideration:
>
> Imagine the earth to be a homogeneous spheroid, entirely covered by an ocean of the same density with itself. It is then evident that, if the whole mass be set in uniform rotation around any axis whatever, the ocean will assume the form of an oblate ellipsoid of revolution, whose smaller axis coincides with that of rotation. Hence, the axes of rotation and of figure will be in perfect coincidence under all circumstances.
>
> To apply a similar reasoning to the case of the earth, imagine that the axis of rotation is displaced by 0."20 from that of greatest moment of inertia, which I shall call the axis of figure. Then, with an ocean of the same density as the earth, its equator would be displaced by the same amount. The ocean level would change in middle latitudes by about one inch at the maximum. But this change would have for its effect a corresponding change in the axis of figure. As the ocean covers only three-fourths of the earth the axis would be displaced by three-fourths of the distance between the two axes were ocean and earth of equal density. But as the density of the earth is some five times as great the actual change would be only one-fifth of this. It would even be less than one-fifth, because the displacement of the ocean equator would be resisted by the attraction of the earth itself. The exact amount of this resistance can not be accurately given, but I think the displacement would thereby

[11] For $f=7/5$ the inequality in gravity at the Equator would be over 1.1 centimeters above or below its mean value, a variation from minimum to maximum about as great as the mean change in gravity between Equator and pole apart from the effect of the centrifugal force. The maximum deflection of the vertical would be about 8 minutes of arc.

[12] "On the dynamics of the earth's rotation with respect to the variation of latitude," Monthly Notices of the Royal Astronomical Society, vol. 52 (1892), p. 336.

be reduced to one-half. I therefore think that one-fourteenth would be an approximate estimate of the displacement of the axis of figure [13] in consequence of the movement of the ocean. As Mr. Chandler's period requires a displacement of two-sevenths the ocean displacement only accounts for one-fourth of the difference.

The remainder is to be attributed to the elasticity of the earth itself. It is evident that the flexure caused by the noncoincidence of the two axes tends to distort the earth into a spheroid of the same form as that which the ocean assumes, and thus to bring the two axes together.

We have now to show how this deformation of the earth changes the time of revolution. Let us imagine ourselves to be looking down upon the North Pole and let P be the actual mean pole of the earth when the two axes are in coincidence and R the end of the axis of rotation. Then, in consequence of the rotation around R, the actual pole will be displaced to a certain point, P'. Now the law of rotation of R is such that it constantly moves around the instantaneous position of P' itself. In other words, the angular motion of R at each moment is that which it would have if P' had remained at rest. Hence the angular motion as seen from P is less than that from P' in the ratio of $P'R : PR$.

But as R rotates, P' continually changes its position and rotates also, remaining on the straight line PR. Thus the time of revolution of R around P is increased in the same ratio.

Newcomb's argument may be illustrated by figure 13. Let R and R_1 be "consecutive" positions of the pole of rotation, separated by an infinitesimal interval of time dt. The corresponding positions of the actual pole of figure, as affected by the yielding of the earth, are P' and P_1' lying on the lines joining R and R_1 with P_1, the point where the pole of figure would be if the pole of rotation were to

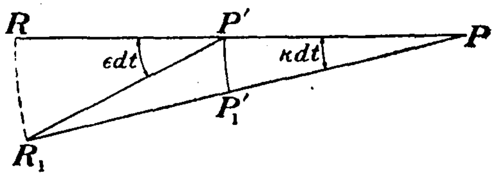

FIG. 13.—Relation between the undisturbed pole of figure, the actual pole of figure, and the pole of rotation.

coincide with P. Let ϵ be the angular velocity of the point R about P' and κ its angular velocity about P. Then for the infinitesimal distance RR_1 we have

$$RR_1 = (PR)\kappa dt = (P'R)\epsilon dt$$

or

$$\kappa : \epsilon = P'R : PR$$

as stated by Newcomb. There are thus three different poles that appear in the theory of the variation of latitude: (1) The pole of the axis of instantaneous rotation; this is the pole that is located by the astronomical observations of the International Latitude Service; in figure 13 it is the point R or R_1. (2) The point that would be the pole of figure if the pole of rotation should happen to coincide with it. This pole may be called the *undisturbed pole of figure* or *undisturbed pole of inertia*. The requirement that the pole of rotation should coincide in imagination with the pole of figure is necessary in order to be rid of the centrifugal force arising from the noncoincidence of the two poles. See under pole (3). The changes in the position of this pole are due solely to changes in the distribution of matter within and on the earth apart from any stress arising from the rotation. It is represented by P in figure 13. (3) Between P and R is

[13] See p. 100. Without the effect of the yielding of the ocean the free period of the latitude variation would, according to the estimate, be some 40 days less than the observed value.

the actual pole of figure, represented by P' or P_1'. The displacement of the actual pole away from the undisturbed pole is due to the slight centrifugal force resulting from the noncoincidence of the pole of rotation with the undisturbed pole of figure and to the yielding of the earth under this centrifugal force, a yielding that tends to deform the earth so as to displace the actual pole of figure nearer to the pole of rotation.

The equations of motion of a slowly changing body have been treated by various authors.[14] Convenient expositions and developments of their work will be found in various textbooks and treatises.[15] There have been several treatments with special reference to the variation of latitude; one of the most recent and thorough is due to Schweydar.[16] For our purposes use will be made of an article by Larmor [17] rather too long to reproduce here, in which he shows by very general reasoning how the problem of the motion of a yielding body may be reduced to that of a rigid one. Larmor proves that for the actual elastic earth we may substitute a rigid earth of about the same volume but with so much of its equatorial bulge removed as would be due to the yielding of the earth under the entire centrifugal force of rotation.[18] This would leave the earth less flattened, and since the differences of the moments $C-A$ and $C-B$ serves to measure the flattening, we must replace these differences by something smaller by about 3/10 of itself. While the relative changes in the differences of the moments are considerable, the change in any one moment is relatively small when compared with the moment itself. The smaller flattening of the equivalent rigid body corresponds to a longer free period. The free period is thus about $303/(1-0.3)=433$ mean solar days.

A similar calculation on the ratio of the axes of the ellipse of polar motion shows that the yielding of the earth makes the ellipse flatter. The effect of removing the centrifugal force is represented by a quantity proportional to $P_2(\sin\phi)$, where P_2 is the Legendre's zonal harmonic of the second degree and ϕ is the latitude. The inequality in the two moments A and B may be represented by another surface spherical harmonic and the two harmonics may be superposed when neither is large, which is the supposition here made. The quantity to be deducted from each difference of moments $C-A$ and $C-B$ is then the same for small differences between A and B. The rigid body equivalent to the yielding body will then have for a ratio of axes of the polar ellipse

$$\sqrt{\frac{(C-A-x)A}{(C-B-x)B}},$$

---

[14] Liouville: Additions à la Connaissance des Temps pour 1859, or Journal des Mathématiques pures et appliquées, vol. 3 (1858). Gylden: Recherches sur la rotation de la Terre; Actes de la Société royale des Sciences d' Upsal, vol. 8 (1871). Thomson: Appendix C to Darwin's article on the Influence of Geological Changes on the Earth's Axis of Rotation; Philosophical Transactions of the Royal Society of London, pt. 1, vol. 167 (1877), p. 271; or Darwin's Scientific Papers, vol. 3, p. 1.

[15] Routh: Advanced Rigid Dynamics; 5th ed., p. 17. Lamb: Higher Mechanics (Cambridge, Eng., 1920), p. 171. Tisserand: Mécanique Céleste; Vol. II, Chap. XXX. Helmert: Höhere Geodäsie; vol. 2, p. 408.

[16] Die Bewegung der Drehachse der Elastischen Erde in Erdkörper und im Raume: Astronomische Nachrichten, vol. 203 (1916), p. 101.

[17] The relation of the earth's free precessional nutation due to its resistance against tidal deformation, Proceedings of the Royal Society of London, sec. A, vol. 82 (1909), p. 89.

[18] If the entire centrifugal force were removed, it is to be presumed that the earth would in time yield plastically and take on a nearly spherical form. This, however, is not what is meant. The yielding is to be taken as elastic, that is, proportional to the force and not increasing with the time, except for the yielding of the oceanic waters already mentioned. The effect of the rearrangement of the water on the equatorial bulge, or more precisely on $C-A$, must be allowed for in estimating the purely elastic yielding. The material of the earth being what it is, if the centrifugal force were removed, the equatorial bulge would be reduced by about 3/10 of its present value.

instead of

$$\sqrt{\frac{(C-A)A}{(C-B)B}}.$$

The changes in A and B themselves are neglected in comparison with the change in $C-A$ and $C-B$. For the earth the quantity x is about 3/10 of the quantity $\frac{1}{2}$ $(A+B)$. If the ratio $f=(B-A)/[C-\frac{1}{2}(A+B)]$ is equal to 1/46, as deduced by Helmert from gravity observations, then the ratio of the axis for a rigid earth is 1.011 as already given in the table on page 97 and for a yielding earth 1.016; the corresponding eccentricities are, respectively, 0.15 and 0.18. This result of this rough and ready calculation agrees with Schweydar's result as based on a more elaborate method of calculation.

The reduction of 0.3 for the yielding of the earth includes the effect of the yielding of the ocean waters. On account of the irregular distribution of land and water on the globe this effect is not quite symmetrical about the earth's axis of rotation; this dissymmetry affects the ratio, the shape of the ellipse of polar motion, and also the direction of its major axis, so that the meridian in which the latter lies would no longer coincide with the meridian of the larger principal equatorial moment of inertia. The problem of correcting for the irregular distribution of land and water on a rigid earth is somewhat similar to the problem of correcting the equilibrium theory of the tides for the distribution of land and water, and has been practically solved by Darwin and Turner[19] in accordance with the ideas of Lord Kelvin. The work of Darwin and Turner shows that the equilibrium tides at any point in the ocean are almost the same as if the earth were entirely covered with water and a preliminary computation for the analogous problem of the effect of the arrangement of land and water on the motion of the pole shows that this effect is comparatively small also. The problem is further complicated by the elastic yielding of the earth under the irregularly distributed pressure of the ocean water. The results on page 60 are, of course, uncorrected.

The general problem has been treated by Brill[20] but his results do not appear to be immediately applicable to the problem in hand. Brill does, however, reach the conclusion that Newcomb's estimate (p. 99) of the effect of the yielding of the ocean water on the free period of latitude variation, though confessedly a rough estimate, is in close agreement with Brill's own more elaborate calculations. An attempt will be made to evaluate the correction for a distribution of land and water more closely conforming to the actual one than the one used for simplicity in the preliminary estimate, and also to estimate the effect due to the elastic yielding of the earth under the irregular distribution of water pressure. The results, when applied as corrections to the quantities specifying the shape and position of the ellipse of free polar motion, should render the results as to the moments of inertia of the earth derived from the corrected ellipse directly comparable with those from gravity observations.

When the moments of inertia are unequal the actual pole of figure (P' in figure 13) is no longer exactly on the line RP joining the

[19] "On the correction of the equilibrium theory of tides for the continents," Proceedings of the Royal Society of London, vol. 40 (1886), p. 303, or Darwin's Scientific Papers, Vol. I (Cambridge, England, 1907, p. 328). The theory does not treat the self-attraction of the water, or rather it supposes the self-attraction to be the same as if the earth were completely covered with water.

[20] Über die Elastizität der Erde (doctor's thesis), Göttingen, 1908.

undisturbed pole of figure with the pole of rotation. Schweydar computes that when the pole of rotation describes an ellipse having a ratio of axes equal to 1.016, the actual pole of figure describes an ellipse in which the ratio is 1.038, a somewhat flatter ellipse than that described by the pole of rotation.

So far the discussion has been of the free vibration only. Owing to changes taking place in the distribution of matter in the earth the undisturbed poles of figure shift slightly. The most conspicuous changes in the distribution of masses are approximately periodic with an annual period, and are due to changes of like period in barometric pressures, rainfall, ocean currents, etc.

Corresponding to terms of any period in the motion of the undisturbed pole of inertia there will be terms of like period in the motion of the pole of rotation. The portion of the polar motion expressed by terms of the same period as those that express the motion of the undisturbed pole of figure are said to represent the forced motion of the pole of rotation. The form of path represented by the motion in the free period is always the same, that is, it is always an ellipse of fixed eccentricity lying in a fixed direction or a circle as the case may be, regardless of the form of the forced motion, but the size of the ellipse or circle and the initial phase of the free motion do depend on the forced motion.

When a term is present in the forced motion with a period approximately equal to the free period, the familar phenomenon of "resonance" occurs and the amplitude of the free motion is greatly increased. The annual term, which represents the principal part of the forced motion in the case of the earth, is near enough in period to the 432.5 days of the free period to cause a considerable increase in the amplitude of the free motion. The precise amount of the increase depends on initial conditions, but the amplitude of the free motion is probably five or six times as great as it would be if the forced motion has a much longer period. It is an interesting speculation to consider whether, at some time in the past, the free period owing to a different flattening of the earth and different internal conditions might not have approximated closely to a year. The free motion of the pole would then become large, and various peculiar conditions might conceivably arise which, for a yielding body, can not be followed by the present theory, since the latter is limited to small displacements of the pole.[21]

The equations connecting the motion of the pole of rotation with that of the undisturbed pole of figure are stated below without proof. Let x and y be the rectangular coordinates of the pole of rotation projected on a plane tangent to the earth at some point near the mean pole, and let ξ and η be the rectangular coordinates of the undisturbed pole of figure referred to the same axes. The equations in question are then

$$\left.\begin{aligned}\frac{dx}{dt}&=\kappa(y-\eta),\\ \frac{dy}{dt}&=-\kappa(x-\xi).\end{aligned}\right\}\qquad(9)$$

[21] For a rigid body the solution of equations (1) in terms of the elliptic functions (p. 94) is perfectly general.

Here $\kappa=\frac{2\pi}{P}$, P being the free period expressed in the same unit as the time t. It should be said that the axes used are similar to those of the International Latitude Service used in Chapter IV in that the y axis is so taken that the rotation of the earth as seen from the North Pole and the free motion of the pole both involve negative rotations. The direction of the x axis, however, is not necessarily toward Greenwich as it is for the x axis of the International Latitude Service. It should be said also that equations (9) apply to the case of equal equatorial moments of inertia, that is, to a circular free path of the pole of rotation when the forced motion is periodic. The more general case of unequal equatorial moments does not appear to have been fully worked out for the forced motion. The expressions for ξ and η are not, however, limited to periodic terms but may be of any nature whatever. The solution of the two equations (9) simultaneously, supposing ξ and η given, evidently leads to a linear differential equation of the second order with constant coefficients and a known second member. Equations (9) are fully discussed by Wanach[23] who gives three forms of the general solution and considers a number of curious special cases that arise from different assumed forms for ξ and η. The two examples that follow are adapted from Wanach.

Example 1.—Suppose that the undisturbed pole of inertia has a progressive motion given by

$$\left.\begin{aligned}\xi&=\xi_0+at,\\ \eta&=\eta_0+bt.\end{aligned}\right\}\quad(10)$$

Evidently ξ_0 and η_0 are the coordinates at the origin of time and a and b the velocities along the coordinate axes. The motion of the pole of rotation is given by

$$\left.\begin{aligned}x&=\xi_0-\frac{b}{\beta}+at+R\cos(\kappa t-\zeta),\\ y&=\eta_0+\frac{a}{\beta}+bt-R\sin(\kappa t-\zeta),\end{aligned}\right\}\quad(11)$$

where

$$\left.\begin{aligned}R\cos\zeta&=x_0-\xi_0+\frac{b}{\kappa},\\ R\sin\zeta&=y_0-\eta_0-\frac{a}{\kappa}.\end{aligned}\right\}\quad(12)$$

The terms at and bt in (11) indicate that the progressive motion of the pole of rotation takes place in the same direction and at the same rate as the progressive motion of the undisturbed pole of figure. The free motion given by the periodic terms in κt is evidently uniform and circular, the amplitude and phase depending on the initial conditions. The effect of the progressive motion on amplitude and phase is shown by the terms b/κ and a/κ in (12).

Let us suppose a progressive motion equal to that derived in Chapter II, namely, 0″.005 a year. The year being the unit of time, $\kappa=2\pi/1.184$. For definiteness we take the motion as entirely along the x axis, so that $a=0''.005$ and $b=0$. The term $-a/\kappa$ is then equal to 0″.00094. This quantity is small in comparison with an amplitude

[23] Die Chandlersche und die Newcombsche Periode der Polbewegung: Zentralbureau der Internationalen Erdmessung, publication no. 34, Berlin, 1919.

of 0″.2000, which is about that of the free motion. The maximum effect on the amplitude is ± 0″.0009 and on the phase about one-third of a degree.

Example 2.—Suppose a forced motion of the undisturbed pole of inertia given by

$$\left.\begin{aligned} \xi &= a \cos (mt-\alpha), \\ \eta &= b \cos (mt-\beta). \end{aligned}\right\} \quad (13)$$

Equations (13) represent a general elliptic motion of the type discussed in section 4 of this appendix, the period being $2\pi/m$. The values of x and y are

$$\left.\begin{aligned} x &= \frac{-\kappa}{m^2-\kappa^2}\,[\kappa a \cos (mt-\alpha) + mb \sin (mt-\beta)] + R \cos (\kappa t - \zeta), \\ y &= \frac{-\kappa}{m^2-\kappa^2}\,[\kappa b \cos (mt-\beta) - ma \sin (mt-\alpha)] - R \sin (\kappa t - \zeta), \end{aligned}\right\} \quad (14)$$

where

$$\left.\begin{aligned} R \cos \zeta &= x_0 + \frac{\kappa}{m^2-\kappa^2}\,(\kappa a \cos \alpha - mb \sin \beta), \\ R \sin \zeta &= y_0 + \frac{\kappa}{m^2-\kappa^2}\,(\kappa b \cos \beta + ma \sin \alpha). \end{aligned}\right\} \quad (15)$$

Equations (15) show the effect of a possible "resonance" between the periods of the free and the forced motion. If the periods approach equality, $m^2-\kappa^2$ decreases toward zero and the amplitude from (15) increases indefinitely.

The pole whose motion is followed by means of the observations of the International Latitude Service is the pole of rotation. When its coordinates x and y are known the coordinates of the undisturbed pole of figure may be deduced from (9). We get

$$\left.\begin{aligned} \xi &= x - \frac{1}{\kappa}\frac{dy}{dt}, \\ \eta &= y + \frac{1}{\kappa}\frac{dx}{dt}. \end{aligned}\right\} \quad (16)$$

The values of ξ and η thus depend on the derivatives of x and y, and when x and y are deduced directly from observation and are therefore affected by errors of various sorts, the derivatives deduced from the erroneous x and y are usually still more uncertain and erroneous. Equations (16) give unsatisfactory results when they are used to follow the motions of the undisturbed pole of figure by using individual observed values of x and y. About all that can be done is to get the mean periodic portions of ξ and η by using instead of the actual values of x and y, their smoothed out values obtained from harmonic constants.[27] If desired, instead of the constants themselves, the 12 or 24 mean results could be used that were put through the process of analysis in order to obtain the harmonic constants.

A numerical illustration of the relation between the pole of rotation and the undisturbed pole of figure is given by figures 14 and 15. The harmonic constants of the pole of rotation are taken from Table 12 below the horizontal line and represent the results when the

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[27] The higher harmonic terms generally have little significance in this connection. (See p. 108.)

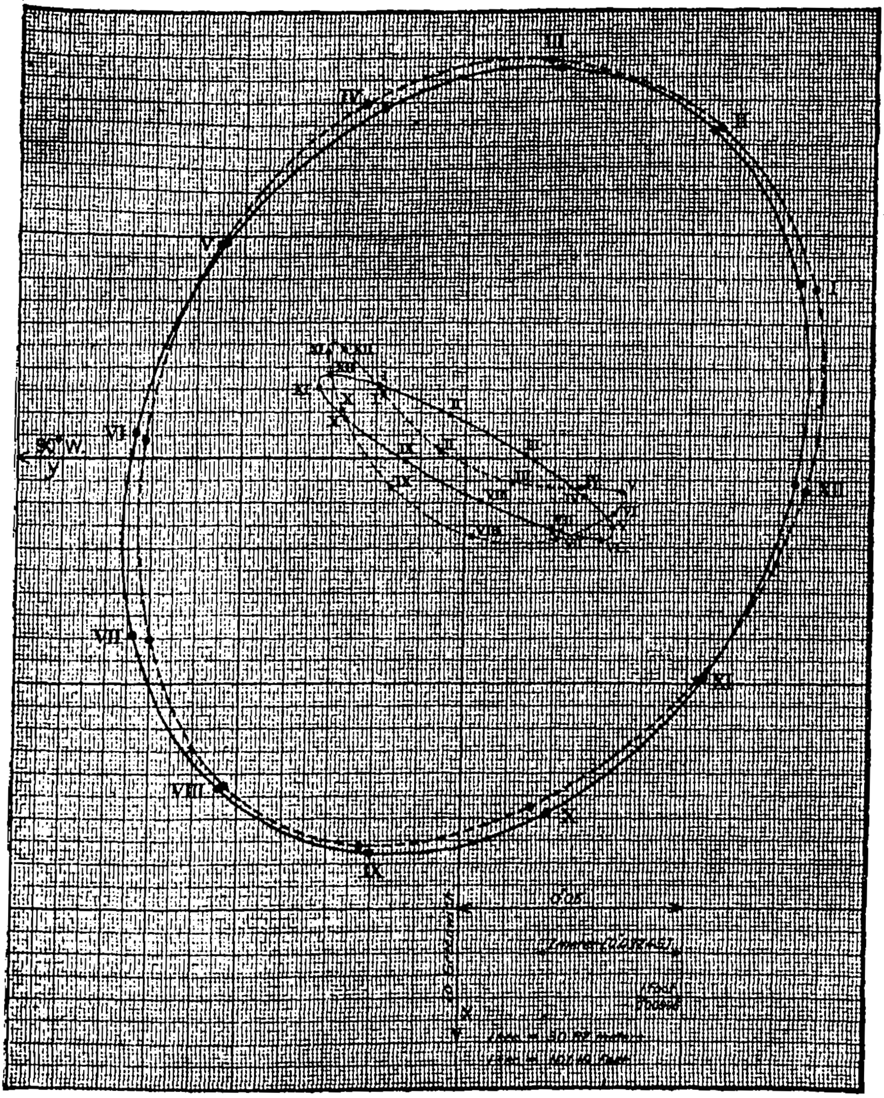

FIG. 14.—Mean annual path of the pole of rotation and of the undisturbed pole of figure: 6 stations, 1900–11.

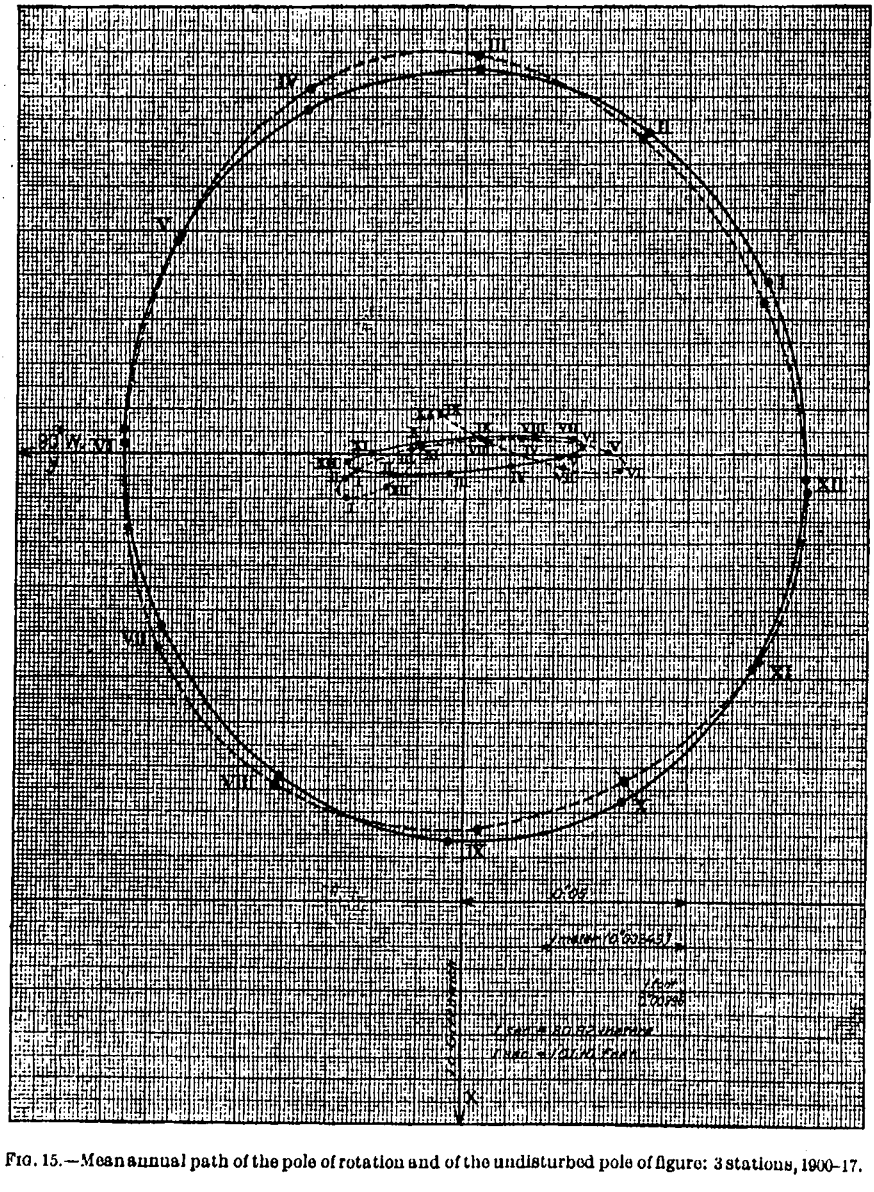

FIG. 15.—Mean annual path of the pole of rotation and of the undisturbed pole of figure: 3 stations, 1900–17.

Kimura term is included in the adjustment. The semiannual terms are taken from page 58. When only the annual terms are included both poles describe ellipses, which are shown in full lines. The smaller and flatter ellipses belong to the undisturbed pole of figure. The major axes of the two ellipses are at right angles; it is easy to show that this is true in the general case. The roman numerals I, II, III, etc., indicate the position of the poles at the beginning of each twelfth of a year, or approximately on the first day of January, February, etc.

If the harmonic constants of the pole of rotation used in the two figures be compared, it will be seen that the chief difference is in the value of β. This changes the direction of the major axis by some 32°, but the ellipses are so nearly circular that the general character of the motion of the pole of rotation is little affected. The effect on the path of the pole of figure is much more noticeable, since the ellipses are flatter; moreover the direction in which the pole of figure describes its elliptic orbit in figure 14 is opposite to what it is in figure 15.

The results of including the semiannual terms is shown by the interrupted lines in the figures. The effect on the path of the pole of rotation is comparatively small, as the paths still resemble ellipses; on the path of the pole of figure, however, the effects are relatively great and there is little resemblance to the original ellipses left, nor do the two dotted curves for the path of the pole of figure at all resemble each other.

For convenience the formula by which the curves were plotted are given here. For figure 14 (series 1900–11, 6 stations):

$$x = -0''.0390 \cos\theta - 0''.0788 \sin\theta + 0''.0011 \cos 2\theta - 0''.0012 \sin 2\theta.$$
$$y = -0''.0738 \cos\theta + 0''.0194 \sin\theta - 0''.0038 \cos 2\theta + 0''.0004 \sin 2\theta.$$

The resulting values of the coordinates of the undisturbed pole of figure are from (16):

$$\xi = -0''.0160 \cos\theta + 0''.0086 \sin\theta + 0''.0020 \cos 2\theta + 0''.0078 \sin 2\theta.$$
$$\eta = +0''.0195 \cos\theta - 0''.0268 \sin\theta - 0''.0010 \cos 2\theta + 0''.0030 \sin 2\theta.$$

For figure 15 (series 1900–17, three stations):

$$x = -0''.0385 \cos\theta - 0''.0774 \sin\theta + 0''.0045 \cos 2\theta - 0''.0007 \sin 2\theta.$$
$$y = -0''.0675 \cos\theta + 0''.0354 \sin\theta + 0''.0007 \cos 2\theta + 0''.0007 \sin 2\theta.$$
$$\xi = +0''.0034 \cos\theta + 0''.0025 \sin\theta + 0''.0062 \cos 2\theta - 0''.0024 \sin 2\theta.$$
$$\eta = +0''.0242 \cos\theta - 0''.0102 \sin\theta + 0''.0024 \cos 2\theta + 0''.0114 \sin 2\theta.$$

The quantity θ represents $\frac{360t}{365.25}$; t being the number of days from the beginning of the year. The values of θ corresponding to the points I, II, III, etc., are respectively 0°, 30°, 60°, etc.

The numerical coefficients of cos θ and sin θ in the expressions for x and y are of course, the values of $a \cos \alpha$, $a \sin \alpha$, etc., corresponding to the values of a, α, b, and β, in Table 12, or to put the matter more precisely, the coefficients in question are values of m, n, p, and q, as deduced by an adjustment of the harmonic constants of the annual component at the several stations. The figures in Table 12 were deduced from the values of m, n, p, and q, given above. The rectangular coordinates, x and y, of the mean annual path of the pole of rotation over a given period may be accurately represented by taking a sufficient number of terms of two Fourier series in θ, but this representation may be valid in a mathematical sense only and then only for x and y themselves. The derivatives of x and y with regard to the time may not be deducible from the term-by-term derivatives of the Fourier series. It is just these derivatives, especially of the higher harmonic terms, that are important in the expressions for ξ and η, the coordinates of the undisturbed pole of figure. Accordingly the expressions for x and y may be quite useless for giving expressions for ξ and η, except for the terms of lower order.

The diversity in the two figures 14 and 15 for the path of the pole of figure, when the semiannual term is included, makes it very doubtful whether the semiannual term used in either figure is an adequate representative of any natural periodic phenomenon. In studying the path of the undisturbed pole of figure it will therefore probably be desirable to limit the expressions for the motion of the pole of rotation to the terms of lowest order; that is, to the annual terms. It has been found that the annual motion of the undisturbed pole of figure is, roughly, what might be expected from the known seasonal shiftings of mass on the earth, due to barometric pressure, snowfall, etc., though it is naturally difficult to evaluate these effects exactly.[24]

It should be remembered in looking at figures 14 and 15, that while they aim to represent the entire average motion of the undisturbed pole of figure, they do not represent the entire motion of the pole of rotation, but omit a very important term, namely, the term corresponding to the free period of about 14 months. It is not necessary to include the free motion in the x and y of the pole of rotation, as used in equation (16) in order to deduce the motion of the undisturbed pole of figure, for even if the free motion were included in the expressions for x and y, it would disappear in the values of ξ and η, the coordinates of the undisturbed pole of figure.

ADDENDUM TO CHAPTER III, PAGE 48.

A recent instance of anomalous changes of latitude has attracted some attention among astronomers. Four European observatories contribute data, but their results do not all show quite the same thing. In Astronomische Nachrichten No. 5134 (vol. 214, 1921) Schnauder reports an apparent increase in the latitude of the Geodetic Institute at Potsdam of some 0."7 in 1921 over its mean value, which had been determined by a long and consistent series of observations, some as recent as 1917. Courvoisier, from the neighboring observatory of Berlin-Babelsberg, reports (through Schnauder) that the latitudes observed there increased from 0."1 above its mean value in 1920.9 to 0."5 above in 1921.8. Boccardi reports in Astronomische Nachrichten No. 5138 (vol. 215, 1921) that he had noted an increase in the latitude of Turin Observatory (Pino Torinese) somewhat smaller in amount but quite abrupt, an increase of 0."3 taking place between 1921.47 and 1921.56. Finally, the Astronomer Royal at Greenwich reports in the Monthly Notices of the Royal Astronomical Society (vol 82, p. 297, 1922) that during 1921 there was no marked irregularity in the latitude of Greenwich.

The 1921 times of maximum and minimum at Greenwich are in good agreement with those computed from the harmonic constants derived from an analysis of the years 1912–1918; the observed range is, however, smaller than the computed range. The rapid increase of latitude at Turin occurs at a time when the harmonic constants would indicate a rapid change in the same direction, but not so rapid as the one actually observed. The data from the German observatories are less detailed than at Greenwich and Turin. It may well be that these changes in Europe in 1921 and the apparent change at Lick Observatory in 1903 are simply rather extreme cases of the same kind of irregularities noted by Przbyllok (see p. 35) in connection with the observations of the International Latitude Service.

[24] See references to articles by Jeffreys and Schweydar in footnote No. 19 on p. 64.

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